Propositional Belief Merging and Belief Negotiation Model

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Abstract

We propose in this paper a new family of belief merging operators, that is based on a game between sources : Until a coherent set of sources is reached, at each round a contest is organized to find out the weakest sources, then those sources has to concede (weaken their point of view). This idea leads to numerous new interesting operators (depending of the exact meaning of *"weakest"* and *"concede"*, that gives the two parameters for this family) and opens new perspectives for belief merging. Some existing operators are also recovered as particular cases. Those operators can be seen as a special case of Booth's Belief Negotiation Models (Booth 2002), but the achieved restriction forms a consistent family of merging operators that worths to be studied on its own.

Introduction

The problem of (propositional) belief merging (Revesz 1997; Lin & Mendelzon 1999; Liberatore & Schaerf 1998; Konieczny & Pino Pérez 1999; 2002a; Konieczny, Lang, & Marquis 2004) can be summarized by the following question: given a set of sources (propositional belief bases) that are mutually inconsistent, how to reach a coherent belief base reflecting the beliefs of the set ?

The idea here is that some/each sources has to concede on some points in order to solve the conflicts. If one has some notion of relative reliability between sources, it is enough and sensible to force the less reliable ones to give up first. There is a lot of different means to do that, which has provided a large literature, e.g. (Cholvy 1993; 1995; 1998; Benferhat *et al.* 1998; Benferhat, Dubois, & Prade 1998). But often we do not have such information, and even if we get it, it remains the more fundamental problem of how to merge sources of equal reliability (Konieczny & Pino Pérez 1999; 2002a).

In this paper we will investigate the merging methods based on a notion of game between the sources. The intuitive idea is simple: when trying to impose its wish, each source will try to form some coalition with near-minded sources. So the source that is the "furthest" from the other ones will certainly be the weakest one. And it will be that source that have to concede first. In this work, we will not focus on how the coalitions form, we only take this idea to designate the *weakest* ones.

So the merging is based on the following game: Until a coherent set of sources is reached, at each round a contest is organized to find out the weakest sources, then those sources has to concede (weaken their point of view).

We can state several intuitions and justifications for the use of such operators. We have already given the first one: coalition with near-minded sources. In a group decision process between rational sources, it can be sensible to expect the sources to look for near-minded sources in order to find help to defend their view, so the "furthest" source is the more likely to have to concede on its view.

A second intuition is the one given by a social pressure on the sources. When confronting several points of view, usually people that have the more exotic views try to change their opinion in order to be accepted by the other members of the group, so opinions that are defended by the least number of sources are usually given up more easily in the negotiation process.

A last intuition that gives the main rationale for that kind of operators is Condorcet's Jury theorem. This theorem states that if all the members of a jury are reliable (in the sense that they have more than a half of chances to find the truth), then listen to the majority is the more rational choice.

After stating some useful definitions and notations in the following section, we will define the new family of operators we propose. The definition will use a notion of weakening and choice functions. We will explore those notions in a subsequent section and we will give some examples of specific operators in order to illustrate their behaviour. We will then look at the logical properties of those operators. Finally, we will look at the links between this work and related works (especially Booth's proposal (Booth 2001; 2002)), before concluding with some open issues and perspectives of this work.

Preliminaries

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional symbols. An interpretation is a function from $\mathcal{P}$ to $\{0, 1\}$. The set of all the interpretations is denoted $\mathcal{W}$. An interpretation ω is a model of a formula φ, noted $\omega \models \varphi$, if and only if it makes it true in the usual classical truth functional way. Let φ be a formula, $mod(\varphi)$ denotes the set of models of φ, i.e. $mod(\varphi) = \{\omega \in \mathcal{W} \mid \omega \models \varphi\}$. Conversely, let X be a set of interpretations, $form(X)$

denotes the formula (up to logical equivalence) whose set of models is X.

A *belief base* φ is a consistent propositional formula (or, equivalently, a finite consistent set of propositional formulae considered conjunctively).

