Identifying the core of logic-based argumentation systems

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***Abstract*—We are interested by argumentation systems which build their arguments from a propositional knowledge base (KB), and evaluate them using Dung's acceptability semantics. We start by showing that such systems are infinite, i.e. from a finite KB, an infinite set of arguments and an infinite set of attacks among them are generated. While the construction of arguments under propositional logic is costly even in the finite case, the fact that those systems are infinite makes them completely useless.**

Then, we provide a procedure which, given an argumentation system, computes its finite sub-system, called *core*. A core considers a finite subset of arguments and a finite subset of attacks, and returns all the results of the original system. This means that a finite subset of arguments is sufficient to draw all the expected conclusions from a KB.

I. INTRODUCTION

Argumentation is a reasoning model based on the construction and evaluation of arguments. It was studied in AI for modeling agents' interactions and for reasoning about defeasible information (see [1] for an overview). For those purposes, different argumentation systems were developed. The most abstract one is that proposed in [5]. It consists of a set of arguments and an attack relation among them. The conflicts are solved using a semantics. Several instantiations of this system exist in the literature. They suppose that a knowledge base containing formulae of a logical language is given, and then define arguments and attacks from the base using the consequence operator of the underlying logic.

It is well-known that argumentation reasoning is computationally costly [6]. For instance, checking whether an argument is in every stable extension is coNP-complete. Besides, when reasoning with a concrete knowledge base, building arguments from the base is also computationally complex. Consider the case of a propositional base. An argument is usually defined as a logical proof containing a consistent subset of the base, called support, and a given statement, called conclusion. Thus, there are at least two tests to be done: a *consistency* test which is an NP-complete problem and an *inference* test (i.e. testing whether the conclusion is a logical consequence of the support) which is a co-NP-complete problem. Hence, finding the components of an argumentation system is a real challenge.

In this paper, we focus on argumentation systems built over propositional knowledge bases. We show that such systems have an infinite set of arguments, an infinite number of attacks and return an infinite number of outputs (which are the conclusions to be drawn from the base). The combination of this infiniteness with the above computational problems makes those systems difficult to deal with. We study then how to reduce the load of computation of arguments and attacks in order to make them applicable in practice. We show that such systems have *cores*. A core is a *finite sub-system* (i.e. with a finite set of arguments and a finite number of attacks) of the original one which is sufficient to return all the results of the former one. The basic idea is that most arguments generated from a knowledge base are either redundant or useless. We avoid the useless arguments in the reasoning process and classify the remaining ones into a finite number of *equivalence classes*. Each class contains equivalent arguments, that is an argument and all its redundant versions. We show that all arguments pertaining to the same class have the same status in the infinite system. Finally, we show that the core of an infinite system is a finite system which takes only one argument per class. This core returns the results of the original infinite system. This result is of great importance since it reduces considerably the complexity of the system.

The paper is structured as follows: Section II recalls the logic-based argumentation system we are interested in, and studies its properties. Section III defines the notion of a core of an argumentation system. Section IV studies the different sources of infiniteness of a set of arguments. Section V identifies the core(s) of an infinite system. Section VI is devoted to some concluding remarks and perspectives. The last section is an appendix containing the proofs of our results.

II. LOGIC-BASED ARGUMENTATION SYSTEMS

Throughout the paper, $(\mathcal{L}, \vdash)$ denotes *propositional logic* where $\mathcal{L}$ is a set of well-formed *formulae* (using the usual connectors $\neg, \wedge, \vee, \rightarrow, \ldots$), $\vdash$ denotes the *classical entailment*, and $\equiv$ is the classical logical equivalence between formulae of $\mathcal{L}$. A *knowledge base* Σ is a *finite* subset of $\mathcal{L}$. We suppose that Σ contains at least one *minimal inconsistent subset* which contains at least one consistent formula. This is equivalent to say that the attack relation of the corresponding argumentation system is not empty. Such assumption discards trivial systems where all the arguments are accepted. From this base, arguments are built as follows:

Definition 1 (Argument): An *argument* is a pair (H, h) s.t. $H \subseteq \Sigma$, H is consistent, $H \vdash h$, and $\nexists H' \subset H$ s.t. H' satisfies the three previous conditions. We call H the *support* of the argument and h its *conclusion*.

