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An abstract argumentation framework for decision making
Un cadre abstrait pour la décision argumentée

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Abstract:

This thesis proposes a novel approach for argumentation-based decision making. It suggests a Dung style general framework that takes as input different arguments and a defeat relation among them, and returns as outputs a status for each option, and a total preordering on a set of options. We study a particular class of this general framework, the one that privileges the option that is supported by the strongest argument, provided that this argument survives to the attacks. The properties of the system are investigated, and the revision of the status of a given option in light of a new argument is studied.

Cette thèse traite le problème de prise de décision sous incertitude. L'idée est d'ordonner un ensemble d'options (décisions) en fonction des conséquences de celles-ci. Nous proposons un modèle basé sur l'argumentation. Ce modèle prend en entrée un ensemble d'options, un ensemble d'arguments supportant ces options, (dits arguments pratiques), un ensemble d'arguments en faveur de croyances (arguments épistémiques) et enfin une relation de contrariété entre les arguments. Le modèle retourne en sortie un pré-ordre total sur l'ensemble d'options et un statut pour chaque option. Ce statut exprime la qualité de l'option. Nous étudions une classe particulière du modèle général. Il s'agit des systèmes dits complets. Dans de tels systèmes, l'ensemble des arguments épistémiques est vide et tous les arguments pratiques sont conflictuels. Cette classe de systèmes favorise l'option qui est supportée par l'argument le plus fort. Les propriétés de ces systèmes sont étudiées ainsi que la révision du statut d'une option à la lumière d'un nouvel argument.

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Contents

Chapter 1

Introduction

Résumé

► La prise de décision a interessé pendant longtemps beaucoup de disciplines comme la philosophie, l'économie et la psychologie. Un problème de décision revient à choisir la "meilleure" ou suffisamment "bonne" action parmi celles qui sont disponibles en fonction des informations sur l'état actuel du monde et les conséquences des actions potentielles. Notez que l'information disponible peut être incomplète ou incertaine.

L'argumentation est une activité verbale et sociale visant à augmenter ou à diminuer l'acceptabilité d'un point de vue controversé pour l'auditeur ou le lecteur, en proposant une constellation de propositions prévues pour justifier (ou réfuter) le point de vue avant un jugement raisonnable. L'argumentation est également considérée comme un modèle de raisonnement basé sur la construction et l'évaluation d'arguments. Ces arguments sont sensés soutenir/expliquer/attaquer des assertions qui peuvent être des décisions, des avis, etc...

Adopter une telle approche dans un problème de décision aurait quelques avantages évidents. En effet, non seulement un "bon" choix sera conseillé à l'utilisateur, mais également les raisons de cette recommandation. La prise de décision basée sur l'argumentation est aussi plus proche de la manière dont les humains délibèrent et prennent leurs décisions.

Dans cette thèse nous nous intéressons à une approche basée sur l'argumentation. Nous proposons un cadre général qui prend en entrée un ensemble d'options, un ensemble d'arguments supportant ces options (dits arguments pratiques), un ensemble d'arguments en faveur de croyances (arguments épistémiques) et enfin une relation de contrariété entre les arguments. Le modèle retourne en sortie un pré-ordre total sur l'ensemble d'options et un statut pour chaque option.
La deuxième principale contribution de cette thèse est une étude d'une classe particulière du cadre général de décision. Dans cette classe, la relation d'attaque entre les arguments est complète, c.-à-d., tous les arguments pratiques s'attaquent entre eux. De plus, l'ensemble des arguments épistémiques est supposé vide. Nous étudions la révision du statut d'une option à la lumière d'un nouvel argument. Nous montrons sous quelles conditions une option peut changer statut. C'est particulièrement important dans des dialogues de négociation parce que les agents choisissent les arguments qui peuvent changer le statut d'une option.◄

1.1 What is decision making?

Decision making, often viewed as a form of reasoning toward action, has raised the interest of many scholars including philosophers, economists, psychologists, and computer scientists for a long time. Any decision problem amounts to select the “best” or sufficiently “good” action(s) that are feasible among different alternatives, given some available information about the current state of the world and the consequences of potential actions. Note that available information may be incomplete or pervaded with uncertainty. Besides, the goodness of an action is judged by estimating, maybe by means of several criteria, how much its possible consequences fit the preferences or the intentions of the decision maker. This agent is assumed to behave in a *rational* way [19, 20, 26], at least in the sense that his decisions should be as much as possible consistent with his preferences. However, we may have a more requiring view of rationality, such as demanding for the conformity of decision maker’s behavior with postulates describing how a rational agent should behave [23].

1.2 Why argumentation in decision making?

Argumentation is a verbal and social activity of reason aimed at increasing (or decreasing) the acceptability of a controversial standpoint for the listener or reader, by putting forward a constellation of propositions intended to justify (or refute) the standpoint before a rational judge. Argumentation is also considered as a reasoning model based on the construction and the evaluation of interacting arguments. Those arguments are intended to support / explain / attack statements that can be decisions, opinions, etc.

Argumentation has developed into an important area of study in artificial intelligence over the last fifteen years, especially in sub-fields such as nonmonotonic reasoning (e.g. [11, 22]) and multiple-source information systems (e.g. [4, 6]). Moreover, it has been shown that such an approach is general enough to capture different existing approaches for nonmonotonic reasoning [13]. Argumentation has also been extensively used for modeling different kinds of dialogs, in particular persuasion (e.g. [5, 17, 21]), inquiry dialogs (e.g. [9]) and information seeking (e.g. [18]).

Adopting such an approach in a decision problem would have some obvious benefits. Indeed, not only would the user be provided with a “good” choice, but also with the reasons underlying this recommendation, in a format that is easy to grasp. Note that each potential choice has usually pros and cons of various strengths. Argumentation-based decision making is expected to be more akin with the way humans deliberate and finally make or understand a choice. Moreover, recently it has been shown that argumentation may play a key role in negotiation dialogs. Indeed, an offer supported by an argument has a better chance to be accepted by its receiver since this argument may influence the preferences of this receiver. These preferences are of course the result of a decision model. Thus, it is important to have an argumentation-based decision model in a negotiation context in order to handle correctly the arguments that agents may receive from other parties during a dialog.

1.3 Contribution of the master thesis

Recently, some decision criteria were articulated in terms of a two-step argumentation process:

1. an inference step in which arguments in favor/against each option are built and evaluated
2. a comparison step in which pairs of alternatives are compared on the basis of “accepted” arguments.

Thus, not only the best alternative is provided to the user but also the reasons justifying this recommendation. However, a two-step approach is not in accordance with the principle of an argumentation system, whose accepted arguments are intended to support the “good” options. Moreover, with such an approach it is difficult to define proof procedures for testing directly whether a given option may be the best one without computing the whole ordering. Finally, it is difficult to analyze how an ordering is revised in light of a new argument.

The contribution of this thesis is twofold: first, we propose a novel approach for argumentation-based decision making. We propose a Dung style general framework that takes as input different arguments and a defeat relation among them, and returns as outputs a status for each option, and a total preordering on a set of options. We study the impact of acceptability semantics on this notion of option status. The second main contribution of this thesis is a study of a particular class of the general decision framework. In this class, the attack relation between arguments is complete, i.e., all arguments supporting options attack each other. In particular, we study the revision of an option status in light of a new argument. We show under which conditions an option can / must change its status. This is particularly important in negotiation dialogs because agents choose the arguments that may change the status of a current option.

1.4 Structure of the document

This master thesis is organized in the following way. The second chapter presents the basic concepts of argumentation. We start by recalling the main steps of an argumentation process, then we introduce the most abstract argumentation framework that exists in the current literature, the one proposed by Dung in [13]. We recall different acceptability semantics and the notion of argument's status. Chapter 3 presents our general argumentation framework for decision making whose outputs are a status for each candidate decision (option) and a total preordering on the whole set of options. The impact of acceptability semantics on status of options is investigated. Chapter 4 studies a particular class of the general framework. We are particularly interested in what we call complete frameworks. The properties of the framework as well as the revision of option status in a light of a new argument are studied. In Chapter 5 we compare our model with existing works on argument-based decision making. The last chapter is devoted to some concluding remarks and future perspectives. All the proofs of the different results are in an Appendix at the end of the document.

Chapter 2

Basics of argumentation

Résumé

► Le but de ce chapitre est de présent les concepts de base de l'argumentation. Nous commençons d'abord par rappeler les étapes principales d'un processus d'argumentation, puis nous présentons le cadre d'argumentation le plus abstrait qui existe dans la littérature courante, celui proposé par Dung dans [13].

L'argumentation suit les étapes suivantes:

1. construction des arguments *pour/contre* des croyances, des options,
2. évaluation de la force de chaque argument,
3. évaluation de l'acceptabilité des arguments,
4. conclusion en utilisant un mécanisme d'inférence.

D'une manière générale, un argument est une raison de croire en une donnée ou de choisir une option parmi différentes alternatives. L'idée fondamentale derrière un modèle basé sur l'argumentation est qu'on conclut sur une information ou option si elle peut être discutée et défendue avec succès contre toute attaques. Considérons l'exemple suivant au sujet de Paul qui a deux arguments en faveur de l'heure. L'argument a indique qu'il est 14h00 puisqu'il peut le voir sur l'horloge et l'argument b indique qu'il est 15h00 puisque son ordinateur affiche cette information. Il est évident qu'il y a un conflit entre ces arguments. Ainsi, nous disons que a attaque b et que b attaque a. Il y a deux mondes possibles pour Paul : un où il est 14h00 et un où il est 15h00. Imaginez maintenant que Paul apprend que l'horloge ne fonctionne pas correctement. Ainsi, un argument c qui déclare que l'horloge ne fonctionne pas attaque l'argument a. S'il n'y a aucun argument qui attaque l'argument c, il est normal de supposer que c est vrai. Ainsi, on conclut que c est bon. Par conséquent, on peut conclure qu'il est 15h00.

Cette section détaille le cadre abstrait d'argumentation proposé par Dung. Dans ce cadre, un argument est une entité abstraite dont la structure et l'origine ne sont pas connues. Le rôle d'un argument est seulement déterminé par sa relation à d'autres arguments. Une telle représentation des arguments permet de se concentrer sur l'acceptabilité des arguments. Nous rappelons donc les différentes sémantiques d'acceptabilité.◄

The aim of this chapter is to present the basic concepts of argumentation. We start first by recalling the main steps of an argumentation process, then we introduce the most abstract argumentation framework that exists in the current literature, the one proposed by Dung in [13].

2.1 Dung's abstract argumentation framework

Argumentation is a reasoning model based on the following steps:

1. Constructing arguments in *favor/against* statements,
2. Evaluating the strength of each argument,
3. Evaluating the acceptability of the interacting arguments,
4. Concluding using an inference mechanism.

Generally speaking, an argument is a reason for believing a statement or for choosing an option among different alternatives. The basic idea behind an argumentation-based model for reasoning is that a statement (which may be either a belief or decision option) is concluded if it can be argued and defended successfully against any attacks. Let us consider the following example about Billy who has two arguments in favor of the actual time. The argument a says that it is 14:00 since he can see it on the clock and the argument b says that it is 15:00 since his computer displays that information.

It is obvious that there is a conflict between those arguments. So, we say that a attacks b and that b attacks a. In this state, there are two possible worlds for Billy: one where it is 14:00 and one where it is 15:00. So, it can be the case that argument a is true and argument b is false or it can be the case that argument b is true and argument a is false. Imagine now that Billy learns that the clock is not working properly. So, an argument c which states that the clock is not working attacks argument a. If there are no arguments which attack argument c, it is natural to suppose that c is true. So, one concludes that c is true. Consequently, a is false and b is true. Hence, one can conclude that it is 15:00.

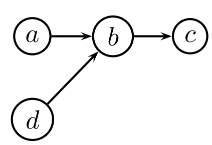
This section details the abstract argumentation framework proposed by Dung in his seminal paper [13]. In that framework, an argument is an abstract entity whose structure and origin are not known. The role of an argument is only determined by its relation to other arguments. Such representation of arguments allows one to focus on the acceptability of arguments.

Definition 1 (Argumentation framework) *An* argumentation framework *is a pair $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ where $\mathcal{A}$ is a set of arguments and $\mathcal{R}$ is a binary relationship between arguments, i.e. $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$, representing attacks among arguments. $(a, b) \in \mathcal{R}$ means that argument a attacks argument b.*

If not explicitly mentioned otherwise, we always refer to an arbitrary but fixed argumentation framework $\langle \mathcal{A}, \mathcal{R} \rangle$. It is worth mentioning that each argumentation framework can be represented by a directed graph, denoted by $\mathcal{G}_{\mathcal{AF}}$, whose nodes are the arguments of $\mathcal{A}$ and arcs are the different attacks of $\mathcal{R}$.

Definition 2 (Graph of an argumentation framework) *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. The graph associated to this system is $\mathcal{G}_{\mathcal{AF}} = (\mathcal{V}, \mathcal{X})$, where $\mathcal{V} = \mathcal{A}$ and $\mathcal{X} = \mathcal{R}$.*

Example 1 *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework such that $\mathcal{A} = \{a, b, c, d\}$ and $\mathcal{R} = \{(a, b), (b, c), (d, b)\}$. The graph associated with this framework is depicted in figure below.*



2.2 The acceptability semantics

Among the conflicting arguments it is important to know which arguments will be kept for inferring conclusions or for choosing options. Dung has defined different acceptability semantics using the notion of *extension* of arguments. An extension is a set of arguments that satisfies two minimal requirements: a coherence condition and a notion of defense.

Definition 3 (Conflict-free) *A set $\mathcal{E}$ of arguments is said to be* conflict-free *if* $\neg((\exists a \in \mathcal{A})$ $(\exists b \in \mathcal{A})$ *such that* $(a \in \mathcal{E}) \wedge (b \in \mathcal{E}) \wedge ((a, b) \in \mathcal{R}))$.

In other words, $\mathcal{E}$ is conflict-free if no argument in $\mathcal{E}$ attacks another argument in $\mathcal{E}$.

Example 2 (Example 1 continued) *Let us consider the argumentation framework of Example 1. It is clear that the set $\{a, d\}$ is conflict-free whereas $\{a, b, c\}$ is not conflict-free.*

Definition 4 (Defense) *An argument a is* defended *by a set $\mathcal{E}$ of arguments iff*

$$(\forall b \in \mathcal{A})((b, a) \in \mathcal{R}) \Rightarrow ((\exists c \in \mathcal{E})\,(c, b) \in \mathcal{R})).$$

We also say that $\mathcal{E}$ defends *a.*

Example 3 (Example 1 continued) *Let us consider again the argumentation framework of Example 1. The set $\{a\}$ (respectively $\{a, d\}$) defends the argument c.*

Let us now introduce different acceptability semantics.

2.2.1 Admissible semantics

A set of arguments is admissible if it is conflict-free and it defends all its elements against any attack.

Definition 5 (Admissible extension) $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ *be an argumentation framework, and let $\mathcal{E} \subseteq \mathcal{A}$. $\mathcal{E}$ is an* admissible extension *iff $\mathcal{E}$ is conflict-free and $(\forall a \in \mathcal{E})$ $\mathcal{E}$ defends a.*

Let us illustrate the above definition through the following simple example.

Example 4 (Example 1 continued) *Let us consider the argumentation framework of Example 1. The admissible sets of this framework are: $\mathcal{E}_0 = \emptyset$, $\mathcal{E}_1 = \{a\}$, $\mathcal{E}_2 = \{d\}$, $\mathcal{E}_3 = \{c, d\}$, $\mathcal{E}_4 = \{a, d\}$, $\mathcal{E}_5 = \{a, c\}$ and $\mathcal{E}_6 = \{a, c, d\}$.*

In [13], it has been shown that the empty set is always an admissible extension.

Property 1 ([13]) *The empty set is an admissible set of any argumentation framework $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$.*

2.2.2 Preferred semantics

Recall the previous example and note that $\emptyset \subseteq \{a\} \subseteq \{a, c\} \subseteq \{a, c, d\}$ and all those sets are admissible. To enforce the agent to take only a maximal set (with respect to set inclusion) of such a sequence, one can use the preferred semantics.

Definition 6 (Preferred extension) *A* preferred extension *of an argumentation framework $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ is a maximal (for set inclusion) admissible extension.*

Example 5 (Example 1 continued) *The only preferred extension is $\{a, c, d\}$.*

Property 2 ([13]) *Let $\langle \mathcal{A}, \mathcal{R} \rangle$ be an arbitrary argumentation framework.*

- *For each admissible set $\mathcal{E}$, there exists a preferred extension $\mathcal{E}'$ s.t. $\mathcal{E} \subseteq \mathcal{E}'$.*
- *$\langle \mathcal{A}, \mathcal{R} \rangle$ possesses at least one preferred extension.*

2.2.3 Stable semantics

The idea behind a stable extension is that it should attack all the arguments which are not in this extension.

Definition 7 (Stable extension) *Let $\mathcal{E} \subseteq \mathcal{A}$. The set $\mathcal{E}$ is a* stable extension *iff it is a preferred extension that attacks any argument in $\mathcal{A} \setminus \mathcal{E}$.*

Property 3 ([13]) *Every stable extension is a preferred extension but the converse is not true.*

Note that argumentation frameworks may not have stable extensions as shown in the following example.

Example 6 *Let $\langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework such that $\mathcal{A} = \{a\}$ and $R = \{(a, a)\}$. The empty set is a preferred extension, however it is not a stable one since the empty set does not attack the argument a.*

When the preferred and stable extensions of an argumentation system coincide, that system is said to be *coherent*.

Definition 8 (Coherent argumentation frameworks) *An argumentation framework $\mathcal{AF}$ is coherent iff its preferred extensions coincide with its stable extensions.*

In [14], it has been proved that when the directed graph associated with an argumentation system has no odd length cycles, then that system is coherent.

Theorem 1 (Coherence condition [14]) *If the graph associated with an argumentation framework $\mathcal{AF}$ has no elementary odd length cycles, then $\mathcal{AF}$ is coherent.*

2.2.4 Grounded semantics

Grounded semantics is the most skeptical semantics proposed by Dung. It gives an argument a unique status since it always returns exactly one extension. This extension is the least fixed point of a characteristic function defined as follows:

Definition 9 (Characteristic function) *Let $\mathcal{E}$ be a conflict-free set of arguments. The* characteristic function*, denoted $\mathcal{F}$, is defined as follows:*

- $\mathcal{F} : 2^{\mathcal{A}} \to 2^{\mathcal{A}}$
- $\mathcal{F}(\mathcal{E}) = \{a \mid a \text{ is defended by } \mathcal{E}\}$

Dung has shown that the above function is *monotonic* w.r.t. set inclusion. He has also shown that if the argumentation framework is finite (i.e. for each argument a there is finitely many arguments which attack a), then the least fixpoint of the function $\mathcal{F}$ can be obtained by iterative application of $\mathcal{F}$ to the empty set.

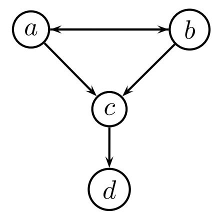
Definition 10 (Grounded extension) *Let $\langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework. The* grounded extension*, denoted* `GE`*, is the least fixpoint of the function $\mathcal{F}$.*

Note that the grounded extension is unique, i.e. an argumentation framework has only one grounded extension. This later contains all the arguments that are not attacked and the ones that are defended directly or indirectly by the non-attacked arguments.

Property 4 ([13]) *The grounded extension of an argumentation framework is a subset of the intersection of all its preferred extensions.*

The following example shows that the intersection of all preferred extensions is not always equal to the grounded extension.

Example 7 *Let us consider the argumentation framework depicted in figure below.*



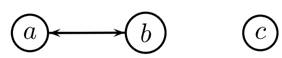
The preferred extensions of this framework are $\mathcal{E}_1 = \{a, d\}$ and $\mathcal{E}_2 = \{b, d\}$. Thus, $\mathcal{E}_1 \cap \mathcal{E}_2 = \{d\}$. However, $\texttt{GE} = \emptyset$.

2.2.5 Complete semantics

The notion of complete extensions captures the kind of agent which believes in everything it can defend.

Definition 11 (Complete extension) *Let $\mathcal{E}$ be a conflict-free set of arguments. The set $\mathcal{E}$ is a* complete extension *iff each argument which is defended by $\mathcal{E}$ belongs to $\mathcal{E}$, i.e., $\mathcal{E} = \mathcal{F}(\mathcal{E})$.*

Example 8 *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework such that $\mathcal{A} = \{a, b, c\}$ and $\mathcal{R} = \{(a, b), (b, a)\}$. The graph associated with this framework is depicted in figure below.*



The complete extensions of this framework are: $\mathcal{E}_0 = \{c\}$, $\mathcal{E}_1 = \{a, c\}$, $\mathcal{E}_2 = \{b, c\}$. Note that $\mathcal{E}_1$ and $\mathcal{E}_2$ are preferred whereas $\mathcal{E}_0$ is not.

Property 5 ([13]) *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework.*

1. *For each admissible extension $\mathcal{E}$, there exists a complete extension $\mathcal{E}'$ such that $\mathcal{E} \subseteq \mathcal{E}'$.*
2. *For each complete extension $\mathcal{E}$, there exists a preferred extension $\mathcal{E}'$ such that $\mathcal{E} \subseteq \mathcal{E}'$.*
3. *The grounded extension is exactly the intersection of all complete extensions.*
4. *Let $\mathcal{E} \subseteq \mathcal{A}$. It holds that: $\mathcal{E}$ is a stable extension $\Rightarrow$ $\mathcal{E}$ is a preferred extension $\Rightarrow$ $\mathcal{E}$ is a complete extension $\Rightarrow$ $\mathcal{E}$ is admissible.*

2.3 Status of arguments

In the previous section, we have shown which arguments may be put together and support a coherent point of view. However, these sets do not say anything on the status of a given argument. In what follows, we define the different status that an argument may have.

Definition 12 (Argument status) *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework, and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its extensions under a given semantics. Let $a \in \mathcal{A}$.*

1. *a is* skeptically accepted *iff there exists at least one non-empty extension and $a \in \mathcal{E}_i$, $\forall \mathcal{E}_i$ with $i = 1, \ldots, n$.*
2. *a is* credulously accepted *iff $\exists \mathcal{E}_i$ such that $a \in \mathcal{E}_i$ and $\exists \mathcal{E}_j$ such that $a \notin \mathcal{E}_j$.*
3. *a is* rejected *iff $\nexists \mathcal{E}_i$ such that $a \in \mathcal{E}_i$.*

A direct consequence of Definition 12 is that an argument is skeptically accepted iff it belongs to the intersection of all extensions, and that it is rejected iff it does not belong to the union of all extensions. Formally:

Property 6 *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework, and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its extensions under a given semantics. Let $a \in \mathcal{A}$.*

1. *a is skeptically accepted iff $a \in \bigcap_{i=1}^{n} \mathcal{E}_i$.*
2. *a is rejected iff $a \notin \bigcup_{i=1}^{n} \mathcal{E}_i$.*

Let $\mathtt{Sc}(\mathcal{AF})$ (respectively $\mathtt{Cr}(\mathcal{AF})$, $\mathtt{Rej}(\mathcal{AF})$) denote the set of all skeptically accepted (respectively credulously accepted, rejected) arguments of the argumentation system $\mathcal{AF}$. It can be shown that these three sets are disjoint. Moreover, their union is the set $\mathcal{A}$ of arguments.