Let $\varphi_1, \ldots, \varphi_n$ be n belief bases (not necessarily different). We call *belief profile* the multi-set Ψ consisting of those n belief bases: $\Psi = (\varphi_1, \ldots, \varphi_n)$ (i.e. two sources can have the same belief base). We note $\bigwedge \Psi$ the conjunction of the belief bases of Ψ, i.e. $\bigwedge \Psi = \varphi_1 \wedge \cdots \wedge \varphi_n$. We say that a belief profile is consistent if $\bigwedge \Psi$ is consistent. The multi-set union will be noted $\sqcup$ and the multi-set inclusion will be noted $\sqsubseteq$. The cardinal of a finite (multi-)set A is noted $\#(A)$ (the cardinal of a finite multi-set is the sum of the numbers of occurrences of each of its elements). Let $\mathcal{E}$ be the set of all finite belief profiles.

Two belief profiles Ψ_1 and Ψ_2 are said to be equivalent ($\Psi_1 \equiv \Psi_2$) if and only if there is a bijection between Ψ_1 and Ψ_2 such that each belief base of Ψ_1 is logically equivalent to its image in Ψ_2.

Belief Game Model

In (Booth 2001; 2002) Richard Booth propose a framework for merging sources of information incrementally. He named this framework *"Belief Negotiation Model"* (BNM) In this work we will use the name *"Belief Game Model"* (BGM) because in our framework there is no room for negociation, so we find it more accurate and it allows us to make a distinction in this paper between Booth's proposal and our. The BGM framework can be seen as a restriction of Booth's BNM framework: the main differences between Booth's proposal and our is that Booth's one take the sources as candidates to weakening, whereas we restrict ourselves to *"points of view"*. That means that in Booth's if one source has to weaken, it can be the case that another source with exactly the same beliefs do not have to weaken too (that is not allowed in our framework). Our proposal add more anonymity by saying that only beliefs decide who has to weaken, not the identity of one source. Similarly, the choice functions are more Markovian in our framework than in Booth's one. We think that those hypothesis are more realistic (and necessary) on a belief merging point of view, whereas Booth's framework allows to model more generalized negotiation schemes, where one can decide for example that each source has to weaken one after the other (see Section *Comparison between BGM and BNM* for a deeper comparison of the two approaches).

Definition 1 *A* choice function *is a function $g : \mathcal{E} \rightarrow \mathcal{E}$ such that:*

- $g(\Psi) \sqsubseteq \Psi$
- *If $\Psi \neq \emptyset$, then $g(\Psi) \neq \emptyset$*
- *If $\bigwedge \Psi \not\equiv \top$, then $\exists \varphi \in g(\Psi)$ s.t. $\varphi \not\equiv \top$*
- *If $\Psi \equiv \Psi'$, then $g(\Psi) \equiv g(\Psi')$*

The choice function aims to find which are the sources that must weaken at a given round. So the two first conditions mean that the sources that will have to weaken are a non-empty subset[1] of the belief profile. As the weakening function aims at weaken the belief base, and as there is no weaker base than a tautological one, the third condition states that at least one non-tautological base must be selected. This condition is necessary to ensures to always reach a result with Belief Game Model. Last condition is an irrelevance of syntax condition. It states that the selection of the bases to weaken does not depend on the particular form of the bases, but only of they informational content. Note that we also have an additional property: anonymity, that means that the result does not depend of the "name" of the source, but only on its point of view. This is due to the fact that we work with multi-sets, that are equivalent by permutation. If one works with an other representation (ordered lists of sources for example), this anonymity property can be given by the last condition, provided that the equivalence between two belief profiles is rightly defined (as in the *Preliminaries* section).

Definition 2 *A* weakening function *is a function $\blacktriangledown : \mathcal{L} \rightarrow \mathcal{L}$ such that:*

- $\varphi \vdash \blacktriangledown(\varphi)$
- *If $\varphi \equiv \blacktriangledown(\varphi)$, then $\varphi \equiv \top$*
- *If $\varphi \equiv \varphi'$, then $\blacktriangledown(\varphi) \equiv \blacktriangledown(\varphi')$*

The weakening function aims to give the new beliefs of a source that has been chosen to be weaken. The two first conditions ensure that the base will be replaced by a strictly weaker one (unless the base is already a tautological one). The last condition is an irrelevance of syntax requirement : the result of the weakening must only depend on the information convey by the base, not on its syntactical form.

We extend the weakening functions on belief profiles as follows: let Ψ' be a subset of Ψ,

$$\blacktriangledown_{\Psi'}(\Psi) = \bigsqcup_{\varphi \in \Psi'} \blacktriangledown(\varphi) \sqcup \bigsqcup_{\varphi \in \Psi \setminus \Psi'} \varphi$$

This means that we weaken only the belief base of Ψ that are in Ψ', the other ones do not change.