Notations: Let $a = (H, h)$ be an argument, $\texttt{Conc}(a) = h$ and $\texttt{Supp}(a) = H$. Let $\mathcal{S} \subseteq \Sigma$, $\texttt{Arg}(\mathcal{S}) = \{a \mid a$ is an argument in the sense of Def. 1 and $\texttt{Supp}(a) \subseteq \mathcal{S}\}$. Let $\mathcal{E} \subseteq \texttt{Arg}(\Sigma)$, $\texttt{Base}(\mathcal{E}) = \cup\texttt{Supp}(a)$ such that $a \in \mathcal{E}$.

The following result shows that the set of all arguments that may be built from a knowledge base is infinite.

Proposition 1: Let Σ be a propositional knowledge base. Then, the set $\texttt{Arg}(\Sigma)$ is infinite.

Since the knowledge base Σ is inconsistent, arguments may be conflicting as well. For the purpose of this paper, we will consider the relation 'undermine' proposed in [7] and largely studied in the literature.

Definition 2 (Assumption attack): Let $a, b \in \texttt{Arg}(\Sigma)$. The argument a *undermines* b, denoted $a\mathcal{R}b$, iff $\exists h \in \texttt{Supp}(b)$ s.t. $\texttt{Conc}(a) \equiv \neg h$.

The arguments of the set $\texttt{Arg}(\Sigma)$ may be undermined by an infinite number of arguments.

Proposition 2: The set $\mathcal{R} \subseteq \texttt{Arg}(\Sigma) \times \texttt{Arg}(\Sigma)$ is infinite.

Let us now define an argumentation system built over Σ.

Definition 3 (Argumentation system): An *argumentation system* (AS) built over a knowledge base Σ is a pair $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ where $\mathcal{R} \subseteq \texttt{Arg}(\Sigma) \times \texttt{Arg}(\Sigma)$ is the attack relation given in Definition 2.

Proposition 2 implies that such an AS is infinite in the sense of the next definition due to [5].

Definition 4 (Finite argumentation system): An AS is *finite* iff each argument is attacked by a finite number of arguments. It is *infinite* otherwise.

It is worth mentioning that, in this sense, an AS may be finite even if its set of arguments is infinite.

Corollary 1: The AS $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ built over a finite knowledge base Σ is infinite.

The arguments of $\texttt{Arg}(\Sigma)$ are evaluated using an acceptability semantics. For the purpose of our paper, we use stable semantics (due to [5]). Similar results can be shown for other well-known semantics, but due to space limitation, they are not included in the paper. Note also that in [4], it has been shown that the ASs given by Def. 3 always have stable extensions.

Definition 5 (Stable semantics): Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS and $\mathcal{E} \subseteq \texttt{Arg}(\Sigma)$.

- $\mathcal{E}$ is *conflict-free* iff $\nexists a, b \in \mathcal{E}$ s.t. $a\mathcal{R}b$.
- $\mathcal{E}$ is a *stable extension* iff it is conflict-free and it attacks every argument in $\texttt{Arg}(\Sigma) \setminus \mathcal{E}$.

Let $\texttt{Ext}(\mathcal{F})$ denote the set of all stable extensions of $\mathcal{F}$.

Since the sets $\texttt{Arg}(\Sigma)$ and $\mathcal{R}$ are infinite, one might expect that the number of extensions is infinite. The following result shows that this is not the case. However, each extension is infinite, that is, it contains an infinite number of arguments.

Proposition 3: Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over a finite knowledge base Σ. $\mathcal{F}$ has a finite number of stable extensions, and each stable extension of $\mathcal{F}$ is infinite.