Property 7 *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework and $\mathtt{Sc}(\mathcal{AF})$, $\mathtt{Cr}(\mathcal{AF})$, $\mathtt{Rej}(\mathcal{AF})$, its sets of arguments.*

1. $\mathtt{Sc}(\mathcal{AF}) \cap \mathtt{Cr}(\mathcal{AF}) = \emptyset$, $\mathtt{Sc}(\mathcal{AF}) \cap \mathtt{Rej}(\mathcal{AF}) = \emptyset$, $\mathtt{Cr}(\mathcal{AF}) \cap \mathtt{Rej}(\mathcal{AF}) = \emptyset$
2. $\mathtt{Sc}(\mathcal{AF}) \cup \mathtt{Cr}(\mathcal{AF}) \cup \mathtt{Rej}(\mathcal{AF}) = \mathcal{A}$.

2.4 Conclusion

In this chapter, we have recalled the basic concepts of an argumentation system. We have namely detailed the different acceptability semantics proposed by Dung in [13]. Note that other semantics have been proposed in the literature by Baroni et al. in [7] and by other researchers. However, these are not studied in this document.

Chapter 3

A general argumentation framework for decision making

Résumé

► Dans ce chapitre nous proposons un cadre général pour la décision argumentée. Nous supposons qu'on a un ensemble $\mathcal{O}$ d'options. Ces options sont distinctes et peuvent être supportés par des arguments. Les arguments supportant des options sont dits pratiques et forment l'ensemble $\mathcal{A}_o$. En plus des arguments pratiques, il y a un ensemble $\mathcal{A}_b$ d'arguments, dit épistémiques, qui supportent des croyances. Arguments épistémiques ont pour rôle de valider / invalider la partie croyance des arguments pratiques. Dans toute la suite du document nous supposons que $\mathcal{A}_o \cap \mathcal{A}_b = \emptyset$ et nous dénotons par $\mathcal{A}$ l'ensemble $\mathcal{A}_o \cup \mathcal{A}_b$.
Chaque option est reliée aux arguments qui la supportent avec la fonction $\mathcal{H}$: $\mathcal{O} \to 2^{\mathcal{A}_o}$. Cette fonction vérifie deux contraintes:

1. $\mathcal{A}_o = \bigcup_{i=1}^{n} \mathcal{H}(o_i)$, $\mathcal{O} = \{o_1, \dots, o_n\}$
2. $(\forall o \neq o')\ \mathcal{H}(o) \cap \mathcal{H}(o') = \emptyset$.

Les arguments n'ont pas forcément la même force. En effet, il se peut qu'un argument soit plus fort qu'un autre car il est construit à partir d'informations sures. Pour capturer cette notion de force, nous considérons trois relations de préférence entre arguments: $\geq_o \subseteq \mathcal{A}_o \times \mathcal{A}_o$, $\geq_b \subseteq \mathcal{A}_b \times \mathcal{A}_b$ et $\geq_m \subseteq \mathcal{A}_b \times \mathcal{A}_o$ qui exprime que les arguments épistémiques sont strictement préférés aux arguments pratiques.

Généralement les arguments peuvent être en conflit. Ces conflits sont capturés par trois telles relations comme suit:

- Soit $\mathcal{R}_b \subseteq \mathcal{A}_b \times \mathcal{A}_b$. Cette relation capture les différents conflits entre les arguments épistémiques. Cette relation est abstrait et son origine n'est pas indiquée.
- Les arguments pratiques peuvent également être en conflit. Ces conflits sont capturés par la relation $\mathcal{R}_o \subseteq \mathcal{A}_o \times \mathcal{A}_o$.
- Les arguments pratiques peuvent être attaqués par des arguments épistémiques. Cependant, on ne permet pas à des arguments pratiques d'attaquer les épistémiques. Cette relation, dénotée par $\mathcal{R}_m$ contient les paires (a, a') où $a \in \mathcal{A}_b$ et $a' \in \mathcal{A}_o$.

Nous combinons les relations $\mathcal{R}_x$ et $\geq_x$, avec $x \in \{b, o, m\}$ afin de définir les relations $\texttt{Def}_x$, comme suit : $(a, b) \in \texttt{Def}_x$ ssi $(a, b) \in \mathcal{R}_x$ et $(b, a) \notin \geq_x$. Alors nous définissons le cadre d'argumentation pour la décision comme suit: $\mathcal{AF} = \langle \mathcal{A}, \texttt{Def} \rangle$ où $\mathcal{A} = \mathcal{A}_b \cup \mathcal{A}_o$ et $\texttt{Def} = \texttt{Def}_b \cup \texttt{Def}_o \cup \texttt{Def}_m$.

Maintenant, on peut choisir une sémantique particulière et calculer toutes les extensions $\mathcal{E}_1, \ldots, \mathcal{E}_n$ sous cette sémantique. En fonction du statut des arguments, on peut calculer un statut pour chaque option. On distingue quatre statuts différents: une option peut être acceptable, rejetée, négociable ou non supportée. Dans notre cadre, un agent préfère toujours des options acceptables à les négociables. Des options négociables sont préférées à non suportée, qui sont à leur tour meilleures que des options rejetées.

Dans le reste du chapitre nous étudions les propriétés du cadre. Nous nous sommes particulièrement intéressés à l'impact de la sémantique d'acceptabilité sur le statut des options ainsi que le pré-ordre fourni. Nous montrons dans quel cas l'agent a plus ou moins d'options acceptables.◄

3.1 Introduction

As said in the introduction, solving a decision problem amounts to defining a preordering, usually a complete one, on a set of possible *choices* on the basis of the different consequences of each decision. Let us illustrate this problem through a simple example borrowed from [15].

Example 9 *The example is about having a surgery* (sg) *or not* ($\neg sg$)*, knowing that the patient has colonic polyps. The knowledge base is:*

- *not having surgery avoids having side-effects,*
- *when having cancer, having surgery avoids loss of life,*
- *the patient has colonic polyps,*
- *having colonic polyps may lead to cancer.*

The preferences of the patient are no side effects, but obviously it is more important for him to not lose his life.

In what follows, let $\mathcal{L}$ denote a logical language. From $\mathcal{L}$, a finite set $\mathcal{O}$ of n distinct *options* is identified. In the above example, the set $\mathcal{O}$ contains two options: sg and $\neg sg$. The options are assumed to be mutually exclusive, and an agent has to choose exactly one of them. The next example highlights this fact.

Example 10 *Let* $\mathcal{O} = \{$*coffee, orange juice*$\}$*. In this case, an agent has to choose between drinking a coffee or drinking orange juice. It is not possible to drink both of them or to not take a drink. If we want to consider this last possibility, we have to change the set of options into* $\mathcal{O}' = \{$*no drink, coffee, orange juice*$\}$*.*

3.2 A general argumentation framework for decision making

3.2.1 Arguments

Like any argumentation framework, our framework takes also as input a set $\mathcal{A}$ of arguments. Two kinds of arguments are distinguished: arguments supporting options, called *practical arguments* and arguments supporting beliefs, called *epistemic arguments*. Argument in favor of an option, built both from agent's beliefs and goals, tries to justify the choice, whereas an argument in favor of a belief, built only from agent's beliefs, tries to destroy arguments in favor of options. Arguments supporting options are collected in the set $\mathcal{A}_o$ and arguments supporting beliefs are collected in the set $\mathcal{A}_b$ such that $\mathcal{A}_o \cap A_b = \emptyset$ and $\mathcal{A} = \mathcal{A}_b \cup \mathcal{A}_o$. In this study, we search for general properties and do not take into account any particular structure of arguments, thus we work with the assumption that the structure of arguments is not known.

Example 11 (Example 9 cont.) *In this example, a = ["the patient has colonic polyps" and "having colonic polyps may lead to cancer"] is considered as an argument for believing that the patient may have cancer. This is an epistemic argument and it involves only beliefs. However, the argument b = ["the patient may have a cancer" and "when having cancer, having a surgery avoids loss of the life"] is an argument for having a surgery. This is a practical argument since it supports the option "having a surgery". Similarly, c = ["not having a surgery avoids having side-effects"] is a practical argument in favor of "not having a surgery".*

In this study, we assume that arguments in $\mathcal{A}_o$ highlight positive features of their conclusions, i.e., they are *in favor* of their conclusions.

Practical arguments are linked to the options they support by a function $\mathcal{H}$ defined as follows:

$$\mathcal{H}\colon \mathcal{O} \to 2^{\mathcal{A}_o} \text{ such that } \forall i, j \text{ if } i \neq j \text{ then } \mathcal{H}(o_i) \cap \mathcal{H}(o_j) = \emptyset \text{ and}$$
$$\mathcal{A}_o = \bigcup_{i=1}^{n} \mathcal{H}(o_i) \text{ with } \mathcal{O} = \{o_1, \ldots, o_n\}.$$

A practical argument a supports only one option o. We say also that o is the conclusion of the practical argument a, and we write $\texttt{Concl}(a) = o$. Note that there may exist options that are not supported by arguments (i.e. $\mathcal{H}(o) = \emptyset$).

Example 12 *Let us assume a set $\mathcal{O} = \{o_1, o_2, o_3, o_4, o_5\}$ of five options, a set $\mathcal{A}_b = \{b_1, b_2, b_3\}$ of three epistemic arguments, and finally a set $\mathcal{A}_o = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ of six practical arguments. The arguments supporting the different options are summarized in table below.*

$\mathcal{H}(o_1)$	$= \{a_1\}$
$\mathcal{H}(o_2)$	$= \{a_2, a_3, a_4\}$
$\mathcal{H}(o_3)$	$= \emptyset$
$\mathcal{H}(o_4)$	$= \{a_5\}$
$\mathcal{H}(o_5)$	$= \{a_6\}$

As pointed out in [2, 24] for instance, arguments may not have the same strength. Some arguments may be stronger than others for different reasons. For instance, because they are built from more certain information. In our particular application, three preference relations between arguments are defined. The first one, denoted by $\geq_b$, is a partial preorder [1] on the set $\mathcal{A}_b$. The second relation, denoted by $\geq_o$, is a partial preorder on the set $\mathcal{A}_o$, i.e., $\geq_o \in \mathcal{A}_o \times \mathcal{A}_o$. Finally, a third preorder, denoted by $\geq_m$ (m for *mixed* relation), captures the idea that any epistemic argument is stronger that any practical argument. The role of epistemic arguments in a decision problem is to validate or to undermine the beliefs on which practical arguments are built. Indeed, decisions should be made under certain information. Thus, $(\forall a \in \mathcal{A}_b)(\forall a' \in \mathcal{A}_o)\ (a, a') \in \geq_m \wedge (a', a) \notin \geq_m$. Note that $(a, a') \in \geq_x$ with $x \in \{b, o, m\}$ means that a is *at least as good as* a'. In what follows, $>_x$ denotes the strict relation associated with $\geq_x$. It is defined as follows: $(a, a') \in >_x$ iff $(a, a') \in \geq_x$ and $(a', a) \notin \geq_x$. Note also that we always assume that $(\forall a \in \mathcal{A}_b)(\forall a' \in \mathcal{A}_o)\ (a, a') \in \geq_m \wedge (a', a) \notin \geq_m$ and that $\geq$ is reflexive and transitive. Sometimes we do not code this fact explicitly in examples. So, one has to add all these preferences and take the transitive-reflexive closure of the obtained relation. We will sometimes write $(a, a') \in \odot$ to refer to one particular of the four possible situations: $(a, a') \in \geq \wedge (a', a) \in \geq$, meaning that the two arguments a and a' are

[1] Recall that a relation is a preorder iff it is *reflexive* and *transitive*.

indifferent for the decision maker, $(a, a') \in >$, meaning that a is strictly preferred to a', $(a', a) \in >$, meaning that a' is strictly preferred to a, $(a, a') \notin \geq \wedge (a', a) \notin \geq$, meaning that the two arguments are incomparable.

Example 13 (Example 12 cont.) *In the previous example, the epistemic arguments b_1, b_2, and b_3 are strictly preferred to the practical ones (a_1, a_2, a_3, a_4, a_5, a_6). Let us now assume that:*

- $(b_1, b_2) \in \geq_b$ *and* $(b_2, b_1) \in \geq_b$*, meaning that the two arguments are equally preferred*
- $(a_1, a_2), (a_2, a_3) \in \geq_o$
- $(\forall i \in \{1, 2, 3\})(\forall j \in \{1, 2, 3, 4, 5, 6\})\ (b_i, a_j) \in >_m$.

Now, we present some useful properties of the preference relation which will be used later.

Property 8 *Let $a, b \in \mathcal{A}$ such that $(a, b) \in \geq_x$ and $(b, a) \in \geq_x$. Then, the following hold.*

1. *If $(a, c) \in \geq_x$, for some argument c, than $(b, c) \in \geq_x$.*
2. *If $(c, a) \in \geq_x$, for some argument c, than $(c, b) \in \geq_x$.*
3. *If $(a, c) \notin \geq_x$, for some argument c, than $(b, c) \notin \geq_x$.*
4. *If $(c, a) \notin \geq_x$, for some argument c, than $(c, b) \notin \geq_x$.*

Property 9 *Let a, b, c be the arguments.*

1. *If $(a, b) \in >_x$ and $(b, c) \in \geq_x$ then $(a, c) \in >_x$.*
2. *If $(a, b) \in \geq_x$ and $(b, c) \in >_x$ then $(a, c) \in >_x$.*

3.2.2 Conflicts among the arguments

Generally arguments may be conflicting. These conflicts are captured by a binary relation on the set of arguments. In what follows, three such relations are distinguished:

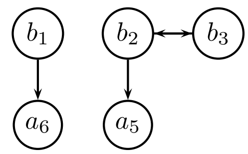
- Let $\mathcal{R}_b \subseteq \mathcal{A}_b \times \mathcal{A}_b$. This relation captures the different conflicts between epistemic arguments. This relation is abstract and its origin is not specified.
- Practical arguments may also be conflicting. These conflicts are captured by the binary relation $\mathcal{R}_o \subseteq \mathcal{A}_o \times \mathcal{A}_o$. Unlike the model proposed in [3] where arguments which support different options are always conflicting, and where arguments supporting the same option are always conflicting too, in our model we relax these two constraints.
- Finally, practical arguments may be attacked by epistemic arguments. The idea is that an epistemic argument may undermine the belief part of a practical argument. However, practical arguments are not allowed to attack epistemic ones. This avoids wishful thinking, i.e., avoids making decisions according to what might be pleasing to imagine instead of by appealing to evidence or rationality. This relation, denoted by $\mathcal{R}_m$, contains pairs (a, a') where $a \in \mathcal{A}_b$ and $a' \in \mathcal{A}_o$.

We suppose that there are no self-attacking arguments, i.e., $(\nexists a \in \mathcal{A})\ (a, a) \in \mathcal{R}_x$, with $x \in \{b, o, m\}$. There is no use of such an argument, because the fact that it is in conflict with itself means that it cannot be in any extension. So, it is rejected. Thus, we prefer not to include it at all.
Before introducing the framework, we need first to combine each preference relation $\geq_x$ (with $x \in \{b, o, m\}$) with the conflict relation $\mathcal{R}_x$ into a unique relation between arguments, denoted $\mathtt{Def}_x$, and called *defeat* relation.

Definition 13 (Defeat relation) *Let $a, b \in \mathcal{A}$. $(a, b) \in \mathtt{Def}_x$ iff $(a, b) \in \mathcal{R}_x$ and $(b, a) \notin \geq_x$.*

Let $\texttt{Def}_b$, $\texttt{Def}_o$ and $\texttt{Def}_m$ denote three defeat relations corresponding to three attack relations. Since arguments in favor of beliefs are always preferred (in the sense of $\geq_m$) to arguments in favor of options, we have $\mathcal{R}_m = \texttt{Def}_m$.

Example 14 (Example 12 cont.) *Let us assume that the following attacks hold among arguments:* $(b_2, b_3), (b_3, b_2) \in \mathcal{R}_b$, $(a_3, a_2) \in \mathcal{R}_o$, (b_1, a_6), $(b_2, a_5) \in \mathcal{R}_m$. *Thus, according to the preference relation between arguments, the following defeats occur:* $(b_2, b_3), (b_3, b_2) \in \texttt{Def}_b$, $(b_1, a_6), (b_2, a_5) \in \texttt{Def}_m$. *The graph associated with system is depicted in figure below:*



3.2.3 The outputs of the framework

In this section we put all the previous ingredients together in order to define the argumentation framework that will return the ordering $\succeq$ on the set $\mathcal{O}$ of options. It is worth mentioning that the proposed framework is preference-based since arguments are linked by a preference relation $\geq_x$, with $x \in \{b, o, m\}$.

Definition 14 (Argumentation framework for decision making) *The argumentation framework for decision making is the pair* $\mathcal{AF} = \langle \mathcal{A}, \texttt{Def} \rangle$ *where* $\mathcal{A} = \mathcal{A}_b \cup \mathcal{A}_o$ *and* $\texttt{Def} = \texttt{Def}_b \cup \texttt{Def}_o \cup \texttt{Def}_m$.
Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ *denote its extensions under a given semantics.*

Let us illustrate the notion of extensions through the following example:

Example 15 (Example 12 cont.) *There are two stable extensions:* $\mathcal{E}_1 = \{b_1,\ b_2,\ a_1,\ a_2,\ a_3,\ a_4\}$ *and* $\mathcal{E}_2 = \{b_1,\ b_3,\ a_1,\ a_2,\ a_3,\ a_4, a_5\}$.
The arguments b_1, a_1, a_2, a_3 *and* a_4 *are skeptically accepted, thus they constitute the set* $\texttt{Sc}(AF)$. *However, the argument* a_6 *is rejected whereas* b_2, b_3, a_5 *are credulously accepted.*

It is worth noticing that the decision framework $\mathcal{AF}$ is the union of two argumentation frameworks: an epistemic framework $\mathcal{AF}_b = \langle \mathcal{A}_b, \texttt{Def}_b \rangle$ and a practical one, $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$. The two frameworks are linked with the relation $\texttt{Def}_m$.

The next result states that the epistemic arguments of each admissible extension of $\mathcal{AF}$ constitute an admissible extension of the epistemic system $\mathcal{AF}_b$.

Theorem 2 *Let* $\mathcal{AF} = (\mathcal{A}_b \cup \mathcal{A}_o, \texttt{Def}_b \cup \texttt{Def}_o \cup \texttt{Def}_m)$ *be a decision system,* $\mathcal{E}_1, \ldots, \mathcal{E}_n$ *its admissible extensions, and* $\mathcal{AF}_b = (\mathcal{A}_b, \texttt{Def}_b)$ *its associated epistemic system. It holds that* $\forall \mathcal{E}_i$, *the set* $\mathcal{E}_i \cap \mathcal{A}_b$ *is an admissible extension of* $\mathcal{AF}_b$.

Let us now formally define the notion of decision framework.

Definition 15 (Decision framework) *Decision framework is a tuple* $\langle \mathcal{O}, \mathcal{A}, \texttt{Def}, \mathcal{H} \rangle$, *where* $\mathcal{O} = \{o_1, \ldots, o_n\}$ *is finite set of mutually exclusive options,* $\langle \mathcal{A}, \texttt{Def} \rangle$ *is an argumentation framework for decision making, and* $\mathcal{H}$ *is the function that for each option returns a set of practical arguments supporting that option, such that* $\forall i, j$ *if* $i \neq j$ *then* $\mathcal{H}(o_i) \cap \mathcal{H}(o_j) = \emptyset$ *and* $\mathcal{A}_o = \bigcup_{i=1}^{n} \mathcal{H}(o_i)$.

Our decision framework returns two outputs:

1. a status for each option showing the quality of the option

2. an ordering among the options.

The status of an option is defined from the status of its arguments. Indeed, an option may have one of four possible statuses: acceptable, negotiable, rejected, non-supported.

Definition 16 (Option status) *Let $o \in \mathcal{O}$.*

- *The option o is* acceptable *for the negotiating agent iff $\exists a \in \mathcal{H}(o)$ such that a is skeptically accepted.*
- *The option o is* rejected *for the negotiating agent iff $\mathcal{H}(o) \neq \emptyset$ and $\forall a \in \mathcal{H}(o)$, a is rejected.*
- *The option o is* negotiable *for the negotiating agent iff $\nexists a \in \mathcal{H}(o)$ such that a is skeptically accepted and $\exists a' \in \mathcal{H}(o)$ such that a' is credulously accepted.*
- *The option is* non-supported *iff it is neither acceptable, nor rejected or negotiable.*

Let $\mathcal{O}_a$ be the set of acceptable options, $\mathcal{O}_n$ the set of negotiable options, $\mathcal{O}_{ns}$ the set of non-supported options and $\mathcal{O}_r$ the set of rejected options.

The following simple property can be shown.

Property 10 *An option $o \in \mathcal{O}$ is non-supported iff $\mathcal{H}(o) = \emptyset$.*

The analysis in the proof of the previous property shows that there can not exist an option with more than one status, e.g. an option o can not be rejected *and* negotiable at the same time. However, an option may *change* its status in light of a new argument as we will show in next sections. So, we have the following property:

Property 11 *Let $o \in \mathcal{O}$. Offer o has exactly one status at the time.*

Example 16 (Example 12 cont.) *The options o_1 and o_2 are acceptable since they are supported by skeptically accepted arguments, the option o_3 is non-supported since it has no argument in its favor, option o_4 is negotiable and finally the option o_5 is rejected.*

Note that there are different subtypes of negotiable options. There are negotiable options that are supported only by credulously accepted arguments, and there are other negotiable options that are supported by some credulously accepted arguments and by some rejected arguments.

In [3], the status of options makes it possible to compare these options, thus to define a preference relation $\succeq$ on the set O. The basic idea is the following: acceptable options are preferred to negotiable ones. Negotiable options are themselves preferred to non-supported options, which in turn are better than rejected options. Options of the same set $\mathcal{O}_x$ with $x \in \{a, n, r, ns\}$ are equally preferred.
In what follows, $\succ$ and $\approx$ denote respectively the strict relation and the equivalence relation associated with $\succeq$. We will denote by $\mathcal{O}_x \succ \mathcal{O}_y$ that each option in $\mathcal{O}_x$ is preferred to any option in $\mathcal{O}_y$. For simplicity reasons, we will use the same notation for comparing options and sets of options.

Definition 17 (Preference between options) *Let $\mathcal{O}$ be a set of options. The following relation holds: $\mathcal{O}_a \succ \mathcal{O}_n \succ \mathcal{O}_{ns} \succ \mathcal{O}_r$, and $(\forall o_i, o_j \in \mathcal{O}_x)$ $(o_i, o_j) \in \succ$ and $(o_j, o_i) \in \succ$.*

Example 17 (Example 12 cont.) *The basic ordering is the following: o_1, $o_2 \succeq o_4 \succeq o_3 \succeq o_5$.*

3.3 Properties of the framework

The aim of this section is to study the impact of the different acceptability semantics on the status of options, consequently on the relation $\succeq$ on the set $\mathcal{O}$ of options. Before starting the study, let us first introduce a useful property that will be used for showing our results.