Definition 3 *A Belief Game Model is a pair $\mathcal{N} = \langle g, \blacktriangledown \rangle$ where g is a choice function and $\blacktriangledown$ is a weakening function.*

The solution to a belief profile Ψ for a Belief Game Model $\mathcal{N} = \langle g, \blacktriangledown \rangle$, noted $\mathcal{N}(\Psi)$, is the belief profile $\Psi_{\mathcal{N}}$, defined as:

- $\Psi_0 = \Psi$
- $\Psi_{i+1} = \blacktriangledown_{g(\Psi_i)}(\Psi_i)$
- *$\Psi_{\mathcal{N}}$ is the first Ψ_i that is consistent*

So the solution to a belief profile is the result of a game on the beliefs of the sources. At each round there is a contest to find out the weakest bases (the losers), and the losers have to concede on their belief by weakening them.

[1]Indeed, all set notions used in this paper (subset, inclusion, union, etc.), are multi-sets one. So here it is strictly speaking sub-muti-set. For the sake of simplicity, and since it can not lead to confusion since we work in this paper only with multi-sets, we will take the set notions, without mentioning the "multi-".

In some cases, the result of the merging has to obey to some constraints (physical constraints, norms, etc...). We will assume that those integrity constraints are encoded as a propositional formula (a belief base), and we will note this base μ. Then we introduce the following notion:

Definition 4 *The solution to a belief profile Ψ for a Belief Game Model $\mathcal{N} = \langle g, \blacktriangledown \rangle$ under the integrity constraints μ, noted $\mathcal{N}_\mu(\Psi)$, is the belief profile $\Psi^\mu_\mathcal{N}$ defined as:*

- $\Psi_0 = \Psi$
- $\Psi_{i+1} = \blacktriangledown_{g(\Psi_i)}(\Psi_i)$
- $\Psi^\mu_\mathcal{N}$ *is the first* Ψ_i *that is consistent with* μ

Often in the following in this paper we will call result of the merging operator (Belief Game Model), the belief base $\bigwedge \Psi^\mu_\mathcal{N} \wedge \mu$. This abuse of notation is not problematic, since this belief base denotes the consensus point obtained by the belief profile $\Psi^\mu_\mathcal{N}$ solution of the Belief Game Model process.

Note that the definition of the Belief Game Model and of the weakening and choice functions ensures that each belief profile Ψ has a solution as soon as the constraints μ are consistent.

Weakening and Choice Functions

In order to define a particular Belief Game Model, we have to choose a choice function and a weakening function. We will give in this section some natural choices for those functions and see what are the resulting BGM operators.

Weakening Function

Let us fist turn out on weakening function. Can we find out a "natural" one ? In fact it is a difficult task, since the exact choice of a weakening function depends on the expected behaviour for the Belief Game Model and depends also on the existence of some "preferential" information. But if we have no such additional information, we have at least two natural candidates : drastic weakening and dilatation.

Definition 5 *Let φ be a belief base. The* drastic weakening function *forget all the information about one source, i.e. :* $\blacktriangledown_\top(\varphi) = \top$.

After this rough function, let us see a more fine grained one. Let us first recall what is the Hamming's distance between interpretations (also called Dalal's distance (Dalal 1988)) since we will use it several times in this paper.

Definition 6 *The Hamming distance between interpretations is the number of propositional symbols on which the two interpretations differ. Let ω and ω' be two interpretations, then*

$$d_H(\omega, \omega') = \#(\{a \in \mathcal{P} \mid \omega(a) \neq \omega'(a)\})$$

Then the dilatation weakening function is defined as :

Definition 7 *Let φ be a belief base. The* dilatation weakening function *is defined as :*

$$mod(\blacktriangledown_\delta(\varphi)) = \{\omega \in \mathcal{W} \mid \exists \omega' \models \varphi \; d_H(\omega, \omega') \leq 1\}$$

Choice Function

Let us turn out now on choice function. The aim of this function is to determine the "losers", that are the sources that have to concede by weakening their beliefs at a given round.

One of the simplest choice function one can choose is identity (denoted g_{id}). It is not the expected behaviour for this function, but it can prove the rationality of our operators if, even in this case, we obtain a sensible merging.