Once the extensions of an AS are identified, a *status* is associated to each argument as follows:

Definition 6 (Status of arguments): Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS, $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its extensions and $a \in \texttt{Arg}(\Sigma)$.

- a is *skeptically* accepted iff $a \in \cap\mathcal{E}_i$, for $i = 1, \ldots, n$.
- a is *credulously* accepted iff $\exists i, j \in \{1, \ldots, n\}$ s.t. $a \in \mathcal{E}_i$ and $a \notin \mathcal{E}_j$.
- a is *rejected* iff $a \notin \cup\mathcal{E}_i$, for $i = 1, \ldots, n$.

The function $\texttt{Status}(a, \mathcal{F})$ returns the status of an argument a in the argumentation system $\mathcal{F}$.

An AS returns two other outputs: the set of inferences to be drawn from the base Σ, and the set of subbases of Σ which are returned by the different extensions.

Definition 7 (Outputs of an AS): Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over a knowledge base Σ.

- $\texttt{Output}(\mathcal{F}) = \{x \in \mathcal{L}$ s.t. $\exists a \in \texttt{Arg}(\Sigma)$, a is skeptically accepted and $x = \texttt{Conc}(a)\}$.
- $\texttt{Bases}(\mathcal{F}) = \{\texttt{Base}(\mathcal{E})$ s.t. $\mathcal{E} \in \texttt{Ext}(\mathcal{F})\}$.

It can be shown that the first output is an infinite set while the second is a finite one.

Proposition 4: Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over a finite knowledge base Σ. $\texttt{Output}(\mathcal{F})$ is infinite, and $\texttt{Bases}(\mathcal{F})$ is finite.

Since the two sets $\texttt{Arg}(\Sigma)$ and $\mathcal{R}$ are both infinite, it is not possible to compute the two outputs of $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$. Even if we focus on a particular conclusion, that is if we are interested in finding out whether a given formula follows from the base, it would be difficult to answer the question since in case the corresponding argument is attacked, then it is certainly attacked by an infinite number of arguments.

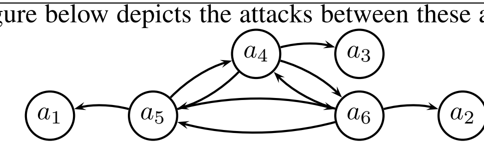
III. CORE OF AN ARGUMENTATION SYSTEM

This section motivates and defines formally the notion of a core of an AS. As said before, it has been shown in [4] that the stable extensions of an argumentation system $(\texttt{Arg}(\Sigma), \mathcal{R})$ return the maximal (for set inclusion) consistent subsets of Σ. Let us consider the following simple example:

Example 1: Let $\Sigma = \{x, \neg y, x \to y\}$ be a propositional knowledge base. Among all arguments of $\texttt{Arg}(\Sigma)$, let us consider the following ones:

$a_1 : \langle\{x\}, x\rangle$	$a_2 : \langle\{\neg y\}, \neg y\rangle$
$a_3 : \langle\{x \to y\}, x \to y\rangle$	$a_4 : \langle\{x, \neg y\}, x \wedge \neg y\rangle$
$a_5 : \langle\{\neg y, x \to y\}, \neg x\rangle$	$a_6 : \langle\{x, x \to y\}, y\rangle$

The figure below depicts the attacks between these arguments.



It can be checked that the sub-system composed only of the six arguments is sufficient to return the three maximal consistent subbases of Σ and to infer all the conclusions of $\texttt{Output}(\mathcal{F})$, where $\mathcal{F}$ is the argumentation system built over Σ by Def. 3. We say that the two systems are *equivalent*.

We propose two definitions of equivalence of ASs. According to the first one, two systems are equivalent if they return equivalent (wrt. $\equiv$) inferences from the same knowledge base Σ. The second definition states that the extensions of the two systems return the same subbases of Σ.