Property 12 *Let $k, n \in \mathcal{N}$, $1 \leq k \leq n$. Let $A_1, \ldots, A_n$ be arbitrary sets. Then:*

- $\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{k} A_i$
- $\bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{n} A_i$

In the particular case where $n \geq 1$ and $k = 1$, we have:

- $\bigcap_{i=1}^{n} A_i \subseteq A_1$
- $A_1 \subseteq \bigcup_{i=1}^{n} A_i$

In what follows, O_y^x will denote the set of options of type x under the semantics y. Thus, $x \in \{a, n, ns, r\}$ whereas $y \in \{ad, c, p, s, g\}$ (for respectively admissible, complete, preferred, stable and grounded). For instance, O_g^a stands for the set of acceptable options under grounded semantics.

3.3.1 Acceptable options

Let us start with admissible semantics. It is worth noticing that under this semantics, there are no skeptically accepted arguments, thus there are no acceptable options. This is due to the fact that the empty set is an admissible extension of the argumentation framework $\mathcal{AF} = \langle \mathcal{A}, \mathtt{Def} \rangle$. Formally:

Property 13 *Let $\mathcal{O}$ be a set of options. $O_{ad}^a = \emptyset$.*

In Chapter 2, the third bullet of Property 5 says that the grounded extension of an argumentation system is exactly the intersection of the complete extensions of the same system. Consequently, it can be shown that acceptable options are the same under the two semantics. Formally:

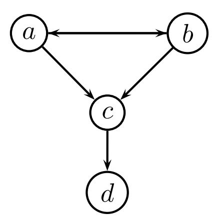
Property 14 *Let $\mathcal{O}$ be a set of options. The equality $O_g^a = O_c^a$ holds.*

The following result shows that acceptable options under grounded semantics are a subset of acceptable options under preferred semantics.

Property 15 *Let $\mathcal{O}$ be a set of options. The inclusion $O_g^a \subseteq O_p^a$ holds.*

The following example shows that the converse is not always true.

Example 18 (Example 7 cont.) *Let $\mathcal{O} = \{o_1\}$, $\mathcal{A} = \{a, b, c, d\}$. Let $\mathcal{H}(o_1) = \{d\}$. The different attacks in the sense of* `Def` *are depicted in the figure below:*



The argumentation system $\langle \mathcal{A}, \mathtt{Def} \rangle$ has two preferred extensions $\mathcal{E}_1 = \{a, d\}$ and $\mathcal{E}_2 = \{b, d\}$. Since d is in both extensions then it is skeptically accepted under preferred semantics. Consequently, the option o_1 is acceptable under preferred semantics. Thus, we have $O_p^a = \mathcal{O} = \{o_1\}$.
However, it can be checked that this argumentation system has an empty grounded extension. Thus, $O_g^a = \emptyset$. So, the option o_1 is rejected under grounded semantics.

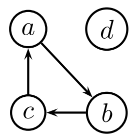
Let us now focus on the link between accepted options under preferred and stable semantics. There are two situations here: the case when the argumentation system has stable extensions and the case where there is no stable extension. The following result shows that the direction of the inclusion differs from one case to another.

Property 16 *Let $\mathcal{O}$ be the set of options, and let $\mathcal{AF} = \langle \mathcal{A}, \mathtt{Def} \rangle$ be the argumentation system for rank-ordering elements of $\mathcal{O}$.*

1. *If $\mathcal{AF}$ has no stable extensions, then $\mathcal{O}_s^a = \emptyset$ and $O_s^a \subseteq O_p^a$.*
2. *If $\mathcal{AF}$ has stable extensions, then $\mathcal{O}_p^a \subseteq \mathcal{O}_s^a$.*

The following example shows that the converse of the inclusion in the first part of the above property is not always true, i.e. presents an argumentation framework for decision making where there is no stable extension and $O_p^a \neq \emptyset$.

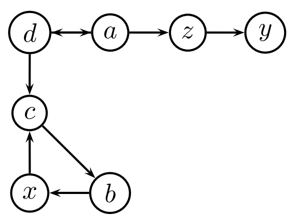
Example 19 *Let $\mathcal{O} = \{o_1\}$, $\mathcal{A} = \{a, b, c, d\}$ and $\mathcal{H}(o_1) = \{d\}$. The different attacks (in the sense of* Def*) are depicted in the figure below:*



It is clear that there is no stable extension in this system, and that there is exactly one preferred extension $\mathcal{E} = \{d\}$. Thus, under preferred semantics, d is skeptically accepted and o_1 is acceptable. So, $\mathcal{O}_p^a = \{o_1\}$ and $\mathcal{O}_s^a = \emptyset$.

The following example shows that the converse of the second part of the above property is not always true.

Example 20 *Let $\mathcal{O} = \{o_1, o_2\}$, $\mathcal{A}_b = \{a, b, c, d, x\}$ and $\mathcal{A}_o = \{y, z\}$. Let $\mathcal{H}(o_1) = \{z\}$ and $\mathcal{H}(o_2) = \{y\}$. Let us assume that* Def *is depicted in the figure below:*



There are two preferred extensions $\mathcal{E}_1 = \{b, d, z\}$ and $\mathcal{E}_2 = \{a, y\}$. It can be checked that $\mathcal{E}_1$ is a stable extension while $\mathcal{E}_2$ is not. The intersection of the two preferred extensions is empty, thus $\mathcal{O}_p^a = \emptyset$. There is an unique stable extension $\mathcal{E}_1$. Thus, its arguments are skeptically accepted. Since $\mathcal{H}(o_1) = \{z\}$, thus o_1 is accepted under stable semantics, and we have $O_s^a = \{o_1\}$.

In summary, we have the following links:

- *Case 1:* There is no stable extension.
  $\mathcal{O}_s^a = \mathcal{O}_{ad}^a = \emptyset \subseteq \mathcal{O}_g^a = \mathcal{O}_c^a \subseteq \mathcal{O}_p^a$.
- *Case 2:* There exists at least one stable extension.
  $\mathcal{O}_{ad}^a = \emptyset \subseteq \mathcal{O}_g^a = \mathcal{O}_c^a \subseteq \mathcal{O}_p^a \subseteq \mathcal{O}_s^a$.

The above result shows that admissible semantics does not provide very rich framework for decision making since there are no acceptable options at all. Grounded semantics accepts very few arguments as expected, because it is very cautious. When the stable extensions exist, this semantics accepts more acceptable options than any other semantics.

3.3.2 Rejected options

This section aims at studying the impact of acceptability semantics on the rejected options.

Property 17 *Let $\mathcal{O}$ be a set of options. It holds that:*

1. $\mathcal{O}^r_{ad} = \mathcal{O}^r_c = \mathcal{O}^r_p$
2. $\mathcal{O}^r_p \subseteq \mathcal{O}^r_g$.

The following example shows that the converse of the inclusion $\mathcal{O}^r_p \subseteq \mathcal{O}^r_g$, proved in the previous property, is not always true.

Example 21 (Example 18 cont.) *Since* $\mathtt{GE} = \emptyset$ *then all the arguments are rejected under grounded semantics.* $\mathcal{O}^r_g = \mathcal{O} = \{o_1\}$. *However, there are two preferred extensions, and the argument d is in both of them. So, d is accepted under preferred semantics. Consequently, the option o_1 is accepted under that semantics.* $\mathcal{O}^r_p = \emptyset$.

Property 18 *Let $\mathcal{O}$ be the set of options, and let $\mathcal{AF}$ be the argumentation system.*

1. *If $\mathcal{AF}$ has no stable extensions, then $\mathcal{O}^r_s = \mathcal{O}$, i.e. all the options are rejected.*
2. *If $\mathcal{AF}$ has stable extensions, then $\mathcal{O}^r_p \subseteq \mathcal{O}^r_s \subseteq \mathcal{O}^r_g$.*

The following two examples show that the converse in the second part of the previous property does not hold.

Example 22 (Example 20 cont.) *Recall that there are two preferred extensions $\mathcal{E}_1 = \{b, d, z\}$ and $\mathcal{E}_2 = \{a, y\}$, and that $\mathcal{E}_1$ is a stable extension while $\mathcal{E}_2$ is not. So, the argument y is credulously accepted under preferred semantics, while it is rejected under stable semantics. The argument z is also credulously accepted under preferred semantics, but it is accepted under stable semantics. So, under preferred semantics, there are no rejected options while under stable semantics there is exactly one rejected option, o_2. In summary, we have $\mathcal{O}^p_r = \emptyset$ and $\mathcal{O}^s_r = \{o_2\}$. So, as we have seen, this is an example where there exists a stable extension and $\mathcal{O}^r_p \neq \mathcal{O}^r_s$.*

Example 23 (Example 18 cont.) *It can be checked that the argumentation system $\langle \mathcal{A}, \mathtt{Def} \rangle$ has exactly two stable extensions: $\mathcal{E}_1 = \{a, d\}$ and $\mathcal{E}_2 = \{b, d\}$. Since d is in both stable extensions then it is skeptically accepted under stable semantics. Consequently, the option o_1 is acceptable under stable semantics. Thus, we have $\mathcal{O}^r_s = \emptyset$. However, it can be checked that this argumentation system has an empty grounded extension. Thus, $O^a_g = \emptyset$. So, the option o_1 is rejected under grounded semantics, $\mathcal{O}^r_g = \mathcal{O} = \{o_1\}$. So, as we have seen, this is an example where there exists a stable extension and $\mathcal{O}^r_s \neq \mathcal{O}^r_g$.*

In summary, the following inclusions hold:

- *Case 1:* There is no stable extension.
  $\mathcal{O}^r_{ad} = \mathcal{O}^r_c = \mathcal{O}^r_p \subseteq \mathcal{O}^r_g \subseteq \mathcal{O}^r_s = \mathcal{O}$.
- *Case 2:* There exists at least one stable extension.
  $\mathcal{O}^r_{ad} = \mathcal{O}^r_c = \mathcal{O}^r_p \subseteq \mathcal{O}^r_s \subseteq \mathcal{O}^r_g$.

In case there is no stable extension, all the options are rejected under stable semantics. In case the system has stable extensions, the grounded semantics rejects the most arguments, and consequently, the most options. We see that admissible, complete and preferred semantics reject exactly the same set of arguments. As expected, the number of rejected options is very high under the grounded semantics. This result is not surprising since grounded semantics is very cautious and accepts very few arguments.

3.3.3 Non-supported options

As mentioned before, an option o is non-supported iff there are no arguments in its favor, i.e. $\mathcal{H}(o) = \emptyset$. It is clear that this is independent of the acceptability semantics. So, we have the following property:

Property 19 *Let $\mathcal{O}$ be the set of options. It holds that:*
$\mathcal{O}^{ns}_{ad} = \mathcal{O}^{ns}_{c} = \mathcal{O}^{ns}_{p} = \mathcal{O}^{ns}_{g} = \mathcal{O}^{ns}_{s}$.

3.3.4 Negotiable options

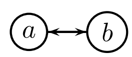
An option o is said to be negotiable if there are no skeptically accepted arguments in its favor, but there is at least one credulously accepted argument in its favor. An agent will always prefer an accepted option to a negotiable one, while it prefers a negotiable option to a non-supported one. Since the number of admissible extensions is always the biggest, this semantics will return the greatest number of credulously accepted arguments, and, consequently, the greatest number of negotiable options. On the other hand, since there is always exactly one grounded extension, there can never exist a negotiable option under this semantics, as formalized in the following property.

Property 20 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^{n}_{g} = \emptyset$.*

Property 21 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^{n}_{g} \subseteq \mathcal{O}^{n}_{s}$.*

The following example shows that the converse is not always true.

Example 24 *Here we provide an example where $\mathcal{O}^{n}_{g} \neq \mathcal{O}^{n}_{s}$. Let us assume a set $\mathcal{O} = \{o_1, o_2\}$ of two options and a set $\mathcal{A}_o = \{a, b\}$ of two practical arguments such that $\mathcal{H}(o_1) = \{a\}$ and $\mathcal{H}(o_2) = \{b\}$. Let us assume that* `Def` *is depicted in the figure below:*



There are exactly two stable extensions: $\{a\}$ and $\{b\}$. There are two negotiable options under stable semantics: o_1 and o_2. However, the grounded extension is an empty set. Thus, all the options are rejected under grounded semantics. So, $\mathcal{O}^{n}_{g} = \emptyset$ and $\mathcal{O}^{n}_{s} = \{o_1, o_2\}$.

Property 22 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^{n}_{s} \subseteq \mathcal{O}^{n}_{p}$.*

The following example shows that the converse is not always true.

Example 25 (Example 20 cont.) *The arguments y and z are both credulously accepted under preferred semantics. Thus, $\mathcal{O}^{n}_{p} = \{o_1, o_2\}$. However, there is exactly one stable extension, namely $\{b, d, z\}$, so z is skeptically accepted while y is rejected. So, $\mathcal{O}^{n}_{s} = \emptyset$.*

Property 23 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^{n}_{p} \subseteq \mathcal{O}^{n}_{c}$.*

The following example shows that the converse is not always true.

Example 26 (Example 7 cont.) *It can be checked that $\mathcal{E}_1 = \{a, d\}$, $\mathcal{E}_2 = \{b, d\}$ and $\mathcal{E}_3 = \emptyset$ are complete extensions of the system $\langle \mathcal{A}, \mathtt{Def} \rangle$. So, the argument d is credulously accepted under complete semantics. However, the system has only two preferred extensions. So, the argument d is skeptically accepted under this semantics. The option o_1 is negotiable under complete semantics and acceptable under preferred semantics. Thus, $\mathcal{O}^{n}_{c} = \{o_1\}$ and $\mathcal{O}^{n}_{p} = \emptyset$.*

Property 24 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^{n}_{c} \subseteq \mathcal{O}^{n}_{ad}$.*

The following example shows that the converse is not always true.

Example 27 *Let* $\mathcal{O} = \{o_1\}$, $\mathcal{H}(o_1) = \{a\}$, $\mathcal{A} = \{a\}$ *and* $\texttt{Def} = \emptyset$. *The argumentation system* $\langle \mathcal{A}, \texttt{Def} \rangle$ *has two admissible extensions:* $\mathcal{E}_1 = \{a\}$ *and* $\mathcal{E}_2 = \emptyset$. *The argument* a *is credulously accepted, thus the option* o_1 *is negotiable under this semantics:* $\mathcal{O}^n_{ad} = \{o_1\}$. *However, there is exactly one extension* $\mathcal{E}_1 = \{a\}$ *under complete semantics. So,* a *is skeptically accepted and* $\mathcal{O}^n_c = \emptyset$.

In summary, the following links hold:
$\mathcal{O}^n_g = \emptyset \subseteq \mathcal{O}^n_s \subseteq \mathcal{O}^n_p \subseteq \mathcal{O}^n_c \subseteq \mathcal{O}^n_{ad}$.
Note that in case there is no stable extension the set $\mathcal{O}^n_s$ is empty, while this is not necessarily true in the general case.

3.4 Conclusion

In this chapter, we have presented a general argumentation framework for decision making. This framework is preference-based since it is grounded on a preference relation between arguments. The framework returns two outputs: a status for each option (acceptable, negotiable, rejected, non-supported) and a preference relation $\succeq$ on the set $\mathcal{O}$ of possible options. In this chapter we have studied the properties of the framework regarding the impact of the different acceptability semantics on the output of the decision framework. In particular, we have shown that an agent can have different sets of acceptable options if it uses different acceptability semantics. We have shown in which case the agent has more or less acceptable options.

Chapter 4

Complete frameworks

Résumé

► Dans ce chapitre, nous étudions une classe particulière de systèmes de décision, appelés systèmes complets. Il s'agit d'un cas particulier de notre cadre général où l'ensemble des arguments épistémiques est vide et où tous les arguments pratiques s'attaquent mutuellement. Nous montrons que tels systèmes sont cohérents, c.-à-d., leurs extensions stables coïncident avec les extensions préférées. Nous montrons aussi que ces systèmes possèdent toujours une extension non vide. Ce résultat est d'une grande importance puisqu'il assume que parmi toute les options de l'ensemble $\mathcal{O}$, il existe au moins une qui peut-être choisie. Nous caractérisons aussi les arguments acceptables de ces systèmes. Nous montrons que les extensions préférés contiennent uniquement des arguments qui se défendent seuls contre toute attaque. Nous montrons aussi comment le statut d'une option peut changer lorsqu'un nouvel argument est reçu.◄

4.1 Complete decision frameworks

The aim of this section is to study a particular class of the general framework proposed in the previous chapter. We are particularly interested in what we call complete frameworks. The idea behind a complete framework is that:

- the set of epistemic arguments is empty (i.e., $\mathcal{A}_b = \emptyset$),
- the attack relation $\mathcal{R}_o$ between practical arguments is "complete" in the sense that all practical arguments are conflicting.

Definition 18 (Complete relation) *The relation $R_o \subseteq \mathcal{A}_o \times \mathcal{A}_o$ is* complete *iff* $(\forall a \in \mathcal{A}_o)(\forall b \in \mathcal{A}_o)$ *such that $a \neq b$, it holds that $(a, b) \in \mathcal{R}_o$ and $(b, a) \in \mathcal{R}_o$.*

Throughout the chapter, we will study complete decision framework. The decision framework is complete iff its argumentation framework is complete.

Definition 19 (Complete argumentation framework for decision making) *A* Complete argumentation framework for decision making *is a pair $\langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ where $\mathtt{Def}_o$ is defined over a complete conflict relation $\mathcal{R}_o$ and a preference relation $\geq_o$.*

4.2 General properties of the framework

The aim of this section is to study the general properties of complete frameworks. In what follows, we will show that the graph associated to $\mathcal{AF}_o$ has no *elementary odd-length cycles.* Before presenting formally the result, let us first define what is an elementary cycle.

Definition 20 (Elementary cycle) *Let $X = \{a_1, \ldots, a_n\}$ be a set of arguments of $\mathcal{A}_o$. X is an* elementary cycle *iff:*

1. *$\forall i \leq n-1$, $(a_i, a_{i+1}) \in \mathtt{Def}_o$ and $(a_n, a_1) \in \mathtt{Def}_o$*
2. *$\nexists X' \subseteq X$ such that X' satisfies condition 1.*

Let us illustrate this notion of elementary cycles through the following simple example.

Example 28 *Consider two following sets of arguments and attacks (in the sense of* Def*) between them:*

In part (1) of the above figure, the set $\{a, b, c, d\}$ forms an elementary cycle. However, in part (2), the set $\{a, b, c, d\}$ is not an elementary cycle since its subset $\{c, d\}$ already satisfies the first condition of Definition 20.

A first result states that when the preference relation $\geq_o$ is a partial pre-order, the graph associated to the corresponding argumentation system has no elementary odd-length cycles. This result is important since the existence of odd-length cycles prevents the existence of stable extensions. Consequently, no option among elements of $\mathcal{O}$ is suggested. This is not suitable since in most practical cases, an agent wants to choose in anyway one solution.

Theorem 3 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making. The graph $\mathcal{G}_{\mathcal{AF}_o}$ has no elementary odd-length cycles.*

The previous result lets us not only characterizing the structure of graphs associated with the system $\mathcal{AF}_o$, but also proving other interesting results concerning the extensions under the well-known acceptability semantics, in particular stable one. Indeed, we will show that the practical system $\mathcal{AF}_o$ is coherent (i.e. its stable semantics and preferred ones coincide). Formally:

Theorem 4 *The system $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ is coherent, i.e. each preferred extension is a stable one.*

Since preferred and stable extensions coincide, in the rest of this chapter we will use the term *extension* to refer to *preferred/stable extension.*

In addition to the above result, we will show that there are *non-empty* extensions. This result is of great importance since it ensures that among all the different options of $\mathcal{O}$, one of them will be for sure proposed as a candidate.

Theorem 5 *The system $\mathcal{AF}_o$ has at least one* non-empty *preferred/stable extension.*

The next result characterize extensions of preference-based argumentation systems. An important property of such systems is that their admissible arguments coincide with *self-defending* arguments, a notion that is formally defined as follows.

Definition 21 (Self defense) *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $a \in \mathcal{A}_o$. The argument a is self-defending iff $(\forall x \in \mathcal{A}_o)\ (x, a) \in \mathtt{Def}_o \Rightarrow (a, x) \in \mathtt{Def}_o$.*

The following basic result holds.

Theorem 6 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $a \in \mathcal{A}$. There exists an extension $\mathcal{E}$ such that $a \in \mathcal{E}$ iff a is self-defending.*

The following result guarantees that two arguments which appear in two distinct extensions always attack each other, in the sense of $\mathtt{Def}_o$.

Theorem 7 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its extensions. Let $a, b \in \mathcal{A}$ be two arguments such that $a, b \in \bigcup_{i=1}^{n} \mathcal{E}_i$ and $\nexists \mathcal{E}_i$ such that $a, b \in \mathcal{E}_i$, for $i = 1, \ldots, n$. Then $(a, b) \in \mathtt{Def}_o$ and $(b, a) \in \mathtt{Def}_o$.*

Now, regarding the links between different extensions, we will show that they are all pairwise disjoint, i.e., they don't have any common argument.

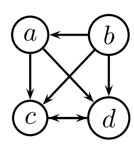
Theorem 8 *The extensions of $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ are pairwise disjoint.*

From the above we immediately obtain the following corollary.

Corollary 1 *The system $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ has a skeptically accepted argument iff it has exactly one extension.*

In this particular system the fact that an argument a is skeptically accepted is closely related to the *in-degree* of that argument in the graph associated to the system. The in-degree of an argument a' in a directed graph is the number of arcs that have a' as a head. In the following $in_{\mathcal{G}}(a)$ denotes the in-degree of argument $a \in \mathcal{A}_o$ in the graph $\mathcal{G}_{\mathcal{AF}_o}$ of $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$.

Example 29 *Consider the following argumentation framework:*



In the picture above, in-degree of argument a is 1, the in-degree of argument b is 0, while that of arguments c and d is 3.

Theorem 9 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making. An argument $a \in \mathcal{A}_l$ is skeptically accepted iff $in_{\mathcal{G}}(a) = 0$.*

We continue by proving more properties which we will need later to analyze the status of options.

Property 25 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, and a be an arbitrary argument. Then:*

1. *a is skeptically accepted iff $(\forall x \in \mathcal{A}_o)\ (a, x) \in \geq_o$.*
2. *a is rejected iff $(\exists x \in \mathcal{A})\ (x, a) \in >_o$.*
3. *a is credulously accepted iff*
   $((\exists x' \in \mathcal{A})\ (a, x') \notin \geq_o) \wedge ((\forall x \in \mathcal{A})\ ((a, x) \notin \geq_o) \Rightarrow (x, a) \notin \geq_o))$.

We will now prove that in this particular system, there are two possible cases: the case where there exists at least one skeptically accepted argument but there are no credulously accepted arguments, and the case where there are no skeptically accepted arguments but there is "at least" one credulously accepted argument. This means that one cannot have a state with both skeptically accepted and credulously accepted arguments. Moreover, it cannot be the case that all the arguments are rejected. Formally:

Theorem 10 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making.The following implications hold:*

1. *If $\mathtt{Sc}(\mathcal{AF}_o) \neq \emptyset$ then $\mathtt{Cr}(\mathcal{AF}_o) = \emptyset$.*
2. *If $\mathtt{Cr}(\mathcal{AF}_o) = \emptyset$ then $\mathtt{Sc}(\mathcal{AF}_o) \neq \emptyset$.*

The next property will make some reasoning easier, because it shows that, in this particular framework, the definition of negotiable options can be simplified.