We will focus on two families of choice functions. The first one is model-based, the second one is formula-based. We think that most of the sensible choice functions belong to one of those families.

Model-Based Choice Functions We will focus here on some modelization of what can be called "social pressure", and can be viewed as a majority principle. Namely, at each round it is the "furthest" sources from the group that will concede. The exact choice of the meaning of "furthest" will fix the chosen operator from this family. Technically we will use a distance between belief bases and an aggregation function to evaluate the distance of a belief base with respect to the others.

We will start from the definition of the distance between two belief bases.

Definition 8 *A (pseudo)distance*[2] d *between two belief bases is a function* $d : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{N}$ *such that:*

- $d(\varphi, \varphi') = 0$ *iff* $\varphi \wedge \varphi' \nvdash \bot$
- $d(\varphi, \varphi') = d(\varphi', \varphi)$

Two examples of a such distances are :

- $d_D(\varphi, \varphi') = \begin{cases} 0 & \text{if } \varphi \wedge \varphi' \nvdash \bot \\ 1 & \text{otherwise} \end{cases}$
- $d_H(\varphi, \varphi') = \min\limits_{\omega \models \varphi, \omega' \models \varphi'} d_H(\omega, \omega')$

Definition 9 *An aggregation function is a total function f associating a nonnegative integer to every finite tuple of nonnegative integers and verifying (non-decreasingness), (minimality) and (identity).*

- *if* $x \leq y$, *then* $f(x_1, \ldots, x, \ldots, x_n) \leq f(x_1, \ldots, y, \ldots, x_n)$. **(non-decreasingness)**
- $f(x_1, \ldots, x_n) = 0$ *if and only if* $x_1 = \ldots = x_n = 0$. **(minimality)**
- *for every nonnegative integer* x, $f(x) = x$. **(identity)**

We say that an aggregation function is symmetric if it also satisfies :

- *For any permutation* σ, $f(x_1, \ldots, x_n) = f(\sigma(x_1, \ldots, x_n))$ **(symmetry)**

Definition 10 *A model-based choice function $g^{d,h}$ is defined as :*

[2] Remark that we miss an important property of distances: we have only $d(\varphi, \varphi') = 0$ *if* $\varphi = \varphi'$, but not the *only if* part. Remark also that we do not require the triangular inequality.

$$g^{d,h}(\Psi) = \{\varphi_i \in \Psi \mid h(d(\varphi_i, \varphi_1), \ldots \ldots, d(\varphi_i, \varphi_n)) \textit{ is maximal } \}$$

where h is an aggregation function, and d is a distance between belief bases.

We say that the model-based choice function is symmetric if the aggregation function is symmetric.

We will focus on some specific aggregation functions in this paper, but we can use different aggregation functions here. In particular we will only focus on symmetrical aggregation functions in this paper (to fit with choice functions requirements) but note that the definition allows non-symmetrical functions. This allows to define operators that are not anonymous, i.e. where each base has not the same importance. So one can use priorities (a weight or a pre-order on the sources) for denoting different level of reliability, different hierarchical importance, etc.

We will use in the following as examples of aggregation functions, two typical ones, the sum (noted Σ) and the maximum (noted max).

Formula-Based Choice Functions All interesting choice functions are not captured in the definition given in the previous section. In particular, a lot of interesting choice functions can be defined by using maximal consistent subsets. Note, however that, conversely to usual formula-based merging operators (Baral *et al.* 1992; Konieczny 2000), we use multi-sets instead of simple sets.

Definition 11 *Let* $\text{MAXCONS}(\Psi)$ *be the set of the* maxcons *of Ψ, i.e. the maximal (with respect to multi-set inclusion) consistent subsets of Ψ. Formally,* $\text{MAXCONS}(\Psi)$ *is the set of all multi-sets M such that:*

- $M \sqsubseteq \Psi$ *and*
- *if* $M \sqsubset M' \sqsubseteq \Psi$, *then* $\bigwedge M' \models \bot$.

Definition 12 *A formula-based choice function g^{mc} is a function of the set of the maxcons of Ψ and the belief base, i.e. :*

$$g^{mc}(\Psi) = \{\varphi_i \in \Psi \mid h(\varphi_i, \text{MAXCONS}(\Psi)) \textit{ is minimal } \}$$

Examples of the use of maxcons are numerous, let us see two of them.