Definition 8 (Equivalent ASs): Let $\mathcal{F}$ and $\mathcal{G}$ be two arbitrary ASs. $\mathcal{F}$ and $\mathcal{G}$ are *EQi-equivalent* iff criterion EQi below holds:

EQ1: For all $x \in \texttt{Output}(\mathcal{F})$, $\exists y \in \texttt{Output}(\mathcal{G})$ s.t. $x \equiv y$ and vice versa. We write $\texttt{Output}(\mathcal{F}) \equiv \texttt{Output}(\mathcal{G})$.

EQ2: $\texttt{Bases}(\mathcal{F}) = \texttt{Bases}(\mathcal{G})$.

Another key notion of a core of an AS is that of *sub-system.*

Definition 9 (Sub-system): For two ASs $\mathcal{F} = (\mathcal{A}_{\mathcal{F}}, \mathcal{R}_F)$ and $\mathcal{G} = (\mathcal{A}_{\mathcal{G}}, \mathcal{R}_{\mathcal{G}})$, $\mathcal{G}$ is a *sub-system* of $\mathcal{F}$ iff

- $\mathcal{A}_{\mathcal{G}} \subset \mathcal{A}_{\mathcal{F}}$ and
- $\mathcal{R}_{\mathcal{G}} = \{(a, b) \in \mathcal{R}_{\mathcal{F}} \mid a \in \mathcal{A}_{\mathcal{G}} \ \& \ b \in \mathcal{A}_{\mathcal{G}}\}$.

Let us now define what is the core of a system.

Definition 10 (Core of an AS): Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over Σ. A *core* of $\mathcal{F}$ is an AS $\mathcal{G}$ s.t.

- $\mathcal{G}$ is finite
- $\mathcal{G}$ is a sub-system of $\mathcal{F}$
- $\mathcal{F}$ and $\mathcal{G}$ are $EQ2$-equivalent

IV. THE SOURCES OF INFINITENESS OF AN AS

There are two main sources for the infiniteness of the set $\texttt{Arg}(\Sigma)$. The first one is the fact of *duplicating* several arguments with the same support and equivalent conclusions. For instance, in Example 1, the arguments $\langle\{x\}, x \vee y\rangle$, $\langle\{x\}, \neg x \rightarrow y\rangle$ and $\langle\{x\}, (\neg x \rightarrow y) \vee (x \vee \neg x)\rangle$ are built from Σ and are in some sense redundant, or *equivalent*. Similar remark holds for the two arguments $\langle\{x\}, x\rangle$ and $\langle\{x\}, x \wedge x\rangle$. In what follows, we show that such arguments have the same status in the argumentation system and behave in the same way wrt the attack relation. Before that, let us first define formally the notion of *equivalence of two arguments*.

Definition 11 (Equivalent arguments): Let $a, b \in \texttt{Arg}(\Sigma)$. The two arguments a and b are *equivalent*, denoted by $a \sim b$, iff $\texttt{Supp}(a) = \texttt{Supp}(b)$ and $\texttt{Conc}(a) \equiv \texttt{Conc}(b)$.

It is worth noticing that the binary relation $\sim$ on $\texttt{Arg}(\Sigma)$ is an *equivalence relation* (i.e. it is reflexive, symmetric and transitive). Note also that in [3], an argument (H, h) is *more conservative* than (H', h') iff $H \subseteq H'$ and $h' \vdash h$. Thus, two arguments are equivalent (in the sense of $\sim$) iff they are more conservative than each other.

The following property shows that an argument behaves in the same way wrt $\mathcal{R}$ as its equivalent arguments.

Property 1: For every $a, a', b, b' \in \texttt{Arg}(\Sigma)$ s.t. $a \sim a'$ and $b \sim b'$, it holds that $a\mathcal{R}b$ iff $a'\mathcal{R}b'$.

The next property shows that arguments having equal supports have the same status.

Property 2: Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over a knowledge base Σ. For all $a, b \in \texttt{Arg}(\Sigma)$, if $\texttt{Supp}(a) = \texttt{Supp}(b)$ then $\texttt{Status}(a, \mathcal{F}) = \texttt{Status}(b, \mathcal{F})$.