Property 26 *Let $o \in \mathcal{O}$. The option o is negotiable iff there is at least one credulously accepted argument in its favor.*

The next property highlights the link between argument status and option status.

Property 27 *The following equivalences hold.*

1. *There is at least one skeptically accepted argument iff there is at least one acceptable option.*
2. *There is at least one credulously accepted argument iff there is at least one negotiable option.*

The consequence of the previous two results is the following theorem. It proves the fact that there are two cases: the first case where there are some acceptable options but no negotiable options, and the second case where there are some negotiable options but no acceptable options.

Theorem 11 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making. The following holds: $\mathcal{O}_a \neq \emptyset \Leftrightarrow \mathcal{O}_n = \emptyset$.*

If an argument a is rejected, then there is some argument x such that x defeats a and a does not defend itself. The next property shows that arguments that defeat a cannot be all rejected.

Property 28 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $a \in \mathcal{A}_o$. If $a \in \mathtt{Rej}(\mathcal{AF}_o)$ then $(\exists x' \in \mathcal{A}_o)$ such that $x' \notin \mathtt{Rej}(\mathcal{AF}_o) \wedge (x', a) \in >_o$.*

The next property proves that if there is exactly one non-rejected argument, then it is skeptically accepted. It is important because it guaranties that it cannot be the case that all the options are rejected.

Property 29 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $e \in \mathcal{A}_o$. If $\mathcal{A}_o \setminus \{e\} \subseteq \mathtt{Rej}(\mathcal{AF}_o)$ then $e \in \mathtt{Sc}(\mathcal{AF}_o)$.*

We will now show that all the skeptically accepted arguments in this particular system are equally preferred.

Property 30 *Let $a, b \in \mathtt{Sc}(\mathcal{AF}_o)$. Then $(a, b) \in \geq_o$ and $(b, a) \in \geq_o$.*

We will now show that an arbitrary argument e is in the same relation with all accepted arguments. Recall that we use the notation $(e, a) \in \odot$ to refer to one particular relation between the arguments e and a.

Property 31 *Let e be an arbitrary argument.*
If $((\exists a \in \mathtt{Sc}(\mathcal{AF}_o))$ such that $(a, e) \in \odot)$ then $((\forall a \in \mathtt{Sc}(\mathcal{AF}_o))\ (a, e) \in \odot)$.

Let us now take a look at credulously accepted arguments. While all the skeptically accepted arguments are in the same class with respect to the preference relation $\geq_o$, this is not always the case with credulously accepted arguments. The next property shows that credulously accepted arguments are either incomparable or indifferent with respect to $\geq_o$.

Property 32 *$\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete framework and $\mathtt{Cr}(\mathcal{AF}_o)$ its credulously accepted arguments. Then $(\forall a, b \in \mathtt{Cr}(\mathcal{AF}_o)$ it holds that*

$$((a, b) \in \geq_o \wedge (b, a) \in \geq_o) \vee ((a, b) \notin \geq_o \wedge (b, a) \notin \geq_o).$$

The next property shows that if a' is credulously accepted then there exists another credulously accepted argument a'' such that they are incomparable in the sense of preference relation.

Property 33 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, and $\mathtt{Cr}(\mathcal{AF}_o) \neq \emptyset$. Then it holds that: $(\forall a' \in \mathtt{Cr}(\mathcal{AF}_o))$ $(\exists a'' \in \mathtt{Cr}(\mathcal{AF}_o))$ $(a', a'') \notin \geq_o \wedge (a'', a') \notin \geq_o$.*

4.3 Revising the status of an option

The aim of this section is to study the revision of the status of a given option in light of a new argument. In other words, given a complete argumentation framework for decision making $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ and an option $o \in \mathcal{O}$, what is the new status of o if new argument $e \notin \mathcal{A}_o$ is received. It is clear that addition of a new practical argument e to the set $\mathcal{A}_o$ may cause some change in $\mathtt{Def}_o$ since the new argument may interact with the existing ones. In what follows, we will denote by $\mathcal{AF}_o \oplus e$ the new argumentation system, whose arguments are $\mathcal{A}_o \cup \{e\}$. In the first subsection, we study the revision of the status of arguments, while in the second one we show the impact of argument status revision to option status.

4.3.1 Revising the status of arguments

The following result shows that the status of a rejected argument will not change when a new argument is received. However, a credulously accepted argument cannot become skeptically accepted. It can either remain credulously accepted or become rejected. Formally:

Property 34 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $e \notin \mathcal{A}_o$.*

1. *If $a \in \mathtt{Rej}(\mathcal{AF}_o)$, then $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$.*
2. *If $a \in \mathtt{Cr}(\mathcal{AF}_o)$, then $a \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.*

The next property is simple but will be very useful later in this chaper.

Property 35 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $e \notin \mathcal{A}_o$.*

1. *If $a \in \mathtt{Sc}(\mathcal{AF}_o)$ then $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ iff $(a, e) \in \geq_o$.*
2. *If $a \notin \mathtt{Rej}(\mathcal{AF}_o)$ then $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$ iff $(e, a) \in >_o$.*

The next property shows that all the skeptically accepted arguments will have the "same destiny" after a new argument arrives.

Property 36 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $a, b \in \mathtt{Sc}(\mathcal{AF}_o)$. Let $e \notin \mathcal{A}_o$.*

1. *If $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ then $b \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.*
2. *If $a \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$ then $b \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$.*

3. *If* $a \in \texttt{Rej}(\mathcal{AF}_o \oplus e)$ *then* $b \in \texttt{Rej}(\mathcal{AF}_o \oplus e)$.

The next theorem analyzes the status of all skeptically accepted arguments after a new argument has arrived.

Theorem 12 *Let* $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ *be a complete argumentation framework for decision making,* $a \in \texttt{Sc}(\mathcal{AF}_o)$ *and* $e \notin \mathcal{A}_o$*. The following holds:*

1. $((a,e) \in \geq_o) \wedge ((a,e) \in \geq_o)$ *iff* $a \in \texttt{Sc}(\mathcal{AF}_o \oplus e) \wedge e \in \texttt{Sc}(\mathcal{AF}_o \oplus e)$
2. $(e,a) \in >_o$ *iff* $a \in \texttt{Rej}(\mathcal{AF}_o \oplus e) \wedge e \in \texttt{Sc}(\mathcal{AF}_o \oplus e)$
3. $(a,e) \in >_o$ *iff* $a \in \texttt{Sc}(\mathcal{AF}_o \oplus e) \wedge e \in \texttt{Rej}(\mathcal{AF}_o \oplus e)$
4. $((a,e) \notin \geq_o) \wedge ((a,e) \notin \geq_o)$ *iff* $a \in \texttt{Cr}(\mathcal{AF}_o \oplus e) \wedge e \in \texttt{Cr}(\mathcal{AF}_o \oplus e)$

Note that, according to Property 31, all skeptically accepted arguments are in the same relation with e as a is. Formally, if a and e are in particular relation which we denote $(a,e) \in \odot$, then $(\forall b \in \mathcal{A}_o)$ $((b \in \texttt{Sc}(\mathcal{AF}_o)) \Rightarrow (b,e) \in \odot)$. Hence, the condition "let $a \in \texttt{Sc}(\mathcal{AF}_o)$ and $(a,e) \in \odot$" in the previous theorem is equivalent to the condition $(\forall a \in \mathcal{A}_o)$ $((a \in \texttt{Sc}(\mathcal{AF}_o)) \Rightarrow (a,e) \in \odot)$.

The Theorem 12 stands as a basic tool for reasoning about the status of new arguments as well as about the changes in the status of other arguments. Once the argument status is known, it is much easier to determine the status of options.

We will now analyze the relation between credulously accepted arguments and new arguments. In the next property, we show that if there are credulously accepted arguments and an argument e is preferred to all of them, then it is strictly preferred to all of them.

Property 37 *Let* $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ *be a complete argumentation framework for decision making,* $\texttt{Cr}(\mathcal{AF}_o) \neq \emptyset$ *and* $e \notin \mathcal{A}_o$*. The following result holds:* $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))$ $(e,a) \in >_o$ *iff* $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))$ $(e,a) \in \geq_o$.

Property 38 *Let* $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ *be a complete argumentation framework for decision making,* $\texttt{Cr}(\mathcal{AF}_o) \neq \emptyset$ *and* $e \notin \mathcal{A}_o$*. The following holds:* $(\forall a \in \texttt{Cr}(A_o))$ $a \in \texttt{Rej}(A_o \oplus e)$ *iff* $(\forall a \in \texttt{Cr}(A_o))$ $(e,a) \in >_o$.

The next theorem is similar to Theorem 12, because it analyzes the status of the arriving argument. The difference is, of course, in the fact that now we suppose that there are no skeptically accepted arguments i.e., there are some credulously accepted arguments.

Theorem 13 *Let* $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ *be a complete argumentation framework for decision making,* $\texttt{Cr}(\mathcal{AF}_o) \neq \emptyset$ *and* $e \notin \mathcal{A}_o$*. Then, the following holds:*

1. $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))$ $(e,a) \in >_o$ *iff* $e \in \texttt{Sc}(\mathcal{AF}_o \oplus e) \wedge \mathcal{A}_o = \texttt{Rej}(\mathcal{AF}_o \oplus e)$.
2. $(\exists a \in \texttt{Cr}(\mathcal{AF}_o))$ $(e,a) \notin >_o \wedge (\nexists a' \in \texttt{Cr}(\mathcal{AF}_o))$ $(a',e) \in >_o$ *iff* $e \in \texttt{Cr}(\mathcal{AF}_o \oplus e)$
3. $(\exists a \in \texttt{Cr}(\mathcal{AF}_o))$ $(a,e) \in >_o$ *iff* $e \in \texttt{Rej}(\mathcal{AF}_o \oplus e) \wedge \mathcal{A}_o = \texttt{Cr}(\mathcal{AF}_o \oplus e)$.

Recall that, according to Property 37, the condition $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))$ $(e,a) \in >_o$ in the previous theorem is equivalent to the condition $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))$ $(e,a) \in \geq_o$. While all the skeptically accepted arguments have the "same destiny" after a new argument arrives, this is not the case with credulously accepted arguments. Some of them may remain credulously accepted while the others may become rejected.

4.3.2 Revising the status of an option

We will now show under which conditions an option can change its status. We start by studying acceptable options.

Theorem 14 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ be a complete argumentation framework for decision making and $o \in \mathcal{O}$ an acceptable option. Suppose that $a \in \texttt{Sc}(\mathcal{AF}_o)$ is an arbitrary skeptically accepted argument and $e \notin \mathcal{A}_o$. Then:*

1. *Option o will stay acceptable iff*
   $((a, e) \in \geq_o) \vee (e \in \mathcal{H}(o)) \wedge ((e, a) \in >_o)$
2. *Option o will become negotiable iff*
   $((a, e) \notin \geq_o) \wedge ((e, a) \notin \geq_o))$
3. *Option o will become rejected iff*
   $(e \notin \mathcal{H}(o)) \wedge (e, a) \in >_o)$

Recall that, according to Property 31, all skeptically accepted arguments are in the same relation with an arbitrary argument. Hence, the condition $(\exists a \in \texttt{Sc}(\mathcal{AF}_o))\ (a, e) \in \odot)$ in the previous theorem is equivalent to the condition $(\forall a \in \texttt{Sc}(\mathcal{AF}_o))\ (a, e) \in \odot)$.

Now we give a similar characterization for negotiable options.

Theorem 15 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ be a complete argumentation framework for decision making and $o \in \mathcal{O}$ an negotiable option. Suppose that $e \notin \mathcal{A}_o$. Then:*

1. *Option o will become acceptable iff*
   $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \texttt{Cr}(\mathcal{A}_o))\ (e, a) \in >)$
2. *Option o will rest negotiable iff*
   $((e \in \mathcal{H}(o)) \wedge (\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ (e, a') \notin >_o \wedge (\nexists a'' \in \texttt{Cr}(\mathcal{AF}_o))\ (a'', e) \in >_o)$
   $\vee$
   $((\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ (a' \in \mathcal{H}(o) \wedge (e, a') \notin >_o))$
3. *Option o will become rejected iff*
   $(e \notin \mathcal{H}(o)) \wedge ((\forall a \in \texttt{Cr}(\mathcal{AF}_o))\ (a \in \mathcal{H}(o)) \Rightarrow (e, a) \in >_o)$.

Note that, according to Property 37, the condition $(\forall a \in \texttt{Cr}(\mathcal{A}_o))\ (e, a) \in >$ in the previous theorem is equivalent to condition $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))\ (e, a) \in \geq_o$.
At the and, we present the ways for a rejected option to change or not its status.

Theorem 16 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ be a complete argumentation framework for decision making and $o \in \mathcal{O}$ an rejected option. Suppose that $e \notin \mathcal{A}_o$. Then:*

1. *Option o will become acceptable iff*
   $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathcal{A}_o)\ (e, a) \in \geq_o)$
2. *Option o will become negotiable iff*
   $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathcal{A}_o)\ (a, e) \notin >_o) \wedge ((\exists a \in \mathcal{A}_o)\ (e, a) \notin >_o)$
3. *Option o will rest rejected iff*
   $(e \notin \mathcal{H}(o)) \vee ((e \in \mathcal{H}(o)) \wedge (\exists a \in \mathcal{A}_o)(a, e) \in >)$

4.4 Conclusion

In this section, we have studied one particular class of argumentation-based decision making frameworks, called complete frameworks. One of the most interesting properties of these frameworks is that the existence of acceptable and negotiable options is mutually exclusive. This is the consequence of the theorem which states that all the extensions are pairwise disjoint. Besides, if there are some skeptically accepted arguments, then they are all in the same class of equivalence with respect to the preference relation $\geq_o$.
Our main goal was to study the sufficient and necessary conditions under which an option changes its status. This is important because at a given step of a negotiation dialog, an agent has to choose the way to accomplish its goals. If it tries to influence another agent to change its preferences over the set of options, it has to be able to know which argument to send with respect to arguments that other agent has and semantics it uses.

Chapter 5

Related work

As said in the introduction, some works have been done on arguing for decision. Quite early, in [16] Brewka and Gordon have outlined a logical approach to decision (for negotiation purposes), which suggests the use of defeasible consequence relation for handling prioritized rules, and which also exhibits arguments for each choice. However, arguments are not formally defined.

In the framework proposed by Fox and Parsons in [15], no explicit distinction is made between knowledge and goals. However, in their examples, values (belonging to a linearly ordered scale) are assigned to formulas which represent goals. These values provide an empirical basis for comparing arguments using a symbolic combination of strengths of beliefs and goals values. This symbolic combination is performed through dictionaries corresponding to different kinds of scales that may be used. In this work, only type of arguments is considered in the style of arguments if favor of beliefs.

In [10], Bonet and Geffner have also proposed an original approach to qualitative decision, inspired from Tan and Pearl [25], based on "action rules" that link a situation and an action with the satisfaction of a *positive* or a *negative* goal. However in contrast with the previous work and the work presented in this paper, this approach does not refer to any model in argumentative inference.

In [1], an abstract and general decision system has been proposed. That system is defined in two steps. At the first step, arguments in favor each option are built. Arguments in favor of beliefs are also allowed. In that setting, practical arguments are not conflicting at all. The idea was to keep all the practical arguments that survive to epistemic attacks. Then, at the second step, options are compared on the basis of a decision principle. This principle is based on the accepted practical arguments. While this approach is general and flexible, it has some drawbacks. These are related to the separation of the two steps.

Chapter 6

Conclusion and future work

Résumé

► Cette thèse a proposé un cadre général pour la prise de décision sous incertitude. Ce cadre est fondé sur le système d'argumentation défini par Dung dans son célèbre papier [13]. L'idée est la suivante: comment ordonner un ensemble d'options en fonction de leurs conséquences. Pour cela, nous supposons que chaque option est supporté par des arguments. Ces arguments sont les raisons d'adopter l'opinion. Les arguments pour les options peuvent être conflictuels et peuvent aussi être attaqués par des arguments épistémiques qui viennent dans ce cas invalider les croyances à partir lesquelles est construit un argument pratique. Les propriétés de ce modelé ont été étudiées, notamment l'impact du choix de la sémantique d'acceptabilité sur le résultat du modèle.
Nous avons aussi étudié une classe particulière du modèle. Cette classe privilégie la décision qui est soutenue par le plus fort argument. Les propriétés de ce modèle sont étudiées et l'impact de l'arrivée d'un nouvel argument sur le statut d'une option est également étudiée.

Il existe plusieurs perspectives par ce travail. Dans un premier temps, nous comptons distinguer deux types d'arguments pratiques: des arguments en faveur des options (somme ceux considérés dans cette thèse) et des arguments contre les options. Un argument contre une option souligne le caractère négatif de l'option. Une autre piste de recherche consiste à étudier comment instancier le cadre général afin de coder d'autres critères de décision. En effet, dans cette thèse, nous avons étudié dans le Chapitre 4 les systèmes complets et nous avons montré que ces systèmes retournent la décision qui est soutenue par le plus fort argument. Cependant d'autres critères ont été définis dans [1]. Nous souhaitons regarder comment les coder dans notre cadre.◄

This master thesis proposed an abstract argumentation-based framework for decision making. The input of the framework is a set of options to be rank-ordered, sets of practical arguments in favor of each option, a set of epistemic arguments, as well as attack and preference relations on the sets of arguments. The framework returns as an output a status for each option and a total preordering on the set of options.

We investigated general properties of the framework, in particular the impact of acceptability semantics on the acceptability of options, which can be a start point for studying outcomes and strategies in automated negotiations where different agents may use different acceptability semantics.

The second part of this thesis is a study of a particular class of this general framework, namely, the class of systems where the attack relation is *complete*, i.e., all arguments are conflicting with each others. The choice of this class of systems has been motivated by real applications, as in automated negotiation and communication. Indeed, in [3] a negotiation model has been proposed. In that model, arguments in favor of different offers are assumed conflicting since they are seen as competitive. Arguments supporting the same option may also be seen as competitive as an agent may seek to find the best one in order to use it during a negotiation. The contribution of the thesis is a study of general properties of this system, as well as the study of the revision of the status of a given option in light of a new argument.

Future research will be guided by two sets of ideas. The first group considers work on proposed decision system which includes the following:

- Although our model is quite general, it only takes into account arguments pro (arguments in favor of a decision). So it can be extended by adding the notion of argument cons (arguments which highlight the negative consequences of particular decision).
- Other particular cases of relation $\mathcal{R}$ are to be studied, for example the *weakly-complete* argumentation frameworks, where two different argument attack each other iff they are in favor of different offers.
- The definition of relation $\mathtt{Def}$ has to be revised in order to make all the extensions of system $\langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ to satisfy certain properties.
- The study of other acceptability semantics that exist in literature will be done in order to use them in this system.
- The system can be expanded in a way to include goals and rejections and link them with options, maybe using a formalism like possibilistic logic to deal with uncertainty of accomplishing a goal when executing an option.
- The hypothesis that the set $\mathcal{O}$ is fixed is not realistic. In practice, when there are no acceptable options humans often try to find new ones.
- Since arguments are seen as very general entities, we didn't give an answer to question how agents can make arguments in favor and / or against options.

The second group of ideas considers negotiation, i.e., group decision making, where different agents may have conflicting interests. The main ideas include:

- The study of how a proposed decision making framework may be used as a mono-agent tool for rank-ordering options in the multi-agent negotiation setting.
- Characterize the situations where negotiation will guide agents to optimal solution and cases where negotiation cannot solve the conflict.
- Study the properties, strategies, possibilities and limitations for negotiation in the setting where different agents use different acceptability semantics.

The first algorithmic study has to be performed in order to prove complexity and find efficient protocols and algorithms for both decision and negotiation.

Appendix

Property 6 *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework, and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its extensions under a given semantics. Let $a \in \mathcal{A}$.*

1. *a is skeptically accepted iff $a \in \bigcap_{i=1}^{n} \mathcal{E}_i$.*
2. *a is rejected iff $a \notin \bigcup_{i=1}^{n} \mathcal{E}_i$.*

Proof. Proof follows directly from Definition 12. □

Property 7 *Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework and $\mathtt{Sc}(\mathcal{AF})$, $\mathtt{Cr}(\mathcal{AF})$, $\mathtt{Rej}(\mathcal{AF})$, its sets of arguments.*

1. *$\mathtt{Sc}(\mathcal{AF}) \cap \mathtt{Cr}(\mathcal{AF}) = \emptyset$, $\mathtt{Sc}(\mathcal{AF}) \cap \mathtt{Rej}(\mathcal{AF}) = \emptyset$, $\mathtt{Cr}(\mathcal{AF}) \cap \mathtt{Rej}(\mathcal{AF}) = \emptyset$*
2. *$\mathtt{Sc}(\mathcal{AF}) \cup \mathtt{Cr}(\mathcal{AF}) \cup \mathtt{Rej}(\mathcal{AF}) = \mathcal{A}$.*

Proof. Let $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ be an argumentation framework

1. Let us prove that three sets mentioned above are pairwise disjoint.
   (a) Assume that $\mathtt{Sc}(\mathcal{AF}) \cap \mathtt{Cr}(\mathcal{AF}) \neq \emptyset$. So, there exists an argument a such that $a \in \mathtt{Sc}(\mathcal{AF})$ and $a \in \mathtt{Cr}(\mathcal{AF})$. Since $a \in \mathtt{Cr}(\mathcal{AF})$ then there exists an extension $\mathcal{E}_i$ such that $a \in \mathcal{E}_i$ and there exists an extension $\mathcal{E}_j$ such that $a \notin \mathcal{E}_j$. Since $a \in \mathtt{Sc}(\mathcal{AF})$, then a is in all extensions. In particular, $a \in \mathcal{E}_j$. Contradiction.
   (b) Assume that $\mathtt{Sc}(\mathcal{AF}) \cap \mathtt{Rej}(\mathcal{AF}) \neq \emptyset$. So, there exists an argument a such that $a \in \mathtt{Sc}(\mathcal{AF})$ and $a \in \mathtt{Rej}(\mathcal{AF})$. Since $a \in \mathtt{Rej}(\mathcal{AF})$ then for all extensions $\mathcal{E}_i$, $a \notin \mathcal{E}_i$. Since $a \in \mathtt{Sc}(\mathcal{AF})$ then there exists at least one non-empty extension and a is in all extensions. Contradiction.
   (c) Assume that $\mathtt{Cr}(\mathcal{AF}) \cap \mathtt{Rej}(\mathcal{AF}) \neq \emptyset$. So, there exists an argument a such that $a \in \mathtt{Cr}(\mathcal{AF})$ and $a \in \mathtt{Rej}(\mathcal{AF})$. Since $a \in \mathtt{Rej}(\mathcal{AF})$ then for all extensions $\mathcal{E}_i$, $a \notin \mathcal{E}_i$. Since $a \in \mathtt{Cr}(\mathcal{AF})$ then there exists an extension $\mathcal{E}_i$ such that $a \in \mathcal{E}_i$. Contradiction.
2. The inclusion $\mathtt{Sc}(\mathcal{AF}) \cup \mathtt{Cr}(\mathcal{AF}) \cup \mathtt{Rej}(\mathcal{AF}) \subseteq \mathcal{A}$ is trivial. Let us now assume that $a \in \mathcal{A}$. If the argumentation system $\mathcal{AF} = \langle \mathcal{A}, \mathcal{R} \rangle$ has no extensions then a is rejected i.e. $a \in \mathtt{Rej}(\mathcal{AF})$.
   Let us now assume that there are exactly n extensions $\mathcal{E}_1, \ldots, \mathcal{E}_n$, with $n \geq 1$. There are three possible cases.
   (a) $a \in \bigcap_{i=1}^{n} \mathcal{E}_i$. This means that $a \in \mathtt{Sc}(\mathcal{AF})$.
   (b) There is at least one extension $\mathcal{E}_i$ such that $a \in \mathcal{E}_i$ and there is at least one extension $\mathcal{E}_j$ such that $a \notin \mathcal{E}_j$. In this case, a is credulously accepted, i.e. $a \in \mathtt{Cr}(\mathcal{AF})$.
   (c) $a \notin \bigcup_{i=1}^{n} \mathcal{E}_i$. This means that $a \in \mathtt{Rej}(\mathcal{AF})$.