Definition 13

$$h^{mc1}(\varphi, \text{MAXCONS}(\Psi)) = \#(\{M \mid M \in \text{MAXCONS}(\Psi) \textit{ and } \varphi \in M\})$$

$$h^{mc2}(\varphi, \text{MAXCONS}(\Psi)) = \max(\{\#(M) \mid M \in \text{MAXCONS}(\Psi) \textit{ and } \varphi \in M\})$$

The first function computes the number of maxcons the belief base belongs to. The second function computes the size of the biggest maxcons the belief base belongs to.

We will note g^{mc1} (respectively g^{mc2}) the formula-based choice function that use h^{mc1} (resp. h^{mc2}).

Instantiating the BGM Framework

In this section we will try to illustrate how interesting the defined Belief Game Model framework is by giving several examples. We will first see some of the simplest operators that we can define with this framework. Then we will illustrate the behaviour of more complex operators on a typical merging example.

Some Simple Examples

Let us first see what operators are obtained with the simplest weakening and dilatation functions (that means that we will either choose the weakening function to be the drastic one, or the choice function to be identity).

- $\langle g_{id}, \blacktriangledown_\top \rangle$: In this case the belief base result of the BGM on Ψ under the constraints μ is the conjunction of all the bases of the profile with the integrity constraints ($\bigwedge \Psi \wedge \mu$) if this conjunction is consistent, and μ otherwise. This operator is called the *basic merging operator* (Konieczny & Pino Pérez 1999).
- $\langle g_{id}, \blacktriangledown_\delta \rangle$: In this case, at each step of the game, each source weaken using dilatation. This gives the well known model-based merging operator $\triangle^{d_H,\max}$ defined in (Revesz 1993; 1997; Konieczny & Pino Pérez 2002a).
- $\langle g^{d_D,\Sigma}, \blacktriangledown_\top \rangle$: Here, the result is the cardinality-maximal consistent subset of Ψ if it is unique and consistent with the constraints μ, and it is simply μ otherwise. This operator is a new one. It is interesting since it can be viewed as a generalized conjunction : it gives the conjunction of all the bases and the constraints if it is consistent, but if it is not, it tries to find the result by doing the least number of repairs (forget of one belief base) of the belief profile. If there is no ambiguity on the correction (i.e. a unique cardinality-maxcons), then it accepts it as the result.
- $\langle g^{d_D,\max}, \blacktriangledown_\top \rangle$: This operator gives as result the conjunction of all the formulas that belongs to all maxcons (also called free formulas in (Benferhat, Dubois, & Prade 1997; 1999)) and the integrity constraints if it is consistent, and μ otherwise.
- $\langle g^{mc1}, \blacktriangledown_\top \rangle$: This operator gives the conjunction of the formulas that belongs to the maximum number of maxcons and the integrity constraints if consistent, and μ otherwise.
- $\langle g^{mc2}, \blacktriangledown_\top \rangle$: In this case, the belief base result of the merging is the conjunction of the belief bases that belong to the biggest maxcons for cardinality and the integrity constraints if consistent, and μ otherwise.

All those operators are not logically independent, some of them are logically stronger than others, as stated in the following proposition.

Proposition 1 *In figure 1 an arrow between an operator A and an operator B ($A \longrightarrow B$) means that operator A is logically stronger (or less cautious) than operator B. Results obtained by transitivity are not represented.*

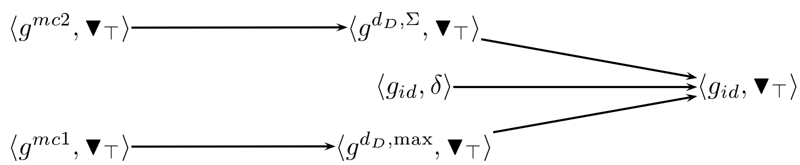

Figure 1: Cautiousness

An Example

We will see on an example (Revesz 1997), what is the behaviour or some BGM operators, namely the operators $\langle g^{d_H,h^\Sigma}, \blacktriangledown_\delta \rangle$, $\langle g^{d_H,h^{\max}}, \blacktriangledown_\delta \rangle$, $\langle g^{mc1}, \blacktriangledown_\delta \rangle$ and $\langle g^{mc2}, \blacktriangledown_\delta \rangle$. Here is the example : There are three sources $\Psi = \{\varphi_1, \varphi_2, \varphi_3\}$ with the following belief bases $Mod(\varphi_1) = \{(1,0,0),(0,0,1),(1,0,1)\}$, $Mod(\varphi_2) = \{(0,1,0),(0,0,1)\}$, $Mod(\varphi_3) = \{(1,1,1)\}$. There are no constraints on the result, so $\mu = \top$.