An easy consequence of the previous result is that equivalent arguments have the same status.

Corollary 2: Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over Σ. For all $a, b \in \texttt{Arg}(\Sigma)$, if $a \sim b$ then $\texttt{Status}(a, \mathcal{F}) = \texttt{Status}(b, \mathcal{F})$.

The second source of infiniteness of $\texttt{Arg}(\Sigma)$ is due to atoms that have no occurrence within Σ but occur in conclusions of arguments. For instance, the two arguments $\langle\{x\}, x \vee z\rangle$ and $\langle\{x\}, x \vee z \vee w\rangle$ belong to the set $\texttt{Arg}(\Sigma)$ (Example 1) although z and w do not occur in Σ. We show that such arguments have no impact on the other arguments of $\texttt{Arg}(\Sigma)$. Let us first introduce some useful notations.

Notations: $\texttt{Atoms}(\Sigma)$ *is the set of atoms occurring in* Σ. $\texttt{Arg}(\Sigma)_{\downarrow}$ *is the subset of* $\texttt{Arg}(\Sigma)$ *that contains only arguments with conclusions based on* $\texttt{Atoms}(\Sigma)$. *For instance, in Example 1,* $\texttt{Atoms}(\Sigma) = \{x, y\}$. *Thus, an argument such as* $(\{x\}, x \vee z \vee w)$ *does not belong to the set* $\texttt{Arg}(\Sigma)_{\downarrow}$.

Note that the set $\texttt{Arg}(\Sigma)_{\downarrow}$ is infinite (due to equivalent arguments). Importantly, its arguments have the same status in the two systems $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ and $\mathcal{F}_{\downarrow} = (\texttt{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{\downarrow})$ where $\mathcal{R}_{\downarrow}$ is of course the restriction of $\mathcal{R}$ to $\texttt{Arg}(\Sigma)_{\downarrow}$.

Proposition 5: Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over Σ and $\mathcal{F}_{\downarrow} = (\texttt{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{\downarrow})$ its sub-system. For all $a \in \texttt{Arg}(\Sigma)_{\downarrow}$, $\texttt{Status}(a, \mathcal{F}) = \texttt{Status}(a, \mathcal{F}_{\downarrow})$.

This result is important since it shows that arguments that use external variables (i.e. variables which are not in $\texttt{Atoms}(\Sigma)$) in their conclusions can be omitted from the reasoning process. Moreover, we show next that their status is still known. It is that of any argument in $\texttt{Arg}(\Sigma)_{\downarrow}$ with the same support.

Proposition 6: Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an AS built over Σ. For all $a \in \texttt{Arg}(\Sigma) \setminus \texttt{Arg}(\Sigma)_{\downarrow}$, $\texttt{Status}(a, \mathcal{F}) = \texttt{Status}(b, \mathcal{F})$ where $b \in \texttt{Arg}(\Sigma)_{\downarrow}$ and $\texttt{Supp}(a) = \texttt{Supp}(b)$.

In sum, Proposition 5 and Proposition 6 clearly show that one can use the sub-system $\mathcal{F}_{\downarrow} = (\texttt{Arg}(\Sigma)_{\downarrow}, \mathcal{R}_{\downarrow})$ instead of $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ without losing any information. However, this system is still infinite due to redundant arguments.

V. IDENTIFYING THE CORE OF AN AS

In a previous section, we have defined a notion of a core of an argumentation system. However, we did not show whether a core always exists and if yes how to compute it. In this section we prove that each system $(\texttt{Arg}(\Sigma), \mathcal{R})$ has a core. In order to identify it, we discard the arguments which use external variables, that is we only focus on arguments of the set $\texttt{Arg}(\Sigma)_{\downarrow}$. Then, among equivalent arguments in $\texttt{Arg}(\Sigma)_{\downarrow}$, only one of them is considered. Recall that a core should be finite; thus, an important question is whether such a subset of arguments is finite? The answer is yes. Before presenting the formal result, let us introduce a useful notation.