□

Property 8 *Let $a, b \in \mathcal{A}$ such that $(a, b) \in \geq_x$ and $(b, a) \in \geq_x$. Then, the following hold.*

1. *If $(a, c) \in \geq_x$, for some argument c, than $(b, c) \in \geq_x$.*
2. *If $(c, a) \in \geq_x$, for some argument c, than $(c, b) \in \geq_x$.*
3. *If $(a, c) \notin \geq_x$, for some argument c, than $(b, c) \notin \geq_x$.*
4. *If $(c, a) \notin \geq_x$, for some argument c, than $(c, b) \notin \geq_x$.*

Proof. The first and the second statement follows directly from the transitivity of the preference relation: $(b, a) \in \geq$ and $(a, c) \in \geq$ imply $(b, c) \in \geq_x$, while $(c, a) \in \geq_x$ and $(a, b) \in \geq_x$ imply $(c, b) \in \geq_x$. Let us now prove the third statement. Suppose that $(b, c) \in \geq_x$. Then, $(a, b) \in \geq_x$ and $(b, c) \in \geq_x$ imply $(a, c) \in \geq_x$. Contradiction. The proof of the fourth statement is similar: suppose that $(c, b) \in \geq_x$. Then, $(c, b) \in \geq_x$ and $(b, a) \in \geq_x$ imply $(c, a) \in \geq_x$. Contradiction.

□

Property 9 *Let a, b, c be the arguments.*

1. *If $(a, b) \in >_x$ and $(b, c) \in \geq_x$ then $(a, c) \in >_x$.*
2. *If $(a, b) \in \geq_x$ and $(b, c) \in >_x$ then $(a, c) \in >_x$.*

Proof.

1. It is obvious that $(a, c) \in \geq_x$, since $\geq_x$ is transitive. Suppose $(c, a) \in \geq_x$. With $(b, c) \in \geq_x$ we have $(b, a) \in \geq_x$ because preference relation is transitive. Contradiction.
2. It is obvious that $(a, c) \in \geq_x$, since $\geq_x$ is transitive. Suppose $(c, a) \in \geq_x$. With $(a, b) \in \geq_x$ we have $(c, b) \in \geq_x$ because preference relation is transitive. Contradiction.

□

Theorem 2 *Let $\mathcal{AF} = (\mathcal{A}_b \cup \mathcal{A}_o, \mathtt{Def}_b \cup \mathtt{Def}_o \cup \mathtt{Def}_m)$ be a decision system, $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its admissible extensions, and $\mathcal{AF}_b = (\mathcal{A}_b, \mathtt{Def}_b)$ its associated epistemic system. It holds that $\forall \mathcal{E}_i$, the set $\mathcal{E}_i \cap \mathcal{A}_b$ is an admissible extension of $\mathcal{AF}_b$.*

Proof. Let $\mathcal{E}_i$ be an admissible extension of $\mathcal{AF}$. Let $\mathcal{E} = \mathcal{E}_i \cap \mathcal{A}_b$. Let us assume that $\mathcal{E}$ is not an admissible extension of $\mathcal{AF}_b$. There are two cases:

1. $\mathcal{E}$ is not conflict-free. This means that $\exists a, a' \in \mathcal{E}$ such that $(a, a') \in \mathtt{Def}_b$. Thus, $\exists a, a' \in \mathcal{E}_i$ such that $(a, a') \in \mathtt{Def}$. This is impossible since $\mathcal{E}_i$ is an admissible extension, thus conflict-free.
2. $\mathcal{E}$ does not defend its elements. This means that $\exists a \in \mathcal{E}$, such that $\exists a' \in \mathcal{A}_b$, $(a', a) \in \mathtt{Def}_b$ and $\nexists a'' \in \mathcal{E}$ such that $(a'', a') \in \mathtt{Def}_b$. Since $(a', a) \in \mathtt{Def}_b$, this means that $(a', a) \in \mathtt{Def}$ with $a \in \mathcal{E}_i$. However, $\mathcal{E}_i$ is admissible, then $\exists a'' \in \mathcal{E}_i$ such that $(a'', a') \in \mathtt{Def}$. Assume that $a'' \in \mathcal{A}_o$. This is impossible since practical arguments are not allowed to defeat epistemic ones. Thus, $a'' \in \mathcal{A}_b$. Hence, $a'' \in \mathcal{E}$. Contradiction.

□

Property 10 *An option $o \in \mathcal{O}$ is non-supported iff $\mathcal{H}(o) = \emptyset$.*

Proof. Let $o \in \mathcal{O}$. Let us assume that $\mathcal{H}(o) \neq \emptyset$. This means that there are two possibilities:

1. all the arguments are rejected, consequently the option is rejected
2. there exists at least one argument, say a, which is not rejected. Since a is not rejected, then:
   (a) a is skeptically accepted. This means that o is accepted.
   (b) a is credulously accepted. This means that the option o is negotiable.

□

Property 12 *Let $k, n \in \mathcal{N}$, $1 \leq k \leq n$. Let A_1, ..., A_n be arbitrary sets. Then:*

- $\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{k} A_i$
- $\bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{n} A_i$

In the particular case where $n \geq 1$ and $k = 1$, we have:

- $\bigcap_{i=1}^{n} A_i \subseteq A_1$
- $A_1 \subseteq \bigcup_{i=1}^{n} A_i$

Proof. Let n and k be arbitrary but fixed integers which satisfy the condition $1 \leq k \leq n$.

- Let us prove the inclusion $\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{k} A_i$. Suppose $x \in \bigcap_{i=1}^{n} A_i$. This means that $x \in A_1 \wedge \ldots \wedge x \in A_k \wedge \ldots \wedge x \in \mathcal{A}_n$. So, $x \in A_1 \wedge \ldots \wedge x \in A_k$, and, consequently, $x \in \bigcap_{i=1}^{k} A_i$.
- Let us prove the inclusion $\bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{n} A_i$. Suppose $x \in \bigcup_{i=1}^{k} A_i$. This means that $x \in A_1 \vee \ldots \vee x \in A_k$. So, $x \in A_1 \vee \ldots \vee x \in A_k \vee \ldots \vee x \in A_n$, and, consequently, $x \in \bigcup_{i=1}^{n} A_i$.

□

Property 13 *Let $\mathcal{O}$ be a set of options. $O^a_{ad} = \emptyset$.*

Proof. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be the admissible extensions of $\langle \mathcal{A}, \texttt{Def} \rangle$. Let us assume that $O^a_{ad} \neq \emptyset$. So, $\exists o \in \mathcal{O}$ such that $o \in O^a_{ad}$. This means that $\exists a \in \mathcal{H}(o)$ such that a is skeptically accepted. According to Property 6, we have $a \in \bigcap_{i=1}^{n} \mathcal{E}_i$. However, according to Property 1, the empty set is an admissible extension. This means that $\exists \mathcal{E}_i = \emptyset$, with $1 \leq i \leq n$. Since $\mathcal{E}_i = \emptyset$, we have $\bigcap_{i=1}^{n} \mathcal{E}_i = \emptyset$. Contradiction with the fact that $a \in \bigcap_{i=1}^{n} \mathcal{E}_i$. □

Property 14 *Let $\mathcal{O}$ be a set of options. The equality $O^a_g = O^a_c$ holds.*

Proof. Let $\langle \mathcal{A}, \mathtt{Def} \rangle$ be an argumentation system. Let $\mathtt{GE}$ be its grounded extension and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its complete extensions.

We will show that $O_g^a \subseteq O_c^a$. Let $o \in \mathcal{O}$. Let us assume that $o \in \mathcal{O}_g^a$ and $o \notin O_c^a$. Since $o \in \mathcal{O}_g^a$, then $\exists a \in \mathcal{H}(o)$ such that a is skeptically accepted under grounded semantics. Thus, $a \in \mathtt{GE}$. Since $o \notin O_c^a$ then $\forall a' \in \mathcal{H}(o)$ a' is not skeptically accepted with respect to complete semantics. According to Property 6, $(\forall a' \in \mathcal{H}(o))$ $a' \notin \bigcap_{i=1}^n \mathcal{E}_i$. According to the third bullet of Property 5, $\bigcap_{i=1}^n \mathcal{E}_i = \mathtt{GE}$. Thus, $(\forall a' \in \mathcal{H}(o))$ $a' \notin \mathtt{GE}$, hence $a \notin \mathtt{GE}$. Contradiction.

We will now show that $O_c^a \subseteq O_g^a$. Let $o \in \mathcal{O}$. Let us assume that $o \in \mathcal{O}_c^a$ and $o \notin O_g^a$. Since $o \in \mathcal{O}_c^a$, then $\exists a \in \mathcal{H}(o)$ such that a is skeptically accepted under complete semantics. Thus, $a \in \bigcap_{i=1}^n \mathcal{E}_i$. Since $o \notin O_g^a$ then $(\forall a' \in \mathcal{H}(o))$ a' is not skeptically accepted with respect to grounded semantics. So, $(\forall a' \in \mathcal{H}(o))$ $a \notin \mathtt{GE}$. According to the third bullet of Property 5, $\bigcap_{i=1}^n \mathcal{E}_i = \mathtt{GE}$. Thus, $(\forall a' \in \mathcal{H}(o))$ $a' \notin \bigcap_{i=1}^n \mathcal{E}_i$, hence $a \notin \bigcap_{i=1}^n \mathcal{E}_i$. Contradiction.

Since $O_g^a \subseteq O_c^a$ and $O_c^a \subseteq O_g^a$, we have $O_g^a = O_c^a$. □

Property 15 *Let $\mathcal{O}$ be a set of options. The inclusion $O_g^a \subseteq O_p^a$ holds.*

Proof. Let $\langle \mathcal{A}, \mathtt{Def} \rangle$ be an argumentation system. Let $\mathtt{GE}$ be its grounded extension and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its preferred extensions. Let o be an option and $\mathcal{H}(o)$ its set of arguments. Since o is accepted under grounded semantics then there exists an argument $a \in \mathcal{H}(o)$ such that a is in the grounded extension. According to Property 4, grounded extension is the subset of the intersection of all preferred extensions. Since, $a \in \mathtt{GE}$ then $a \in \bigcap_{i=1}^n \mathcal{E}_i$. So there is at least one skeptically accepted argument in favor of the option o under preferred semantics, which means that o is accepted under preferred semantics. □

Property 16 *Let $\mathcal{O}$ be the set of options, and let $\mathcal{AF} = \langle \mathcal{A}, \mathtt{Def} \rangle$ be the argumentation system for rank-ordering elements of $\mathcal{O}$.*

1. *If $\mathcal{AF}$ has no stable extensions, then $\mathcal{O}_s^a = \emptyset$ and $O_s^a \subseteq O_p^a$.*
2. *If $\mathcal{AF}$ has stable extensions, then $\mathcal{O}_p^a \subseteq \mathcal{O}_s^a$.*

Proof.

1. If $\mathcal{AF}$ has no stable extensions then there are no skeptically accepted arguments under stable semantics, thus there are no accepted options under this semantics.
2. Let us now assume that $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are stable extensions of $\mathcal{AF}$ and $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_{n+k}$ are preferred extensions that are not stable. Since $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are stable, according to the fourth bullet of Property 5, these are also preferred. According to Property 12, $\bigcap_{i=1}^{n+k} A_i \subseteq \bigcap_{i=1}^n A_i$, thus the set of skeptically accepted arguments under preferred semantics are a subset of the set of skeptically accepted arguments under the stable one. Let us now assume that $\exists o \in \mathcal{O}$ such that $o \in \mathcal{O}_p^a$ and $o \notin \mathcal{O}_s^a$. Since $o \in \mathcal{O}_p^a$ this means that $(\exists a \in \mathcal{H}(o))$ such that $a \in \bigcap_{i=1}^{n+k} A_i$. But, since $\bigcap_{i=1}^{n+k} A_i \subseteq \bigcap_{i=1}^n A_i$, then $a \in \bigcap_{i=1}^n A_i$. Thus, a is skeptically accepted under the stable semantics and $o \in \mathcal{O}_s^a$.

□

Property 17 *Let $\mathcal{O}$ be a set of options. It holds that:*

1. *$\mathcal{O}_{ad}^r = \mathcal{O}_c^r = \mathcal{O}_p^r$*

2. $\mathcal{O}_p^r \subseteq \mathcal{O}_g^r$.

Proof.

1. $\mathcal{O}_{ad}^r = \mathcal{O}_c^r$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be the complete extensions. According to the fourth bullet of Property 5, every complete extension is admissible, so $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are also admissible extensions. However, there can exist one or more admissible extensions which are not complete. So, let $\mathcal{E}_1, \ldots, \mathcal{E}_n, \ldots, \mathcal{E}_{n+k}$ be *all* admissible extensions, i.e. $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are complete and admissible and $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_{n+k}$ are admissible but not complete. Here, we have $k \geq 0$.

   We will show that $O_{ad}^r \subseteq O_c^r$. Let $o \in \mathcal{O}$. Let us assume that $o \in \mathcal{O}_{ad}^r$ and $o \notin O_c^r$. Since $o \in \mathcal{O}_{ad}^r$, then $(\forall a \in \mathcal{H}(o))$ a is rejected under admissible semantics. According to Property 6, $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n+k} \mathcal{E}_i$. Since $o \notin O_c^r$ then $\exists a' \in \mathcal{H}(o)$ such that a' is not rejected with respect to complete semantics. According to Property 6, $a' \in \bigcup_{i=1}^{n} \mathcal{E}_i$. But, according to Property 12, $\bigcup_{i=1}^{n} \mathcal{E}_i \subseteq \bigcup_{i=1}^{n+k} \mathcal{E}_i$. So, if $\exists a' \in \mathcal{H}(o)$ such that $a' \in \bigcup_{i=1}^{n} \mathcal{E}_i$, then $\exists a' \in \mathcal{H}(o)$ such that $a' \in \bigcup_{i=1}^{n+k} \mathcal{E}_i$. Contradiction with the fact that $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n+k} \mathcal{E}_i$.

   We will now show that $O_c^r \subseteq O_{ad}^r$. Let $o \in \mathcal{O}$. Let us assume that $o \in \mathcal{O}_c^r$ and $o \notin O_{ad}^r$. Since $o \in \mathcal{O}_c^r$, then $(\forall a \in \mathcal{H}(o))$ a is rejected under complete semantics. Thus, $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n} \mathcal{E}_i$. Since $o \notin O_{ad}^r$ then $\exists a' \in \mathcal{H}(o)$ such that a' is not rejected with respect to admissible semantics. According to Property 6, $a' \in \bigcup_{i=1}^{n+k} \mathcal{E}_i$. So, if $a' \in \bigcup_{i=1}^{n+k} \mathcal{E}_i$ and $a' \notin \bigcup_{i=1}^{n} \mathcal{E}_i$ then $\bigcup_{i=n+1}^{n+k} \mathcal{E}_i$. Thus $\exists \mathcal{E}_j$ with $n+1 \leq j \leq n+k$ such that $a' \in \mathcal{E}_j$ and $\mathcal{E}_j$ is an admissible extension but not a complete one. According to the first bullet of Property 5, there exists a complete extension $\mathcal{E}_k$, with $1 \leq k \leq n$ such that $\mathcal{E}_j \subseteq \mathcal{E}_k$. So, $a' \in \mathcal{E}_k$. Consequently, $a' \in \bigcup_{i=1}^{n} \mathcal{E}_i$. Contradiction with the fact that $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n} \mathcal{E}_i$.

   Since $O_{ad}^r \subseteq O_c^r$ and $O_c^r \subseteq O_{ad}^r$, we have $O_{ad}^r = O_c^r$.

2. $\mathcal{O}_c^r = \mathcal{O}_p^r$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be the preferred extensions. According to the fourth bullet of Property 5, every preferred extension is complete, so $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are also complete extensions. However, there can exist one or more complete extensions which are not preferred. So, let $\mathcal{E}_1, \ldots, \mathcal{E}_n, \ldots, \mathcal{E}_{n+k}$ be *all* complete extensions, i.e. $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are complete and preferred and $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_{n+k}$ are complete but not preferred. Here, we have $k \geq 0$.

   We will show that $O_c^r \subseteq O_p^r$. Let $o \in \mathcal{O}$. Let us assume that $o \in \mathcal{O}_c^r$ and $o \notin O_p^r$. Since $o \in \mathcal{O}_c^r$, then $(\forall a \in \mathcal{H}(o))$ a is rejected under complete semantics. According to Property 6, $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n+k} \mathcal{E}_i$. Since $o \notin O_p^r$ then $\exists a' \in \mathcal{H}(o)$ a' is not rejected with respect to preferred semantics. According to Property 6, $a' \in \bigcup_{i=1}^{n} \mathcal{E}_i$. But, according to Property 12, $\bigcup_{i=1}^{n} \mathcal{E}_i \subseteq \bigcup_{i=1}^{n+k} \mathcal{E}_i$. So, if $\exists a' \in \mathcal{H}(o)$ such that $a' \in \bigcup_{i=1}^{n} \mathcal{E}_i$, then $\exists a' \in \mathcal{H}(o)$ such that $a' \in \bigcup_{i=1}^{n+k} \mathcal{E}_i$. Contradiction with the fact that $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n+k} \mathcal{E}_i$.

   We will now show that $O_p^r \subseteq O_c^r$. Let $o \in \mathcal{O}$. Let us assume that $o \in \mathcal{O}_p^r$ and $o \notin O_c^r$. Since $o \in \mathcal{O}_p^r$, then $(\forall a \in \mathcal{H}(o))$ a is rejected under preferred semantics. Thus, $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n} \mathcal{E}_i$. Since $o \notin O_c^r$ then $\exists a' \in \mathcal{H}(o)$ such that a' is not rejected with respect to complete semantics. According to Property 6, $a' \in \bigcup_{i=1}^{n+k} \mathcal{E}_i$. So, if $a' \in \bigcup_{i=1}^{n+k} \mathcal{E}_i$ and $a' \notin \bigcup_{i=1}^{n} \mathcal{E}_i$ then $\bigcup_{i=n+1}^{n+k} \mathcal{E}_i$. Thus $\exists \mathcal{E}_j$ with $n+1 \leq j \leq n+k$ such that $a' \in \mathcal{E}_j$ and $\mathcal{E}_j$ is a complete extension but not a preferred one. According to second bullet of Property 5, there exists a preferred extension $\mathcal{E}_k$, with $1 \leq k \leq n$ such that $\mathcal{E}_j \subseteq \mathcal{E}_k$. So, $a' \in \mathcal{E}_k$. Consequently, $a' \in \bigcup_{i=1}^{n} \mathcal{E}_i$. Contradiction with the fact that $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n} \mathcal{E}_i$.

   Since $O_c^r \subseteq O_p^r$ and $O_p^r \subseteq O_c^r$, we have $O_c^r = O_p^r$.

3. $\mathcal{O}_p^r \subseteq \mathcal{O}_g^r$. Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be the preferred extensions, and $\mathtt{GE}$ the grounded extension. Let $o \in \mathcal{O}$. Let us assume that $o \in \mathcal{O}_p^r$ and $o \notin O_g^r$. Since $o \in \mathcal{O}_p^r$, then $\forall a \in \mathcal{H}(o)$ a is rejected under preferred semantics. According to Property 6, $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^n \mathcal{E}_i$. Since $o \notin O_g^r$ then $\exists a' \in \mathcal{H}(o)$ a' is not rejected with respect to grounded semantics. Since there is always exactly one grounded extension, $a' \in \mathtt{GE}$. According to Property 4, grounded extension is a subset of the intersection of all preferred extensions. So, $a' \in \mathtt{GE}$ implies $a' \in \bigcap_{i=1}^n \mathcal{E}_i$. There is always at least one preferred extension, i.e. $i \geq 1$. Consequently, $a' \in \mathcal{E}_1$. According to Property 12, $\mathcal{E}_1 \subseteq \bigcup_{i=1}^n \mathcal{E}_i$. So, if $a' \in \mathcal{E}_1$ then $a' \in \bigcup_{i=1}^n \mathcal{E}_i$. Thus, $a' \in \bigcup_{i=1}^n \mathcal{E}_i$. Contradiction with the fact $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^n \mathcal{E}_i$.

□

Property 18 *Let $\mathcal{O}$ be the set of options, and let $\mathcal{AF}$ be the argumentation system.*

1. *If $\mathcal{AF}$ has no stable extensions, then $\mathcal{O}_s^r = \mathcal{O}$, i.e. all the options are rejected.*
2. *If $\mathcal{AF}$ has stable extensions, then $\mathcal{O}_p^r \subseteq \mathcal{O}_s^r \subseteq \mathcal{O}_g^r$.*

Proof.