- $\langle g^{d_H,h^\Sigma}, \blacktriangledown_\delta \rangle$: As Ψ is not consistent, let us make the first round. $d(\varphi_1,\varphi_2) = 0$, $d(\varphi_1,\varphi_3) = 1$, $d(\varphi_2,\varphi_3) = 2$. So $h^\Sigma_\Psi(\varphi_1) = 1$, $h^\Sigma_\Psi(\varphi_2) = 2$, $h^\Sigma_\Psi(\varphi_3) = 3$. That gives $g^{d_H,h^\Sigma}(\Psi) = \{\varphi_3\}$. So φ_3 is replaced[3] by $\blacktriangledown_\delta(\varphi_3) = form(\{(1,1,1),(1,1,0),(1,0,1),(0,1,1)\})$. We have not reach yet a consistent Ψ, so let us make a further round. Let us first compute the new distances. $d(\varphi_1,\varphi_2) = 0$, $d(\varphi_1,\varphi_3) = 0$, $d(\varphi_2,\varphi_3) = 1$. So $h^\Sigma_\Psi(\varphi_1) = 0$, $h^\Sigma_\Psi(\varphi_2) = 1$, $h^\Sigma_\Psi(\varphi_3) = 1$. That gives $g^{d_H,h^\Sigma}(\Psi) = \{\varphi_2,\varphi_3\}$. So φ_2 is replaced by $\blacktriangledown_\delta(\varphi_2) = form(\{(0,1,0),\ (0,0,1),\ (1,1,0),\ (0,0,0),\ (0,1,1),\ (1,0,1)\})$, and φ_3 is replaced by $\blacktriangledown_\delta(\varphi_3) = form(\{(1,1,1),\ (1,1,0),\ (1,0,1),\ (0,1,1),\ (0,1,0),\ (1,0,0),(0,0,1)\})$. We have reach a consistent belief profile, so the result is $Mod(\Psi_{\langle g^{d_H,h^\Sigma}, \blacktriangledown_\delta \rangle}) = \{(0,0,1),(1,0,1)\}$.
- $\langle g^{d_H,h^{\max}}, \blacktriangledown_\delta \rangle$: As Ψ is not consistent, let us make the first round. $d(\varphi_1,\varphi_2) = 0$, $d(\varphi_1,\varphi_3) = 1$, $d(\varphi_2,\varphi_3) = 2$. So $h^{\max}_\Psi(\varphi_1) = 1$, $h^{\max}_\Psi(\varphi_2) = 2$, $h^{\max}_\Psi(\varphi_3) = 2$. That gives $g^{d_H,h^{\max}}(\Psi) = \{\varphi_2,\varphi_3\}$. So φ_2 is replaced by $\blacktriangledown_\delta(\varphi_2) = form(\{(0,1,0),\ (0,0,1),\ (1,1,0),\ (0,0,0),\ (0,1,1),\ (1,0,1)\})$, and φ_3 is replaced by $\blacktriangledown_\delta(\varphi_3) = form(\{(1,1,1),(1,1,0),(1,0,1),(0,1,1)\})$. The obtained profile is consistent, so the result is $Mod(\Psi_{\langle g^{d_H,h^{\max}}, \blacktriangledown_\delta \rangle}) = \{(1,0,1)\}$.
- $\langle g^{mc1}, \blacktriangledown_\delta \rangle$: Ψ is not consistent, and $\textsc{maxcons}(\Psi) = \{\{\varphi_1,\varphi_2\},\{\varphi_3\}\}$. So $h^{mc1}_\Psi(\varphi_1) = h^{mc1}_\Psi(\varphi_2) = h^{mc1}_\Psi(\varphi_3) = 1$, and $g^{mc1}(\Psi) = \Psi$. So we weaken the three bases, that gives respectively $\blacktriangledown_\delta(\varphi_1) = form(\{(1,0,0),\ (0,0,1),\ (1,0,1),\ (0,0,0),\ (1,1,0),\ (0,1,1),(1,1,1)\})$, $\blacktriangledown_\delta(\varphi_2) = form(\{(0,1,0),(0,0,1),\ (1,1,0),\ (0,0,0),\ (0,1,1),\ (1,0,1)\})$, and $\blacktriangledown_\delta(\varphi_3) = form(\ \{(1,1,1),(1,1,0),(1,0,1),\ (0,1,1)\})$. This belief profile is consistent, and the resulting base is $Mod(\Psi_{\langle g^{mc1}, \blacktriangledown_\delta \rangle}) = \{(1,0,1),(1,1,0),(0,1,1)\}$.
- $\langle g^{mc2}, \blacktriangledown_\delta \rangle$: Ψ is not consistent, and we have $\textsc{maxcons}(\Psi) = \{\{\varphi_1,\ \varphi_2\},\ \{\varphi_3\}\}$. So $h^{mc2}_\Psi(\varphi_1) = h^{mc2}_\Psi(\varphi_2) = 2$ and $h^{mc2}_\Psi(\varphi_3) = 1$, and $g^{mc2}(\Psi) = \{\varphi_3\}$. So φ_3 is replaced by $\blacktriangledown_\delta(\varphi_3) = form(\{(1,1,1),\ (1,1,0),\ (1,0,1),\ (0,1,1)\})$. The belief profile is still not consistent, so one needs one more round. Now we have $\textsc{maxcons}(\Psi) = \{\{\varphi_1,\varphi_2\},\{\varphi_1,\varphi_3\}\}$. So $h^{mc2}_\Psi(\varphi_1) = h^{mc2}_\Psi(\varphi_2) = h^{mc2}_\Psi(\varphi_3) = 2$, and $g^{mc2}(\Psi) = \Psi$. So we weaken the three bases, that gives respectively $\blacktriangledown_\delta(\varphi_1) = form(\{(1,0,0),\ (0,0,1),\ (1,0,1),\ (0,0,0),\ (1,1,0),\ (0,1,1),\ (1,1,1)\})$, $\blacktriangledown_\delta(\varphi_2) = form(\{(0,1,0),\ (0,0,1),\ (1,1,0),\ (0,0,0),\ (0,1,1),\ (1,0,1)\})$, and $\blacktriangledown_\delta(\varphi_3) = form(\ \{(1,1,1),\ (1,1,0),\ (1,0,1),\ (0,1,1)\})$. The belief profile is consistent, and the resulting base is $Mod(\Psi_{\langle g^{mc2}, \blacktriangledown_\delta \rangle}) = \{(0,0,1),(1,0,1),(1,1,0),(0,1,1)\}$.