Notations: *Let* $\mathcal{A} \subseteq \texttt{Arg}(\Sigma)$. *Then,* $\mathcal{A}/\sim$ *stands for the set of all equivalence classes of* $\mathcal{A}$ *wrt relation* $\sim$.

The following result proves that the set $\texttt{Arg}(\Sigma)_{\downarrow}$ is partitioned into a finite number of equivalence classes wrt the equivalence relation $\sim$ given in Definition 11.

Proposition 7: It holds that $|\texttt{Arg}(\Sigma)_{\downarrow}/\sim| \leq 2^n \cdot 2^{2^m}$, where $n = |\Sigma|$ and $m = |\texttt{Atoms}(\Sigma)|$.

This result is of great importance since it shows that it is possible to partition an infinite set of arguments into a finite number of classes. Note that each class may contain an infinite number of arguments. An example of such infinite class is the one which contains (but is not limited to) all the arguments having $\{x\}$ as a support and $x, x \wedge x, \ldots$ as conclusions.

Let us now define a *set of representatives* of a set given an equivalence relation on it. It is a subset of a set which contains exactly one argument of each equivalence class.

Definition 12 (Set of representatives): Let X be a set and $\sim$ an equivalence relation on it. We say that $X_c \subseteq X$ is a *set of representatives* of X if $\forall C \in X/\sim$, $\exists! a \in C \cap X_c$.
Note that for a set $\mathcal{A}_c$ of representatives of $\texttt{Arg}(\Sigma)_\downarrow$, it holds that $|\mathcal{A}_c| = |\texttt{Arg}(\Sigma)_\downarrow/\sim|$ meaning that $\mathcal{A}_c$ is finite. Note also that the set $\texttt{Arg}(\Sigma)_\downarrow$ has an infinite number of sets of representatives.

The following result shows that an AS whose set of arguments is a set of representatives is *finite*. This means that each argument is attacked by a finite number of arguments. Furthermore, the set of arguments $\mathcal{A}_c$ is finite as well.

Proposition 8: Given $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$, if $(\mathcal{A}_c, \mathcal{R}_c)$ is a sub-system of $\mathcal{F}$ where $\mathcal{A}_c$ is a set of representatives of $\texttt{Arg}(\Sigma)_\downarrow$ then the argumentation system $(\mathcal{A}_c, \mathcal{R}_c)$ is finite.
The fact that $(\mathcal{A}_c, \mathcal{R}_c)$ is a *finite sub-system* of the original argumentation system $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ is not sufficient to guarantee that it is a core of $\mathcal{F}$. In order to be so, the two systems should be equivalent wrt. either EQ1 or EQ2. The following result shows that they are equivalent wrt. EQ2.

Proposition 9: Let $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system built over Σ. If $\mathcal{G} = (\mathcal{A}_\mathcal{G}, \mathcal{R}_\mathcal{G})$ and $\mathcal{H} = (\mathcal{A}_\mathcal{H}, \mathcal{R}_\mathcal{H})$ are two sub-systems of $\mathcal{F}$ s.t. $\mathcal{A}_\mathcal{G} \subseteq \mathcal{A}_\mathcal{H}$ and $\mathcal{A}_\mathcal{G}$ is a set of representatives of $\texttt{Arg}(\Sigma)_\downarrow$ then both properties below hold:

- For all $a \in \mathcal{A}_\mathcal{G}$, $\texttt{Status}(a, \mathcal{F}) = \texttt{Status}(a, \mathcal{G}) = \texttt{Status}(a, \mathcal{H})$.
- $\texttt{Bases}(\mathcal{F}) = \texttt{Bases}(\mathcal{G}) = \texttt{Bases}(\mathcal{H})$.

This result is of great importance since it shows that a subset of arguments is sufficient to return the expected results from a knowledge base.