1. If $\mathcal{AF}$ has no stable extensions then there are no skeptically accepted arguments under stable semantics, thus there are no accepted options under this semantics.
2. We will show that if there is at least one stable extension, then $\mathcal{O}_p^r \subseteq \mathcal{O}_s^r$.
   Let us assume that $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are stable extensions of $\mathcal{AF}$ and $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_{n+k}$ are preferred extensions that are not stable. Since $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are stable, according to the fourth bullet of Property 5 these are also preferred. So, stable extensions are $\mathcal{E}_1, \ldots, \mathcal{E}_n$ and preferred extensions are $\mathcal{E}_1, \ldots, \mathcal{E}_{n+k}$.
   Let us now assume that $\exists o \in \mathcal{O}$ such that $o \in \mathcal{O}_p^r$ and $o \notin \mathcal{O}_s^r$. Since $o \in \mathcal{O}_p^r$, then $(\forall a \in \mathcal{H}(o))$ a is rejected under preferred semantics. According to Property 6 we have $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n+k} A_i$. Since $o \notin \mathcal{O}_s^r$, then $(\exists a' \in \mathcal{H}(o))$ a' is not rejected under stable semantics. According to Property 6 we have $(\exists a' \in \mathcal{H}(o))$ such that $a' \in \bigcup_{i=1}^n A_i$. According to Property 12, $\bigcup_{i=1}^n A_i \subseteq \bigcup_{i=1}^{n+k} A_i$. So, with $a' \in \bigcup_{i=1}^n A_i$, we have $a' \in \bigcup_{i=1}^{n+k} A_i$. Contradiction with the fact that $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^{n+k} A_i$. So, $\mathcal{O}_p^r \subseteq \mathcal{O}_s^r$.
   We will now show that if there is at least one stable extension, then $\mathcal{O}_s^r \subseteq \mathcal{O}_g^r$.
   Let us assume that $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are stable extensions of $\mathcal{AF}$ and $\mathcal{E}_{n+1}, \ldots, \mathcal{E}_{n+k}$ are complete extensions that are not stable. Since $\mathcal{E}_1, \ldots, \mathcal{E}_n$ are stable, according to the fourth bullet of Property 5 these are also complete. So, stable extensions are $\mathcal{E}_1, \ldots, \mathcal{E}_n$ and complete extensions are $\mathcal{E}_1, \ldots, \mathcal{E}_{n+k}$.
   Let us now assume that $\exists o \in \mathcal{O}$ such that $o \in \mathcal{O}_s^r$ and $o \notin \mathcal{O}_g^r$. Since $o \in \mathcal{O}_p^r$, then $(\forall a \in \mathcal{H}(o))$ a is rejected. According to Property 6 we have $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^n A_i$. Since $o \notin \mathcal{O}_s^r$, then $(\exists a' \in \mathcal{H}(o))$ such that a' is not rejected under grounded semantics. Thus, $(\exists a' \in \mathcal{H}(o))$ such that $a' \in \mathtt{GE}$. According to the third bullet of Property 5, grounded extension is the intersection of all complete extensions. Hence, if $a' \in \mathtt{GE}$ then $a' \in \bigcap_{i=1}^{n+k} A_i$. So, $a' \in \bigcap_{i=1}^{n+k} A_i$. According to Property 12, $\bigcap_{i=1}^{n+k} A_i \subseteq \bigcap_{i=1}^n A_i$. Consequently, $a' \in \bigcap_{i=1}^n A_i$. Recall that we supposed that $\exists \mathcal{E}_1$ such that $\mathcal{E}_1$ is a stable extension, i.e. $n \geq 1$. According to Property 12, $a' \in \bigcap_{i=1}^n A_i$ implies $a' \in \mathcal{E}_1$. using the same property one more time, $a' \in \mathcal{E}_1$ implies $a' \in \bigcup_{i=1}^n A_i$. Contradiction with the fact $(\forall a \in \mathcal{H}(o))$ $a \notin \bigcup_{i=1}^n A_i$. So, $\mathcal{O}_s^r \subseteq \mathcal{O}_g^r$.

□

Property 20 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^n_g = \emptyset$.*

Proof. Let us suppose that $\mathcal{O}^n_g \neq \emptyset$. So, $\exists o \in \mathcal{O}$ such that $\exists a \in \mathcal{H}(o)$ such that a is credulously accepted. According to Definition 12, $\exists \mathcal{E}_i$ such that $a \in \mathcal{E}_i$ and $\exists \mathcal{E}_j$ such that $a \notin \mathcal{E}_j$, where $\mathcal{E}_i$ and $\mathcal{E}_j$ are different grounded extensions. But, there is always exactly one grounded extension $\mathtt{GE}$. So, $\mathcal{E}_i = \mathcal{E}_j = \mathtt{GE}$. Thus, we have $a \in \mathtt{GE}$ and $a \notin \mathtt{GE}$. Contradiction. □

Property 21 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^n_g \subseteq \mathcal{O}^n_s$.*

Proof. According to Property 20, $\mathcal{O}^n_g = \emptyset$. So, $\mathcal{O}^n_g \subseteq \mathcal{O}^n_s$. □

Property 22 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^n_s \subseteq \mathcal{O}^n_p$.*

Proof. Let $o \in \mathcal{O}^n_s$ and assume that $o \notin \mathcal{O}^n_p$. Since $o \in \mathcal{O}^n_s$ then $\exists a \in \mathcal{H}(o)$ such that a is credulously accepted. So, there exist two stable extensions S_i, S_j such that $a \in S_i$ and $a \notin S_j$. According to the fourth bullet of Property 5, S_i and S_j are also preferred extensions. Consequently, $a \in \mathcal{O}^n_p$. □

Property 23 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^n_p \subseteq \mathcal{O}^n_c$.*

Proof. Let $o \in \mathcal{O}^n_p$. This means that $\exists a \in \mathcal{H}(o)$ such that there exist two preferred extensions S_i, S_j such that $a \in S_i$ and $a \notin S_j$. However, according to the fourth bullet of Property 5, S_i and S_j are also complete extensions. Thus, $o \in \mathcal{O}^n_c$. □

Property 24 *Let $\mathcal{O}$ be the set of options. It holds that $\mathcal{O}^n_c \subseteq \mathcal{O}^n_{ad}$.*

Proof. Let $o \in \mathcal{O}^n_c$. So, $\exists a \in \mathcal{H}(o)$ such that there exist two complete extensions S_i, S_j such that $a \in S_i$ and $a \notin S_j$. On the other hand, according to the fourth bullet of Property 5, S_i and S_j are also admissible extensions. Thus, $o \in \mathcal{O}^n_{ad}$. □

Theorem 3 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making. The graph $\mathcal{G}_{\mathcal{AF}_o}$ has no elementary odd-length cycles.*

Proof. Let $a_1, \ldots, a_{2n+1}$ be arguments of $\mathcal{A}_o$. Let us assume that there is an elementary odd-length cycle between these arguments, i.e. $\forall i \leq 2n$, $(a_i, a_{i+1}) \in \mathtt{Def}_o$, and $(a_{2n+1}, a_1) \in \mathtt{Def}_o$. Since the cycle is elementary, then $\nexists a_i, a_{i+1}$ such that $(a_i,\, a_{i+1}) \in \mathtt{Def}_o$ and $(a_{i+1},\, a_i) \in \mathtt{Def}_o$. Thus, $(a_i, a_{i+1}) \in >_o$, $\forall i \leq 2n$. Thus, $(a_1, a_2) \in >_o \ldots (a_{2n}, a_{2n+1}) \in >_o, (a_{2n}, a_1) \in >_o$. Since the relation $>_o$ is transitive, then we have both $(a_1, a_{2n+1}) \in >_o$ and $(a_{2n+1}, a_1) \in >_o$. Contradiction. □

Theorem 4 *The system $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ is coherent, i.e. each preferred extension is a stable one.*

Proof. According to Theorem 3, the graph associated with $\mathcal{AF}_o$ has no elementary odd-length cycles. Moreover, according to Theorem 1, if the graph associated with an argumentation system has no elementary odd-length cycles then it is coherent. □

Theorem 5 *The system $\mathcal{AF}_o$ has at least one* non-empty *extension.*

Proof. According to Theorem 3, the graph associated with $\mathcal{AF}_o$ has no elementary odd-length cycles. Besides, Berge has proved in [8] that a graph without elementary odd-length cycles has a maximal *non-empty* kernel. Moreover, Doutre has shown in [12] that a maximal kernel corresponds exactly to a preferred extension. Thus $\mathcal{AF}_o$ has at least one non-empty preferred extension. □

Theorem 6 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ be a complete argumentation framework for decision making and $a \in \mathcal{A}$. There exists an extension $\mathcal{E}$ such that $a \in \mathcal{E}$ iff a is self-defending.*

Proof. $\Rightarrow$ Assume that $\mathcal{E}$ is an extension of $\mathcal{AF}_o$, $a \in \mathcal{E}$, $a' \in \mathcal{A}_o$, $(a', a) \in \texttt{Def}_o$, $(a, a') \notin \texttt{Def}_o$. Then, there must be $a'' \in \mathcal{E}$ such that $(a'', a') \in \texttt{Def}_o$. Then $(a, a'') \notin \texttt{Def}_o$ and $(a'', a) \notin \texttt{Def}_o$ because otherwise $\mathcal{E}$ is self-conflicting.

Since the relation $\mathcal{R}$ is complete, then $(a'', a) \in \mathcal{R}$ and $(a, a'') \in \mathcal{R}$. Moreover, the facts $(a, a'') \notin \texttt{Def}_o$ and $(a'', a) \notin \texttt{Def}_o$ mean (according to Definition 13) that $(a, a'') \in \geq_o$ and $(a'', a) \in \geq_o$.

Similarly, since $\mathcal{R}$ is complete, then $(a', a) \in \mathcal{R}$. Thus, the facts $(a', a) \in \texttt{Def}_o$ and $(a, a') \notin \texttt{Def}_o$ mean (according to Definition 13) that $(a', a) \in >_o$.

From $(a', a) \in >_o$ and $(a, a'') \in \geq_o$, according to Property 8, we have $(a', a'') \in >_o$, therefore $(a'', a') \notin \texttt{Def}_o$, contradiction.

$\Leftarrow$ Assume that argument a is self-defending. Then $\{a\}$ is an admissible extension of $\mathcal{AF}_o$, and therefore must belong to some preferred extension of $\mathcal{AF}_o$ (according to Property 2). □

Theorem 7 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ be a complete argumentation framework for decision making and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ its extensions. Let $a, b \in \mathcal{A}$ be two arguments such that $a, b \in \bigcup_{i=1}^{n} \mathcal{E}_i$ and $\nexists \mathcal{E}_i$ such that $a, b \in \mathcal{E}_i$, for $i = 1, \ldots, n$. Then $(a, b) \in \texttt{Def}_o$ and $(b, a) \in \texttt{Def}_o$.*

Proof. We proceed by case analysis. Assume first that $(a, b) \in \texttt{Def}_o$ and $(b, a) \notin \texttt{Def}_o$. Then b is not self-defending, which by Theorem 6 means that it cannot belong to any extension of $\mathcal{AF}_o$, contradiction.

Now assume that $(a, b) \notin \texttt{Def}_o$ and $(b, a) \notin \texttt{Def}_o$. Since both a and b belong to some extension, they are self-defending. Therefore set $\{a, b\}$ is admissible which contradicts the assumption that there is no extension of $\mathcal{AF}_o$ which contains both. Hence, it must be the case that $(a, b) \in \texttt{Def}_o$ and $(b, a) \in \texttt{Def}_o$. □

Theorem 8 *The extensions of $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ are pairwise disjoint.*

Proof. Assume that $\mathcal{E}_1$, $\mathcal{E}_2$ are two extensions that are not disjoint. Then, there exists $x \in \mathcal{E}_1 \cap \mathcal{E}_2$. Moreover, since extensions are distinct, then there must be two arguments a and b such that $a \in \mathcal{E}_1$, $a \notin \mathcal{E}_2$, $b \in \mathcal{E}_2$, $b \notin \mathcal{E}_1$. Since $\mathcal{E}_1, \mathcal{E}_2$ are stable extensions, they are conflict-free. Consequently, $(a,x),(x,a),(b,x),(x,b) \notin \mathtt{Def}_o$. Since $\mathcal{R}$ is complete, $(a,x),(x,a),(b,x),(x,b) \in \mathcal{R}_o$. According to definition of $\mathtt{Def}_o$, it must be the case that $(a,x),(x,a),(b,x),(x,b) \in \geq_o$. Transitivity of preference relation implies $(a,b),(b,a) \in \geq_o$. This means that $(a,b) \notin \mathtt{Def}_o$ and $(b,a) \notin \mathtt{Def}_o$, which contradicts with Theorem 7. □

Corollary 1 *The system $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ has a skeptically accepted argument iff it has exactly one extension.*

Proof. The proof follows directly from Theorem 8 and Definition 12. Indeed, an argument cannot belong to more than one extension. □

Theorem 9 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making. An argument $a \in \mathcal{A}_l$ is skeptically accepted iff $ing_{\mathcal{G}}(a) = 0$.*

Proof. $\Rightarrow$ Let $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be the stable extensions of $\mathcal{AF}_o$ and $a \in \bigcap_{i=1}^n \mathcal{E}_i$. Assume that there exists $a' \in \mathcal{A}_o$ such that $(a', a) \in \mathtt{Def}_o$. Since there is a conflict between these two arguments, then a' is not in any extension, $a' \notin \bigcup_{i=1}^n \mathcal{E}_i$. By Theorem 7, we know that $(a, a') \in \mathtt{Def}_o$. Since $a' \notin \bigcup_{i=1}^n \mathcal{E}_i$, there must be $b_1 \in \mathcal{A}_o$ such that $(b_1, a') \in \mathtt{Def}_o$ and $(a', b_1) \notin \mathtt{Def}_o$. Clearly $b_1 \notin \bigcup_{i=1}^n \mathcal{E}_i$ because otherwise $(a, b_1) \in \geq_o$ and $(b_1, a') \in \geq_o$ which by transitivity implies $(a, a') \in \geq_o$ which contradicts the fact that $(a', a) \in \mathtt{Def}_o$. Therefore, it must be the case that $b_1 \notin \bigcup_{i=1}^n \mathcal{E}_i$. Moreover, it holds that for all $e \in \bigcup_{i=1}^n \mathcal{E}_i$, if $(e, b_1) \in \mathtt{Def}_o$ than $(b_1, e) \in \mathtt{Def}_o$. To prove this, consider the case where there exists $e \in \bigcup_{i=1}^n \mathcal{E}_i$ such that $(e, b_1) \in \mathtt{Def}_o$ and $(b_1, e) \notin \mathtt{Def}_o$. Then, $(a, e) \in \geq_o$, $(e, b_1) \in \geq_o$, $(b_1, a') \in \geq_o$, which, by the transitivity of preference, implies that $(a, a') \in \geq_o$, a contradiction.

The above means that we can construct a sequence of arguments $b_1, \ldots, b_k \notin \bigcup_{i=1}^n \mathcal{E}_i$ such that:

1. $(b_i, b_{i-1}) \in \mathtt{Def}_o$ and $(b_{i-1}, b_i) \notin \mathtt{Def}_o$
2. For all b_i, $1 \leq i \leq n$, and all $e \in \bigcup_{i=1}^n \mathcal{E}_i$, if $(e, b_i) \in \mathtt{Def}_o$ than $(b_i, e) \in \mathtt{Def}_o$.

Consider a maximal such sequence and its first element b_k.

We will now prove that that all the arguments in this sequence are different. Suppose that $(\exists i, j \in \{1, \ldots, n\})$ $b_i = b_j$. Without loss of generality, suppose that $i > j$. Then, because of transitivity of the relation $>_o$, we have $(b_i, b_j) \in >_o$. On the other hand, $b_i = b_j$, so $(b_i, b_i) \in >_o$. This implies that $(b_i, b_i) \in \geq_o$ and $(b_i, b_i) \notin \geq_o$. Contradiction. Hence, all the arguments in this sequence are different. Since there is a finite number of arguments and all the arguments in the sequence are different, the sequence is finite.

For an arbitrary element $c \in \mathcal{A} \setminus (\bigcup_{i=1}^n \mathcal{E}_i)$ one of the following will hold:

1. $(c, b_k) \in \mathtt{Def}_o$ and $(b_k, c) \notin \mathtt{Def}_o$
2. $(c, b_k) \notin \mathtt{Def}_o$ or $(b_k, c) \in \mathtt{Def}_o$.

(because the second statement is negation of the first one).

In case (1), since $b_1, \ldots, b_k$ was a maximal sequence that verified first two conditions (those for constructing a maximal sequence $b_1, \ldots, b_n$), it must be the case that there exists $e \in \bigcup_{i=1}^n \mathcal{E}_i$ such that $(e, c) \in \mathtt{Def}_o$ and $(c, e) \notin \mathtt{Def}_o$. So, $(a, e) \in \geq$, $(e, c) \in \geq$, $(c, b_k) \in \geq$, $(b_k, a') \in \geq$, which by transitivity implies $(a, a') \in \geq$. Contradiction. Then, for all $c \in$

$\mathcal{A} \setminus (\bigcup_{i=1}^{n} \mathcal{E}_i)$ it holds that $(c, b_k) \notin \texttt{Def}_o$ or $(b_k, c) \in \texttt{Def}_o$. So, b_k is self-defending, which, by Theorem 6 means that there must be some extension that contains b_k. This contradicts the fact that $b_k \notin \bigcup_{i=1}^{n} \mathcal{E}_i$. Therefore, such a sequence cannot be constructed, and therefore there is no argument $a' \in \mathcal{A}_o$ such that $(a', a) \in \texttt{Def}_o$. Contradiction. Thus, $in_{\mathcal{G}}(a) = 0$.

$\Leftarrow$ Suppose that $in_{\mathcal{G}}(a) = 0$. Suppose now that stable extensions are $\mathcal{E}_1, \ldots, \mathcal{E}_n$. Let us prove that a is in every extension, and thus skeptically accepted. Suppose the converse, i.e., suppose that there exists an extension $\mathcal{E}_i$ such that $a \notin \mathcal{E}_i$. Since $\mathcal{E}_i$ is stable than it defeats all arguments that don't belong to itself. So, $\mathcal{E}_i$ defeats a. Contradiction, since we supposed $in_{\mathcal{G}}(a) = 0$. □

Property 25 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ be a complete argumentation framework for decision making. Let a be an arbitrary argument. Then:*

1. *a is skeptically accepted iff $(\forall x \in \mathcal{A}_o)$ $(a, x) \in \geq_o$.*
2. *a is rejected iff $(\exists x \in \mathcal{A})$ $(x, a) \in >_o$.*
3. *a is credulously accepted iff $((\exists x' \in \mathcal{A})$ $(a, x') \notin \geq_o) \wedge ((\forall x \in \mathcal{A})$ $((a, x) \notin \geq_o) \Rightarrow (x, a) \notin \geq_o))$.*

Proof.

1. $\Rightarrow$ Suppose that a is skeptically accepted. According to Theorem 9, $(\nexists x \in \mathcal{A}_o)$ $(x, a) \in \texttt{Def}_o$. Suppose that there is an argument x' such that $(a, x') \notin \geq_o$. Since $\mathcal{R}_o$ is complete, then $(x', a) \in \mathcal{R}_o$. Thus, according to definition of $\texttt{Def}_o$, we have $(x', a) \in \texttt{Def}_o$. Contradiction with the fact $(\nexists x \in \mathcal{A}_o)$ $(x, a) \in \texttt{Def}_o$.

   $\Leftarrow$ Let us now suppose that $(\forall x \in \mathcal{A}_o)$ $(a, x) \in \geq_o$ and that a is not skeptically accepted. Since a is not skeptically accepted, according to Theorem 9, $(\exists x')$ $(x, a) \in \texttt{Def}_o$. Since $(x', a) \in \texttt{Def}_o$ then, according to definition of $\texttt{Def}_o$, $(a, x') \notin \geq_o$. Contradiction with the fact $(\forall x \in \mathcal{A}_o)$ $(a, x) \in \geq_o$.
2. $\Rightarrow$ Suppose that a is rejected. Then, there is no extension $\mathcal{E}$ such that $a \in \mathcal{E}$. Then, according to Theorem 6, a is not self-defending. So, $(\exists x' \in \mathcal{A}_o)$ $((x', a) \in \texttt{Def}_o \wedge (a, x') \notin \texttt{Def}_o)$. Since $(x', a) \in \mathcal{R}_o$ and $(x', a) \in \texttt{Def}_o$ then, according to definition of $\texttt{Def}_o$, we have $(a, x') \notin \geq_o$. Since $(a, x') \in \mathcal{R}_o$ and $(a, x') \notin \texttt{Def}_o$ then, according to definition of $\texttt{Def}_o$, we have $(x', a) \in \geq_o$. According to definition of $>_o$, $(a, x') \notin \texttt{Def}_o$ and $(x', a) \in \geq_o$ give $(x', a) \in >_o$.

   $\Leftarrow$ Suppose now that $(\exists x' \in \mathcal{A}_o)$ $(x', a) \in >_o$. Since the relation $\mathcal{R}_o$ is complete, we have $(x', a) \in \mathcal{R}_o$. According to definition of $>_o$, we have $(a, x') \notin \geq_o$. These two facts, together with the the definition of $\texttt{Def}_o$ imply $(x', a) \in \texttt{Def}_o$. The fact that $(x', a) \in \geq_o$ implies that, according to definition of $\texttt{Def}_o$, $(a, x') \notin \texttt{Def}_o$. So, $(x', a) \in \texttt{Def}_o$ and $(a, x') \notin \texttt{Def}_o$ which means that a is not self-defending. According to Theorem 6, there is no extension $\mathcal{E}$ such that $a \in \mathcal{E}$. So, a is rejected.
3. $\Rightarrow$ Let us suppose that a is credulously accepted. According to Definition 12, there is at least one extension $\mathcal{E}_i$ such that $a \in \mathcal{E}_i$. According to Theorem 6, since a is in $\mathcal{E}_i$ then a is self-defending. Suppose now that $(a, x') \notin \geq_o$. So, $(x', a) \in \texttt{Def}_o$. Since a is self-defending, we have $(a, x') \in \texttt{Def}_o$. So, $(x', a) \notin \geq_o$. Hence, $((\forall x \in \mathcal{A})$ $((a, x) \notin \geq_o) \Rightarrow (x, a) \notin \geq_o))$. We will now prove that $((\exists x' \in \mathcal{A})$ $(a, x') \notin \geq_o)$. Since a is not skeptically accepted, then, according to Theorem 9, $(\exists y' \in \mathcal{A}_o)$ $(y', a) \in \texttt{Def}_o$. This means that $(a, y') \notin \geq_o$. So, we proved that $((\exists x' \in \mathcal{A})$ $(a, x') \notin \geq_o) \wedge ((\forall x \in \mathcal{A})$ $((a, x) \notin \geq_o) \Rightarrow (x, a) \notin \geq_o))$.

   $\Leftarrow$ Let us now suppose that $((\exists x' \in \mathcal{A})$ $(a, x') \notin \geq_o) \wedge ((\forall x \in \mathcal{A})$ $((a, x) \notin \geq_o) \Rightarrow (x, a) \notin \geq_o))$. We have $((\exists x' \in \mathcal{A})$ $(a, x') \notin \geq_o)$, so $(x', a) \in \texttt{Def}_o$. Thus, according to

Theorem 9, a is not skeptically accepted. Suppose now that a is rejected. That means that $((\exists x' \in \mathcal{A})\ ((x', a) \in \mathtt{Def}_o)$ and $((a, x') \notin \mathtt{Def}_o)$. The fact $((x', a) \in \mathtt{Def}_o)$ implies $((a, x') \notin \geq_o)$. According to the assumption $((\forall x \in \mathcal{A})\ ((a, x) \notin \geq_o) \Rightarrow (x, a) \notin \geq_o))$, we have $((x', a) \notin \geq_o)$. Thus, $((a, x') \in \mathtt{Def}_o)$. Contradiction. Since a is neither skeptically accepted nor rejected, it is credulously accepted.

□

Theorem 10 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making. The following implications hold:*

1. *If $\mathtt{Sc}(\mathcal{AF}_o) \neq \emptyset$ then $\mathtt{Cr}(\mathcal{AF}_o) = \emptyset$.*
2. *If $\mathtt{Cr}(\mathcal{AF}_o) = \emptyset$ then $\mathtt{Sc}(\mathcal{AF}_o) \neq \emptyset$.*

Proof.

1. Since the framework has a skeptically accepted argument, then according to Corollary 1, it has only one extension, say $\mathcal{E}$. Suppose that $(\exists a \in \mathcal{A}_o)$ a is credulously accepted. According to Definition 12, there are two different extensions $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $a \in \mathcal{E}_1$ and $a \notin \mathcal{E}_2$. Contradiction with the fact that there is exactly one extension.
2. According to Theorem 5, the argumentation framework $\mathcal{AF}_o$ has at least one non-empty extension $\mathcal{E}_1$. Let $a \in \mathcal{E}_1$ be an arbitrary argument which belongs to this extension. Since $a \in \mathcal{E}_1$, according to Definition 12, a is skeptically accepted or credulously accepted. Since we have supposed that there are no credulously accepted arguments, then a is skeptically accepted.