Proposition 10: The AS $(\mathcal{A}_c, \mathcal{R}_c)$, where $\mathcal{A}_c$ is a set of representatives of $\texttt{Arg}(\Sigma)_\downarrow$, and $\mathcal{R}_c = \mathcal{R}|_{\mathcal{A}_c}$ is a core of the argumentation system $(\texttt{Arg}(\Sigma), \mathcal{R})$.

As already said, an AS has an infinite number of cores. These cores return equivalent results.

Notation: *Let $\mathcal{E}, \mathcal{E}' \subseteq \texttt{Arg}(\Sigma)$. We write $\mathcal{E} \simeq \mathcal{E}'$ iff for all $a \in \mathcal{E}$, there exists $a' \in \mathcal{E}'$ s.t. $a \sim a'$ and vice versa. For two argumentation systems $\mathcal{F}$ and $\mathcal{G}$, we write $\texttt{Ext}(\mathcal{F}) \simeq \texttt{Ext}(\mathcal{G})$ iff for all $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$, there exists $\mathcal{E}' \in \texttt{Ext}(\mathcal{G})$ s.t. $\mathcal{E} \simeq \mathcal{E}'$ and vice versa.*

Proposition 11: Let $\mathcal{G}$ and $\mathcal{H}$ be two cores of an AS $\mathcal{F}$. It holds that: $\texttt{Output}(\mathcal{G}) \simeq \texttt{Output}(\mathcal{H})$ and $\texttt{Ext}(\mathcal{G}) \simeq \texttt{Ext}(\mathcal{H})$.
What about the the set of conclusions $\texttt{Output}(\mathcal{F})$ that may be drawn from the knowledge base Σ? The next result shows that it is equal to the closure under the classical entailment $\vdash$ of the set of outputs returned by any core of $\mathcal{F}$.

Proposition 12: Let $\mathcal{F}$ be an AS built over a knowledge base Σ and let $\mathcal{G}$ be one of its cores. $\texttt{Output}(\mathcal{F}) = \{x \in \mathcal{L}$ s.t. $\texttt{Output}(\mathcal{G}) \vdash x\}$.

Note that no core is equivalent to the original AS $\mathcal{F} = (\texttt{Arg}(\Sigma), \mathcal{R})$ wrt EQ1. This is because the set $\texttt{Output}(\mathcal{G})/\equiv$ of any core $\mathcal{G}$ is finite while $\texttt{Output}(\mathcal{F})/\equiv$ is infinite (due to conclusions containing atoms not occurring in Σ). However, the previous result shows that it is possible to compute the output of the original argumentation system from the output of one of its cores.

An important question now is how to choose a core? A simple solution would be to pick exactly one formula from each set of logically equivalent formulae. Since a lexicographic order on set $\mathcal{L}$ is usually available, we can take the first formula from that set according to that order. Instead of defining a lexicographic order, one could also choose to take the disjunctive (or conjunctive) normal form of a formula.

VI. CONCLUSION

This paper has tackled an important problem which is the identification of the core of an argumentation system. We have shown that under propositional logic, argumentation systems are infinite and involve infinite sets of arguments. However, those arguments are not all pertinent for computing the outputs of a system. We have shown that there exist key finite subsets each of which is sufficient to return all the expected results. Note that the set of arguments and attacks induced by a logical knowledge base is semantical in nature. It is of great practical importance to determine a syntactical counterpart to be effectively used for computing stable extensions,... The situation is reminiscent of that for theory revision (introduced by [2] with syntactical counterparts first due to [9]).

There are a number of ways to extend this work. One future direction consists of generalizing the notion of core of an AS to other attack relations. In this paper, we have studied an argumentation system which uses the relation *undermine*. However, other attack relations were discussed in [3] in the case of propositional logic. A second idea would be to extend this work to other logics, such as algebraic logics (e.g. logics with conditionals of the form $\alpha_1 \leftarrow \alpha_2, \ldots, \alpha_n$ such that replacement of equivalents holds) and various underlying rule-based systems [8].

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