□

Property 26 *Let $o \in \mathcal{O}$. The option o is negotiable iff there is at least one credulously accepted argument in its favor.*

Proof. $\Rightarrow$ Trivial, according to Definition 16.

$\Leftarrow$ Let a be an credulously accepted argument in favor of o. Since there exists at least one credulously accepted argument, Theorem 10 implies that there are no skeptically accepted arguments. In particular, there are no skeptically accepted arguments in favor of o. According to Definition 16, o is negotiable. □

Property 27 *The following equivalences hold.*

1. *There is at least one skeptically accepted argument iff there is at least one acceptable option.*
2. *There is at least one credulously accepted argument iff there is at least one negotiable option.*

Proof.

1. $\Rightarrow$ Suppose that there is at least one skeptically accepted argument a. Since all the arguments are practical arguments, a is in favor of some option o. Then, according to Definition 16, o is acceptable.

   $\Leftarrow$ Let us now suppose that there is at least one acceptable option o. Then, according to Definition 16, there is at least one skeptically accepted argument a such that $a \in \mathcal{H}(o)$.

2. $\Rightarrow$ Suppose that there is at least one credulously accepted argument a. Then, according to Theorem 10, there are no skeptically accepted arguments. Since all the arguments are practical arguments, a is in favor of some option o. Since there are no skeptically accepted arguments at all, there are no skeptically accepted arguments in favor of option o. So, there is at least one credulously accepted argument a in favor of option o and there are no skeptically accepted arguments in favor of option o. According to Definition 16, o is negotiable.

   $\Leftarrow$ Let us now prove the last part of the property. Suppose that there is at least one negotiable option o. Then, according to Definition 16, there is at least one credulously accepted argument a in its favor.

$\square$

Theorem 11 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making. The following holds: $\mathcal{O}_a \neq \emptyset \Leftrightarrow \mathcal{O}_n = \emptyset$.*

Proof. $\Rightarrow$ Let $\mathcal{O}_a \neq \emptyset$. According to Property 27, there is at least one skeptically accepted argument. Then, according to Theorem 10, there are no credulously accepted arguments. Using Property 27, we conclude that there are no negotiable options.

$\Leftarrow$ Let $\mathcal{O}_n = \emptyset$. According to Property 27, there are no credulously accepted arguments. Then, according to Theorem 10, there is at least one skeptically accepted argument. The Property 27 implies that there is at least one acceptable option. $\square$

Property 28 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $a \in \mathcal{A}_o$. If $a \in \mathtt{Rej}(\mathcal{AF}_o)$ then $(\exists x' \in \mathcal{A}_o)$ such that $x' \notin \mathtt{Rej}(\mathcal{AF}_o) \wedge (x', a) \in >_o$.*

Proof. Let $a \in \mathcal{A}_o$. Assume that a is rejected. Thus, from Property 25, there is at least one argument z_0 such that $(z_0, a) \in >_o$. Since z_0 is rejected, there exists at least one argument z_1 such that $(z_1, z_0) \in >_o$. Now, we can construct the sequence of arguments $z_0, \ldots, z_k$ such that $(\forall i \in \{1, \ldots, k\})$ $(z_i, z_{i-1}) \in >_o$. Let $z_0, \ldots, z_n$ be a maximal such a sequence. We will now prove that that all the arguments in this sequence are different. Suppose that $(\exists i, j \in \{0, \ldots, n\})$ $z_i = z_j$. Without loss of generality, suppose that $i > j$. Then, because of transitivity of the relation $>_o$, we have $(z_i, z_j) \in >_o$. On the other hand, $z_i = z_j$, so $(z_i, z_i) \in >_o$. This implies that $(z_i, z_i) \in \geq_o$ and $(z_i, z_i) \notin \geq_o$. Contradiction. Hence, all the arguments in this sequence are different. Since there is a finite number of arguments and all the arguments in the sequence are different, the sequence is finite. So, let z_n be the last argument in this sequence. Note that, because of the transitivity of relation $>_o$, it holds that $(z_n, x) \in >_o$. The argument z_n can be rejected or not. Suppose that it is rejected. Then, the fact that it is rejected implies that $(\exists z_{n+1})$ $(z_{n+1}, z_n) \in >_o$. Contradiction with the fact that the sequence which ends with z_n is maximal. Suppose that z_n is not rejected. So, $(z_n, x) \in >_o$ and z_n is not rejected. Contradiction with the fact $(\forall x \in \mathcal{A}_o)$ $(x, a) \in >_o \Rightarrow x \in \mathtt{Rej}(\mathcal{AF}_o)$. In both cases we have a contradiction, so the assumption was false. Hence, $(\exists x' \in \mathcal{A}_o)$ $(x', a) \in >_o \wedge x' \notin \mathtt{Rej}(\mathcal{AF}_o)$. $\square$

Property 29 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $e \in \mathcal{A}_o$. If $\mathcal{A}_o \setminus \{e\} \subseteq \mathtt{Rej}(\mathcal{AF}_o)$ then $e \in \mathtt{Sc}(\mathcal{AF}_o)$.*

Proof. Suppose that e is not skeptically accepted. Then, e is credulously accepted or rejected.

1. Suppose that e is rejected. According to Theorem 28, $(\exists x' \in \mathcal{A}_o)\ x' \notin \mathtt{Rej}(\mathcal{AF}_o)$ and $(x', e) \in >_o$. Contradiction with the fact that all the arguments are rejected.
2. Suppose that $e \in \mathtt{Cr}(\mathcal{AF}_o)$. According to Property 25, $((\exists x' \in \mathcal{A})\ (e, x') \notin \geq_o) \wedge ((\forall x \in \mathcal{A})\ ((e, x) \notin \geq_o) \Rightarrow (x, e) \notin \geq_o))$. Since there are no self-attacking arguments, we have $x' \neq e$. Since $x' \neq e$ and all the arguments except e are rejected, then x' is rejected. According to Theorem 28, $(\exists y' \in \mathcal{A}_o)$ such that y' is not rejected and $(y', x') \in >_o$. Since y' is not rejected and all the arguments except e are rejected, then $y' = e$. Since $(y', x') \in >_o$ and $y' = e$, then $(e, x') \in >_o$. Since $(e, x') \in >_o$ then $(e, x') \in \geq_o$. Contradiction with the fact $(e, x') \notin \geq_o$.

□

Property 30 *Let $a, b \in \mathtt{Sc}(\mathcal{AF}_o)$. Then $(a, b) \in \geq_o$ and $(b, a) \in \geq_o$.*

Proof. Since the system has a skeptically accepted argument, according to Corollary 1, there is exactly one extension $\mathcal{E}$. Since both a and b are accepted, then $a, b \in \mathcal{E}$. Since $\mathcal{E}$ is conflict-free, $(a, b) \notin \mathtt{Def}_o$ and $(b, a) \notin \mathtt{Def}_o$. The fact $(a, b) \notin \mathtt{Def}_o$ implies $(b, a) \in \geq_o$ and, similarly, $(b, a) \notin \mathtt{Def}_o$ implies $(a, b) \in \geq_o$. □

Property 31 *Let e be an arbitrary argument.*
If $((\exists a \in \mathtt{Sc}(\mathcal{AF}_o))$ such that $(a, e) \in \odot)$ then $((\forall a \in \mathtt{Sc}(\mathcal{AF}_o))\ (a, e) \in \odot)$.

Proof. Let us suppose that $((\exists a \in \mathcal{A}_o)\ a \in \mathtt{Sc}(\mathcal{AF}_o) \wedge (a, e) \in \odot)$. Let b be an arbitrary accepted argument. According to Property 30, $(a, b) \in \geq_o$ and $(b, a) \in \geq_o$. According to Property 8, $(a, e) \in \odot$ implies $(b, e) \in \odot$. □

Property 32 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $\mathtt{Cr}(\mathcal{AF}_o)$ its credulously accepted arguments. Then $(\forall a, b \in \mathtt{Cr}(\mathcal{AF}_o)$ it holds that*

$$((a, b) \in \geq_o \wedge (b, a) \in \geq_o) \vee ((a, b) \notin \geq_o \wedge (b, a) \notin \geq_o).$$

Proof. Suppose that the $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))(\exists b \in \mathtt{Cr}(\mathcal{AF}_o))\ \neg((a, b) \in \geq_o \wedge (a, b) \in \geq_o) \wedge \neg((a, b) \notin \geq_o \wedge (a, b) \notin \geq_o)$. Then, either $(a, b) \in >_o$ or $(b, a) \in >_o$ Without loss of generality, we can suppose that $(a, b) \in >_o$. Then, with $(a, b) \in \mathcal{R}_o$, we have $(a, b) \in \mathtt{Def}_o$ and $(b, a) \notin \mathtt{Def}_o$. So, the argument b is not self-defending. According to Theorem 6, there is no extension $\mathcal{E}$ such that $b \in \mathcal{E}$. Consequently, b is not credulously accepted. Contradiction with the fact $b \in \mathtt{Cr}(\mathcal{AF}_o)$. □

Property 33 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $\mathtt{Cr}(\mathcal{AF}_o) \neq \emptyset$. Then it holds that: $(\forall a' \in \mathtt{Cr}(\mathcal{AF}_o))\ (\exists a'' \in \mathtt{Cr}(\mathcal{AF}_o))\ (a', a'') \notin \geq_o \wedge (a'', a') \notin \geq_o$.*

Proof. Suppose the converse. Then $(\exists a' \in \mathtt{Cr}(\mathcal{AF}_o))\ (\forall a \in \mathtt{Cr}(\mathcal{AF}_o))\ \neg\ ((a, a') \notin \geq_o \wedge (a', a) \notin \geq_o)$. Recall the result of the Property 32 which states that $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))(\forall b \in \mathtt{Cr}(\mathcal{AF}_o))\ ((a, b) \in \geq_o \wedge (b, a) \in \geq_o) \vee ((a, b) \notin \geq_o \wedge (b, a) \notin \geq_o)$. So, if for two credulously accepted arguments a and a' it holds that $\neg\ ((a, a') \notin \geq_o \wedge (a', a) \notin \geq_o)$, then it must be the case that $((a, a') \in \geq_o \wedge (a', a) \in \geq_o)$. So, $(\exists a' \in \mathtt{Cr}(\mathcal{AF}_o))\ (\forall a \in \mathtt{Cr}(\mathcal{AF}_o))\ ((a, a') \in \geq_o \wedge$

$(a', a) \in \geq_o)$. Let $b, c \in \mathtt{Cr}(\mathcal{AF}_o)$. Since $(b, a') \in \geq_o$ and $(a', c) \in \geq_o$, then, because of the transitivity of the preference relation, $(b, c) \in \geq_o$. Similarly, since $(c, a') \in \geq_o$ and $(a', b) \in \geq_o$, then $(c, b) \in \geq_o$. So, all the credulously accepted arguments are in the same class of equivalence with respect to $\geq_o$. This means that there is no attack in the sense of $\mathtt{Def}_o$ between the arguments of $\mathtt{Cr}(\mathcal{AF}_o)$. So, $\mathtt{Cr}(\mathcal{AF}_o)$ is admissible. Since there are some credulously accepted arguments, according to Definition 12, there are at least two different non-empty preferred extensions $\mathcal{E}_1$ and $\mathcal{E}_2$. Since there are some credulously accepted arguments, then, according to Theorem 10, there are no skeptically accepted arguments. Since all the arguments in $\mathcal{E}_1$ and $\mathcal{E}_2$ are in some extension, they are not rejected. Since there are no skeptically accepted arguments, they are credulously accepted. The Theorem 8 states that $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$. All the arguments that are not in $\mathtt{Cr}(\mathcal{AF}_o)$ are not credulously accepted. Since there are no skeptically accepted arguments, they are rejected. Let us prove that $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathtt{Cr}(\mathcal{AF}_o)$. If $\neg(\mathcal{E}_1 \subseteq \mathtt{Cr}(\mathcal{AF}_o))$ then there is some argument which is credulously accepted (since it is in $\mathcal{E}_1$) and in the same time it is rejected (since it is not in $\mathtt{Cr}(\mathcal{AF}_o)$). Contradiciton. So, $\mathcal{E}_1 \subseteq \mathtt{Cr}(\mathcal{AF}_o)$. The same proof for $\mathcal{E}_2$. So, $\mathcal{E}_1$ and $\mathcal{E}_2$ are preferred extensions and $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$ and $\mathcal{E}_1 \neq \emptyset$ and $\mathcal{E}_2 \neq \emptyset$. Since $\mathcal{E}_2 \neq \emptyset$, then $\mathcal{E}_1 \neq \mathtt{Cr}(\mathcal{AF}_o)$. So, $\mathcal{E}_1$ is preferred and $\mathtt{Cr}(\mathcal{AF}_o)$ is admissible and $\mathcal{E}_1 \subseteq \mathtt{Cr}(\mathcal{AF}_o)$ and $\mathcal{E}_1 \neq \mathtt{Cr}(\mathcal{AF}_o)$. Contradiction, because, according to Definition 6, a preferred extension is a maximal admissible extension. □

Property 34 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $e \notin \mathcal{A}_o$.*

1. *If $a \in \mathtt{Rej}(\mathcal{AF}_o)$, then $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$.*
2. *If $a \in \mathtt{Cr}(\mathcal{AF}_o)$, then $a \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.*

Proof.

1. Let $a \in \mathcal{A}_o$. Assume that a is rejected in $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$. According to Property 25, $\exists x \in \mathcal{A}_o$ such that $(x, a) \in >_o$. Let $e \notin \mathcal{A}_o$. $\mathcal{AF}_o \oplus e$ is an argumentation system such that its set of arguments is $\mathcal{A}_o \cup \{e\}$. So, $a, x \in \mathcal{A}_o \cup \{e\}$, which (according to Property 25) means that a is rejected in $\mathcal{AF}_o \oplus e$.
2. Assume that a is credulously accepted in $\mathcal{AF}_o$. Thus, according to Property 25, $\exists x \in \mathcal{A}_o$ such that $(a, x) \notin \geq_o$. It is clear that $a, x \in \mathcal{A}_o \cup \{e\}$. Assume that a is skeptically accepted in the system $\mathcal{AF}_o \oplus e$. According to Property 25, $(\forall x \in \mathcal{A}_o \cup \{e\})$ $(a, x) \in \geq_o$. Contradiction with the fact $(a, x) \notin \geq_o$.

□

Property 35 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $e \notin \mathcal{A}_o$.*

1. *If $a \in \mathtt{Sc}(\mathcal{AF}_o)$ then $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ iff $(a, e) \in \geq_o$.*
2. *If $a \notin \mathtt{Rej}(\mathcal{AF}_o)$ then $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$ iff $(e, a) \in >_o$.*

Proof. 1. Let $a \in \mathtt{Sc}(\mathcal{AF}_o)$.
$\Rightarrow$ Suppose that $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ and $(a, e) \notin \geq_o$. Since the attack relation $\mathcal{R}_o$ is complete, then $(a, e) \in \mathcal{R}_o$ and $(e, a) \in \mathcal{R}_o$. With $(a, e) \notin \geq_o$, we have $(e, a) \in \mathtt{Def}_o$. Since $(e, a) \in \mathtt{Def}_o$, according to Property 25, we have that $a \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Contradiction.
$\Leftarrow$ Let $(a, e) \in \geq_o$. Since $a \in \mathtt{Sc}(\mathcal{AF}_o)$, according to Property 25, $(\forall x \in \mathcal{A}_o)$ $(a, x) \in \geq_o$. Suppose that $a \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Then, according to Property 25, $(\exists x' \in \mathcal{A}_o \cup \{e\})$

$(a, x') \notin \geq_o$. We will prove that $x' \notin \mathcal{A}_o$. Suppose the converse, i.e., suppose that $x' \in \mathcal{A}_o$. Since $(\forall x \in \mathcal{A}_o)$ $(a, x) \in \geq_o$, then $(a, x') \in \geq_o$. Contradiction, so it must be the case that $x' \notin \mathcal{A}_o$. With $x' \in \mathcal{A}_o \cup \{e\}$ and $x' \notin \mathcal{A}_o$ we have $x' = e$, and, consequently, $(a, e) \notin \geq_o$. Contradiction.

2. Let $a \in A_o \setminus \mathtt{Rej}(\mathcal{AF}_o)$.
   $\Rightarrow$ Let a become rejected. Since $a \notin \mathtt{Rej}(\mathcal{AF}_o)$, then, according to Property 25, $(\nexists x \in \mathcal{A}_o)$ $(x, a) \in >_o$. Since $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, then, according to Property 25, $(\exists y \in \mathcal{A}_o \cup \{e\})$ $(y, a) \in >_o$. We will prove that $y = e$. Suppose not. Then, $y \in \mathcal{A}_o$ and $(y, a) \in >_o$. Contradiction with the fact $(\nexists x \in \mathcal{A}_o)$ $(x, a) \in >_o$. So, $y = e$ and, consequently, $(e, a) \in >_o$.
   $\Leftarrow$ Let $(e, a) \in >_o$. Since $(e, a) \in >_o$, then, according to Property 25, a is rejected.

□

Property 36 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making and $a, b \in \mathtt{Sc}(\mathcal{AF}_o)$. Let $e \notin \mathcal{A}_o$.*

1. *If $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ then $b \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.*
2. *If $a \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$ then $b \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$.*
3. *If $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$ then $b \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$.*

Proof.

1. Since $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$, then, according to Property 35, $(a, e) \in \geq_o$. According to Property 31, $(b, e) \in \geq_o$. According to Property 35, $b \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.
2. Since $a \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$, then, according to Property 35, $(a, e) \notin \geq_o$. Since $a \notin \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, then, according to Property 35, $(e, a) \notin >_o$. According to Property 31, $(b, e) \notin \geq_o$ and $(e, b) \notin >_o$. Since $(b, e) \notin \geq_o$, then, according to Property 35, $b \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Since $(e, b) \notin >_o$, then, according to Property 35, $b \notin \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Hence, according to Property 7, $b \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$.
3. Since $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, then, according to Property 35, $(e, a) \in >_o$. According to Property 31, $(e, b) \in >_o$. According to Property 35, $b \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$.

□

Theorem 12 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, $a \in \mathtt{Sc}(\mathcal{AF}_o)$ and $e \notin \mathcal{A}_o$. The following holds:*

1. *$((a, e) \in \geq_o) \wedge ((a, e) \in \geq_o)$ iff $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e) \wedge e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$*
2. *$(e, a) \in >_o$ iff $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e) \wedge e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$*
3. *$(a, e) \in >_o$ iff $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e) \wedge e \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$*
4. *$((a, e) \notin \geq_o) \wedge ((a, e) \notin \geq_o)$ iff $a \in \mathtt{Cr}(\mathcal{AF}_o \oplus e) \wedge e \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$*

Proof.

1. Let $((a,e) \in \geq_o) \wedge ((e,a) \in \geq_o)$. Let us start by proving that $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Suppose not. So, a changed its status. According to Property 35, $(a,e) \notin \geq_o$. Contradiction. Thus, $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.

   We will now prove that $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Suppose not. Then, according to Property 25, $(\exists x' \in \mathcal{A}_o \cup \{e\})$ $(e,x) \notin \geq_o$. Since we proved that $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$, then, according to Property 25, $(\forall x \in \mathcal{A}_o \cup \{e\})$ $(a,x) \in \geq_o$. In particular, $(a,x') \in \geq_o$. Since $(e,a) \in \geq_o$ and $(a,x') \in \geq_o$, the transitivity of the preference relation $\geq_o$ implies that $(e,x') \in \geq_o$. Contradiction. So, $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.

2. Let $(e,a) \in >_o$. According to Property 25, $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, since there is now at least one argument which is strictly preferred to it.

   Let us now prove that $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Suppose not. Then, according to Property 25, $(\exists x' \in \mathcal{A}_o \cup \{e\})$ $(e,x') \notin \geq_o$. Since there are no self-attacking arguments, we have $x' \neq e$. So, $x' \in \mathcal{A}_o$. Since $a \in \mathtt{Sc}(\mathcal{AF}_o)$, it holds that $(\forall x \in \mathcal{A}_o)$ $(a,x) \in \geq_o$. In particular, $(a,x') \in \geq_o$. So, $(e,a) \in >_o$ and $(a,x') \in \geq_o$. According to Property 9, $(e,x') \in >_o$. Consequently, $(e,x') \in \geq_o$. Contradiction with the fact $(e,x') \notin \geq_o$. So, $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.

3. $(a,e) \in >_o$. We will prove that $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Suppose not. So, a changed its status. According to Property 35, $(a,e) \notin \geq_o$. Contradiction with the fact $(a,e) \in >_o$. So, $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$.

   We will now prove that $e \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Since $(a,e) \in >_o$, then, according to Property 25, $e \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$.

4. Let $((a,e) \notin \geq_o) \wedge ((a,e) \notin \geq_o)$. We will prove that $a \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$. Suppose that $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. So, according to Property 25, $(\forall x \in \mathcal{A}_o \cup \{e\})$ $(a,x) \in \geq_o$. But, $(a,e) \notin \geq_o$. Contradiction. So, $a \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Suppose that $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Then, according to Property 25, $(\exists x' \in \mathcal{A}_o \cup \{e\})$ $(x',a) \in >_o$. $a \in \mathtt{Sc}(\mathcal{AF}_o)$. So, according to Property 25, $(\forall x \in \mathcal{A}_o)$ $(a,x) \in \geq_o$. Suppose that $x' \in \mathcal{A}_o$. Then, $(x',a) \in >_o$ and $(a,x') \in \geq_o$. Contradiction, so $x' \notin \mathcal{A}_o$. The fact that $x' \in \mathcal{A}_o \cup \{e\}$ and $x' \notin \mathcal{A}_o$ implies that $x' = e$. So, $(e,a) \in >_o$. Contradiction. Hence, $a \notin \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Since we proved that $a \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ and $a \notin \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, then, according to Property 7 $a \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$.

   Let us now prove that $e \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$. Suppose that $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. According to Property 25, $(\forall x \in \mathcal{A}_o)$ $(e,x) \in \geq_o$. But, $(e,a) \notin \geq_o$. Contradiction. So, $e \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Suppose now that $e \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Then, according to Property 25, $(\exists y' \in \mathcal{A})$ $(y',e) \in >_o$. Since $(y',e) \in >_o$ then $(e,y') \notin \geq_o$. Since $\geq_o$ is reflexive, then $y' \neq e$. So, $y' \in \mathcal{A}_o$. $a \in \mathtt{Sc}(\mathcal{AF}_o)$. So, according to Property 25, $(\forall x \in \mathcal{A}_o)$ $(a,x) \in \geq_o$. Since $y' \in \mathcal{A}_o$, then $(a,y') \in \geq_o$. So, we have $(a,y') \in \geq_o$ and $(y',e) \in >_o$. Thus, according to Property 9, $(a,e) \in >_o$. Contradiction. Since we proved that $e \notin \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ and $e \notin \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, then, according to Property 7, it must be the case that e is credulously accepted.

□

Property 37 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, $\mathtt{Cr}(\mathcal{AF}_o) \neq \emptyset$ and $e \notin \mathcal{A}_o$. The following result holds: $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e,a) \in >_o$ iff $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e,a) \in \geq_o$.*

Proof. $\Rightarrow$ Trivial, according to definition of $>_o$.

$\Leftarrow$ Let us suppose that $(\exists a' \in \mathtt{Cr}(\mathcal{AF}_o))$ $((e,a') \notin >_o \wedge (e,a') \in \geq_o)$. So, according to definition of $>_o$, $(a',e) \in \geq_o$. According to Property 33, $(\exists a'' \in \mathtt{Cr}(\mathcal{AF}_o))$ $((a',a'') \notin \geq_o \wedge (a'',a') \notin \geq_o)$. Since $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e,a) \in \geq_o$, then, in particular, $(e,a'') \in \geq_o$. With $(a',e) \in \geq_o$ and $(e,a'') \in \geq_o$ we have $(a',a'') \in \geq_o$. Contradiction. □

Property 38 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, $\mathtt{Cr}(\mathcal{AF}_o) \neq \emptyset$ and $e \notin \mathcal{A}_o$. The following holds: $(\forall a \in \mathtt{Cr}(A_o))$ $a \in \mathtt{Rej}(A_o \oplus e)$ iff $(\forall a \in \mathtt{Cr}(A_o))$ $(e, a) \in >_o$.*

Proof. $\Rightarrow$ Let all the credulously accepted arguments become rejected. Suppose that $a' \in \mathtt{Cr}(\mathcal{AF}_o)$. According to Property 35, since $a' \in \mathtt{Cr}(\mathcal{AF}_o)$ and $a' \in \mathtt{Rej}(\mathcal{A}_o \oplus e)$, it holds that $(e, a') \in >_o$.

$\Leftarrow$ Let $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o$. Suppose that $a' \in \mathtt{Cr}(\mathcal{AF}_o)$. According to Property 25, since $(e, a') \in >_o$ then $a' \in \mathtt{Rej}(\mathcal{A}_o \oplus e)$. □

Theorem 13 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, $\mathtt{Cr}(\mathcal{AF}_o) \neq \emptyset$ and $e \notin \mathcal{A}_o$. Then, the following holds:*

1. *$(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o$ iff $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e) \wedge \mathcal{A}_o = \mathtt{Rej}(\mathcal{AF}_o \oplus e)$.*
2. *$(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin >_o \wedge (\nexists a' \in \mathtt{Cr}(\mathcal{AF}_o))$ $(a', e) \in >_o$ iff $e \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$*
3. *$(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(a, e) \in >_o$ iff $e \in \mathtt{Rej}(\mathcal{AF}_o \oplus e) \wedge \mathcal{A}_o = \mathtt{Cr}(\mathcal{AF}_o \oplus e)$.*

Proof. During the proof, we will sometimes use the following fact. Since, according to Property 37, $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o$ is equivalent to $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in \geq_o$, then the negation of $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o$ is equivalent to negation of $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in \geq_o$. So, $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin >_o$ is equivalent to $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin \geq_o$.

1. $\Rightarrow$ Let $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o$. Let $a \in \mathtt{Cr}(\mathcal{AF}_o)$. Since $(e, a) \in >_o$, then, Property 25 implies that $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. So, $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Since, according to Property 34, rejected arguments cannot change their status, then $\mathcal{A}_o \subseteq \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. So, as the consequence of Property 29, we have that e is skeptically accepted.

   $\Leftarrow$ Let $a \in \mathtt{Cr}(\mathcal{AF}_o)$. Since $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, then, according to Property 35, it holds that $(e, a) \in >_o$. Since $a \in \mathtt{Cr}(\mathcal{AF}_o)$ was arbitrary, we have $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o$.
2. $\Rightarrow$ Since $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin >_o$ then $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin \geq_o$. Since $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin \geq_o$, then, according to Property 25, e is not skeptically accepted. Since $(\nexists a'' \in \mathtt{Cr}(\mathcal{AF}_o))$ $(a'', e) \in >_o$, then, according to the same property, e is not rejected. Since e is neither skeptically accepted nor rejected, according to Property 7, it is credulously accepted.

   $\Leftarrow$ Let e be credulously accepted. Since e is credulously accepted, according to Property 7, it is neither skeptically accepted, nor rejected. Since e is not rejected, then, according to Property 25, $(\nexists a'' \in \mathtt{Cr}(\mathcal{AF}_o))$ $(a'', e) \in >_o$. Since e is not skeptically accepted, then, according to the same property, $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin \geq_o$. Since $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin \geq_o$ then $(\exists a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \notin >_o$.
3. $\Rightarrow$ Let $(\exists a'' \in \mathtt{Cr}(\mathcal{AF}_o))$ $(a'', e) \in >_o$. According to Property 25, e is rejected. Let us now prove that $\mathtt{Cr}(\mathcal{AF}_o) \subseteq \mathtt{Cr}(\mathcal{AF}_o \oplus e)$. Suppose not. So, $(\exists a' \in \mathtt{Cr}(\mathcal{AF}_o))$ such that a' changes its status. Since, according to Property 34, no argument can become skeptically accepted, then a' becomes rejected. According to Property 35, it holds that $(e, a') \in >_o$. Since $(a'', e) \in >_o$ and $(e, a') \in >_o$ then $(a'', a') \in >_o$. Since the preference relation between the arguments does not change, this means that $(a'', a') \in >_o$ was true in the moment when a' and a'' were both credulously accepted. Contradiction with Property 32. So, we proved that e is rejected and that no other argument changes its status.

$\Leftarrow$ Let e be rejected. So, according to Theorem 28, $(\exists a' \in \mathcal{A}_o)$ such that $(a', e) \in >_o$ and $a' \notin \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Since $a' \neq e$ then $a' \in \mathcal{A}_o$. So, $a \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$. Since $a \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$, then, according to Property 34, $a \notin \mathtt{Rej}(\mathcal{AF}_o)$. Since $\mathtt{Sc}(\mathcal{AF}_o) = \emptyset$, then $a \in \mathtt{Cr}(\mathcal{AF}_o)$. So, $(\exists a' \in \mathtt{Cr}(\mathcal{AF}_o))$ $(a', e) \in >_o$.

□

Theorem 14 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, and $o \in \mathcal{O}$ an acceptable option. Suppose that $a \in \mathtt{Sc}(\mathcal{AF}_o)$ is an arbitrary skeptically accepted argument and $e \notin \mathcal{A}_o$. Then:*

1. *Option o will stay acceptable iff*
   $((a, e) \in \geq_o) \vee (e \in \mathcal{H}(o)) \wedge ((e, a) \in >_o)$
2. *Option o will become negotiable iff*
   $((a, e) \notin \geq_o) \wedge ((e, a) \notin \geq_o))$
3. *Option o will become rejected iff*
   $(e \notin \mathcal{H}(o)) \wedge (e, a) \in >_o)$

Proof.

1. $\Rightarrow$ According to Definition 16, option o was acceptable, so there was already at least one skeptically accepted argument a' in its favor before that agent received the argument e. Suppose that the option o remains acceptable. Since, according to Theorem 34, no argument can become skeptically accepted, then either some skeptically accepted argument in favor of o remained skeptically accepted or e is skeptically accepted and e is in favor of o. Let us explore the first possibility. So, $\exists a'' \in \mathcal{H}(o) \cap \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. The argument a'' remained skeptically accepted, so, according to Property 36, a will remain skeptically accepted as well. Since $(a'', e) \in \geq_o$ and, according to Property 31, all the skeptically accepted arguments are in the same relation with e, then $(a, e) \in \geq_o$. Suppose now that $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e) \cap \mathcal{H}(o)$. Since e is skeptically accepted, according to Theorem 12, we have $(e, a) \in \geq_o$. If $(a, e) \in \geq_o$ then the first part of the disjunction is true, i.e., $(a, e) \in \geq_o$. If $(a, e) \notin \geq_o$ then $(e, a) \in >_o$. So, the second part of the disjunction is true, i.e., $(e, a) \in >_o \wedge e \in \mathcal{H}(o)$.

   $\Leftarrow$ Suppose now that $(a, e) \in \geq_o \vee ((e, a) \in >_o \wedge e \in \mathcal{H}(o))$. Suppose that the first part of the disjunction is true, i.e., $(a, e) \in \geq_o$. According to Theorem 12, $a \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Consequently, o remains acceptable. Suppose now that the second part of the disjunction is true, i.e., $(e, a) \in >_o \wedge e \in \mathcal{H}(o)$. Since $(e, a) \in >_o$, then, according to Theorem 12, $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Since $e \in \mathcal{H}(o)$ then o is acceptable.
2. $\Rightarrow$ Since the option o becomes negotiable, according to the Definition 16, there is at least one credulously accepted argument in its favor. The Property 34 states that rejected arguments cannot become credulously accepted. So, either an skeptically accepted argument a' in favor of o has become credulously accepted or e is credulously accepted and e is in favor of o. The first possibility, with respect to the Theorem 12, implies that $(a, e) \notin \geq_o$ and $(e, a) \notin \geq_o$. The second possibility, according to the same theorem, leads to the same conclusion: $(a, e) \notin \geq_o$ and $(e, a) \notin \geq_o$, which ends the proof.

   $\Leftarrow$ Let $(a, e) \notin \geq_o \wedge (e, a) \notin \geq_o$. The Theorem 12, together with the fact that $(a, e) \notin \geq_o$ $\wedge$ $(e, a) \notin \geq_o$ leads to the conclusion that $a, e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Since $\mathtt{Cr}(\mathcal{AF}_o \oplus e) \neq \emptyset$, according to Property 27, $\mathtt{Sc}(\mathcal{AF}_o \oplus e) = \emptyset$. So, there will be no skeptically accepted arguments in favor of o, and there will be at least one credulously accepted argument in its favor. According to the Definition 16, o becomes negotiable.

3. $\Rightarrow$ Let o be an acceptable option that becomes rejected. The option o was acceptable, so, according to Definition 16, there were at least one skeptically accepted argument a' in its favor. Since o has become rejected, according to the same definition, $\mathcal{H}(o) \subseteq \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, so a' must have become rejected. So, a' was not rejected but it is rejected now. Let a'' be an arbitrary skeptically accepted argument. $a' \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, so, according to Property 36, $a'' \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Since a'' has become rejected, the Property 35 implies that $(e, a'') \in >_o$. Let us now prove that $e \notin \mathcal{H}(o)$. Suppose that the converse, $e \in \mathcal{H}(o)$, is true. The fact $(e, a) \in >_o$, according to the Theorem 12, implies that e is skeptically accepted. Since $e \in \mathcal{H}(o)$, then there is at least one skeptically accepted argument in favor of the option o, which, according to Definition 16, contradicts the fact that o became rejected. So, the assumption $e \in \mathcal{H}(o)$ is false. Hence, $e \notin \mathcal{H}(o)$.
   $\Leftarrow$ Let $(e, a) \in >_o \wedge e \notin \mathcal{H}(o)$. The fact $(e, a) \in >_o$, according to the Theorem 12, implies that $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$ and $a \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Let a' be the arbitrary skeptically accepted argument. According to Property 36, a' will become rejected, too. So, an arbitrary skeptically accepted argument becomes rejected. This means that all skeptically accepted arguments will become rejected, $\mathtt{Sc}(\mathcal{AF}_o) \subseteq \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Since $\mathtt{Sc}(\mathcal{AF}_o) \neq \emptyset$, according to Theorem 10, $\mathtt{Cr}(\mathcal{AF}_o) = \emptyset$. According to Property 34, rejected arguments cannot change their status. Since there were no credulously accepted arguments and all skeptically accepted arguments became rejected and all the rejected arguments remain rejected, we conclude that all the arguments except e are rejected, $\mathcal{A}_o \subseteq \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. Recall that $e \notin \mathcal{H}(o)$. All the arguments in favor of o are rejected. Since there is at least one argument in favor of o and all the arguments in its favor are rejected, according to Definition 16, o is rejected.

□

Theorem 15 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \mathtt{Def}_o \rangle$ be a complete argumentation framework for decision making, and $o \in \mathcal{O}$ an negotiable option. Suppose that $e \notin \mathcal{A}_o$. Then:*

1. *Option o will become acceptable iff*
   $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathtt{Cr}(\mathcal{A}_o))\ (e, a) \in >)$
2. *Option o will rest negotiable iff*
   $((e \in \mathcal{H}(o)) \wedge (\exists a' \in \mathtt{Cr}(\mathcal{AF}_o))\ (e, a') \notin >_o \wedge (\nexists a'' \in \mathtt{Cr}(\mathcal{AF}_o))\ (a'', e) \in >_o)$
   $\vee$
   $((\exists a' \in \mathtt{Cr}(\mathcal{AF}_o))\ (a' \in \mathcal{H}(o) \wedge (e, a') \notin >_o))$
3. *Option o will become rejected iff*
   $(e \notin \mathcal{H}(o)) \wedge ((\forall a \in \mathtt{Cr}(\mathcal{AF}_o))\ (a \in \mathcal{H}(o)) \Rightarrow (e, a) \in >_o)$.

Proof.

1. $\Rightarrow$ Let o become acceptable. According to Definition 16, this means that there will be at least one skeptically accepted argument in its favor. According to Property 34, no argument cannot become skeptically accepted. So, in order to make o become acceptable, agent must receive a new argument in favor of o. Hence, $e \in \mathcal{H}(o)$ and $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Since e is skeptically accepted, then, according to Theorem 10, $\mathtt{Cr}(\mathcal{AF}_o \oplus e) = \emptyset$. So, all the credulously accepted arguments have changed their status. With respect to Property 34, they are all rejected. So, all arguments in $\mathcal{A}_o \setminus \{e\}$ are rejected. The Property 29 states that in this case, e must be skeptically accepted. Since $\mathtt{Cr}(\mathcal{AF}_o) \subseteq \mathtt{Rej}(\mathcal{AF}_o \oplus e)$, then, according to Property 38, $(\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o$.
   $\Leftarrow$ Let $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathtt{Cr}(\mathcal{AF}_o))\ (e, a) \in >_o)$. The fact $((\forall a \in \mathtt{Cr}(\mathcal{AF}_o))$ $(e, a) \in >_o)$ is, according to the Property 38, equivalent to $\mathtt{Cr}(\mathcal{AF}_o) \subseteq \mathtt{Rej}(\mathcal{AF}_o \oplus e)$. So,

all the credulously accepted arguments have become rejected. There were no skeptically accepted arguments. According to the Property 34, all the rejected arguments remain rejected. So, all the arguments except e are rejected. According to the Property 29, $e \in \texttt{Sc}(\mathcal{AF}_o \oplus e)$. Since $(e \in \mathcal{H}(o))$, then there is exactly one accepted argument in favor of the option o. According to Definition 16, o is acceptable.

2. $\Rightarrow$ Let o stay negotiable. According to Property 26, this means that there is at least one credulously accepted argument in favor of o. If $((\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ a' \in \mathcal{H}(o) \wedge (e, a') \notin >_o)$ then that fact ends the proof. Suppose that $((\nexists a \in \texttt{Cr}(\mathcal{AF}_o))\ a \in \mathcal{H}(o) \wedge (e, a) \notin >_o)$. According to Property 34, all the rejected arguments remain rejected. Since $((\forall a \in \texttt{Cr}(\mathcal{AF}_o))\ a \in \mathcal{H}(o) \Rightarrow (e, a) \in >_o)$, this means that for all the credulously accepted arguments in favor of o, it holds that $(e, a) \in >_o$. According to Property 25, this means that all the credulously accepted arguments in favor of o will become rejected. Since o remains negotiable, according to Property 26, this means that there is at least one credulously accepted argument in its favor. So, it must be the case that $e \in \texttt{Cr}(\mathcal{AF}_o \oplus e)$ and $e \in \mathcal{H}(o)$. According to Theorem 13, since e is credulously accepted then $(\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ (e, a') \notin >_o \wedge (\nexists a'' \in \texttt{Cr}(\mathcal{AF}_o))\ (a'', e) \in >_o$.

   $\Leftarrow$ Let $(e \in \mathcal{H}(o)) \wedge (\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ (e, a') \notin >_o \wedge (\nexists a'' \in \texttt{Cr}(\mathcal{AF}_o))\ (a'', e) \in >_o)$ or $((\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ a' \in \mathcal{H}(o) \wedge (e, a') \notin >_o)$. Suppose that $(e \in \mathcal{H}(o)) \wedge (\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ (e, a') \notin >_o \wedge (\nexists a'' \in \texttt{Cr}(\mathcal{AF}_o))\ (a'', e) \in >_o)$. According to Theorem 13, $e \in \texttt{Cr}(\mathcal{AF}_o \oplus e)$. Since $e \in \mathcal{H}(o)$, according to the Property 26, o is negotiable. Let us now suppose that $(\exists a' \in \texttt{Cr}(\mathcal{AF}_o))\ a' \in \mathcal{H}(o) \wedge (e, a') \notin >_o$ is true. The fact $(e, a') \notin >_o$ and Property 35 imply that $a' \notin \texttt{Rej}(\mathcal{AF}_o \oplus e)$. Since, according to the Property34, no argument cannot become skeptically accepted, a' is neither rejected nor skeptically accepted. According to the Proposition 7, it is credulously accepted. The Property 26 implies that o is negotiable.

3. $\Rightarrow$ Since o becomes rejected, according to Definition 16, that means that $\mathcal{H}(o) \subseteq \texttt{Rej}(\mathcal{AF}_o \oplus e)$. Suppose that $(\exists a' \in \mathcal{H}(o) \cap \texttt{Cr}(\mathcal{AF}_o))\ (e, a') \notin >_o$. According to Property 35, $a \notin \texttt{Rej}(\mathcal{AF}_o \oplus e)$. So, there is at least one argument in favor of o which is not rejected. According to Definition 16, o is not rejected. Contradiction. Suppose now that $e \in \mathcal{H}(o)$. Since o is rejected, then $e \in \texttt{Rej}(\mathcal{AF}_o \oplus e)$. Since e is rejected, according to Property 28, $(\exists x' \in \mathcal{A}_o)\ x' \notin \texttt{Rej}(\mathcal{AF}_o \oplus e)$ and $(x', e) \in >_o$. Since o was negotiable, $\mathcal{H}(o) \cap \texttt{Cr}(\mathcal{AF}_o) \neq \emptyset$. Let $a'' \in \mathcal{H}(o) \cap \texttt{Cr}(\mathcal{AF}_o)$. It holds that $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))\ (a \in \mathcal{H}(o)) \Rightarrow ((e, a) \in >_o)$. In particular, $(e, a'') \in >_o$. It also holds that $(x', e) \in >_o$. The Property 9 implies that $(x', a'') \in >_o$. So, a'' was not self-defending in $\mathcal{AF}_o$ (before the agent has received the argument e), so $a'' \in \texttt{Rej}(\mathcal{AF}_o)$. Contradiction. So, $e \notin \mathcal{H}(o)$.

   $\Leftarrow$ Since $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))\ (a \in \mathcal{H}(o) \Rightarrow (e, a) \in >_o)$, then, as a consequence of Property 35, $(\forall a \in \texttt{Cr}(\mathcal{AF}_o))\ (a \in \mathcal{H}(o)) \Rightarrow a \in \texttt{Rej}(\mathcal{AF}_o \oplus e)$. So, $\texttt{Cr}(\mathcal{AF}_o) \cap \mathcal{H}(o) \subseteq \texttt{Rej}(\mathcal{AF}_o \oplus e)$ and, according to the Property 34, $\texttt{Rej}(\mathcal{AF}_o) \subseteq \texttt{Rej}(\mathcal{AF}_o \oplus e)$. So, since $e \notin \mathcal{H}(o)$, all the arguments in favor of o are rejected. Since o was negotiable, then $\mathcal{H}(o) \neq \emptyset$. So, according to Definition 16, o becomes rejected.

□

Theorem 16 *Let $\mathcal{AF}_o = \langle \mathcal{A}_o, \texttt{Def}_o \rangle$ be a complete argumentation framework for decision making, and $o \in \mathcal{O}$ an rejected option. Suppose that $e \notin \mathcal{A}_o$. Then:*

1. *Option o will become acceptable iff*
   $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathcal{A}_o)\ (e, a) \in \geq_o)$

2. *Option o will become negotiable iff*
   $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathcal{A}_o)\ (a, e) \notin >_o) \wedge ((\exists a \in \mathcal{A}_o)\ (e, a) \notin >_o)$

3. *Option o will rest rejected iff*
$(e \notin \mathcal{H}(o)) \vee ((e \in \mathcal{H}(o)) \wedge (\exists a \in \mathcal{A}_o)(a,e) \in >)$

Proof.

1. $\Rightarrow$ Suppose that option o becomes acceptable. This means that there is at least one skeptically accepted argument in its favor. Since it was rejected, and, according to Property 34, all rejected arguments remain rejected, it must be that $e \in \mathcal{H}(o)$ and $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. The Property 25 now implies that $(\forall a \in \mathcal{A}_o)$ $(e,a) \in \geq_o$.

   $\Leftarrow$ Suppose that $e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathcal{A}_o)$ $(e,a) \in \geq_o$. According to Property 25, $e \in \mathtt{Sc}(\mathcal{AF}_o \oplus e)$. Since $e \in \mathcal{H}(o)$, we have one skeptically accepted argument in favor of option o, hence it is acceptable.

2. $\Rightarrow$ Suppose that option o becomes negotiable. According to Property 26, there is at least one credulously accepted argument in its favor. Since it was rejected, and, according to Property 34, all rejected arguments remain rejected, it must be that $e \in \mathcal{H}(o)$ and $e \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$. From Property 25, we have $((\forall a \in \mathcal{A}_o)$ $(a,e) \notin >_o) \wedge ((\exists a \in \mathcal{A}_o)$ $(e,a) \notin >_o)$.

   $\Leftarrow$ Suppose that $(e \in \mathcal{H}(o)) \wedge ((\forall a \in \mathcal{A}_o)$ $(a,e) \notin >_o) \wedge ((\exists a \in \mathcal{A}_o)$ $(e,a) \notin >_o)$. According to Property 25, $e \in \mathtt{Cr}(\mathcal{AF}_o \oplus e)$. Since $e \in \mathcal{H}(o)$, we have one credulously accepted argument in favor of option o, which together with Property 26 means that o is negotiable.

3. $\Rightarrow$ Suppose that option o stays rejected. This means that all arguments in its favor are rejected. If $e \notin \mathcal{H}(o)$ the proof is over. Let us suppose that $e \in \mathcal{H}(o)$. Since $e \in \mathtt{Rej}(\mathcal{AF}_o \oplus e)$ then the Property 25 implies that $(\exists a \in \mathcal{A}_o)(a,e) \in >$.

   $\Leftarrow$ Let $(e \notin \mathcal{H}(o)) \vee ((e \in \mathcal{H}(o)) \wedge (\exists a \in \mathcal{A}_o)(a,e) \in >)$. If $e \notin \mathcal{H}(o)$, then, according to Property 34, all rejected arguments remain rejected, so the option remains rejected. If $e \notin \mathcal{H}(o)$ then $(\exists a \in \mathcal{A}_o)(a,e) \in >$. Property 25 implies that e is rejected, so with $\mathcal{H}(o) \subseteq \mathtt{Rej}(\mathcal{AF}_o)$ we have that o is rejected.

$\square$

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