As one can note, on this example the four operators give different (non trivial) results. As all these operators take dilatation as weakening functions, we sometimes have the interpretation $(1,1,0)$ as model of the base result of the merging, whereas it is a model of none of the initial belief bases. This means that, conversely to usual formula-based merging operators (Baral *et al.* 1992; Konieczny 2000; Konieczny, Lang, & Marquis 2004), the result of the BGM does not (always) imply the disjunction of the belief bases of the profile.

Logical Properties

Some work in belief merging aims at finding sets of axiomatic properties operators may exhibit the expected behaviour (Revesz 1993; 1997; Liberatore & Schaerf 1998; Konieczny & Pino Pérez 1998; 1999; 2002b). We focus here on the characterization of Integrity Constraints (IC) merging operators (Konieczny & Pino Pérez 1999; 2002a).

Definition 14 (IC merging operators) $\triangle$ *is an* IC merging operator *if and only if it satisfies the following postulates:*

(IC0) $\triangle_\mu(\Psi) \models \mu$

(IC1) *If μ is consistent, then $\triangle_\mu(\Psi)$ is consistent*

(IC2) *If $\bigwedge \Psi$ is consistent with μ, then $\triangle_\mu(\Psi) \equiv \bigwedge \Psi \wedge \mu$*

(IC3) *If $\Psi_1 \equiv \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(\Psi_1) \equiv \triangle_{\mu_2}(\Psi_2)$*

[3] In order to avoid unnecessary notations, we do not use subscripts to denote the different weakening steps of the bases, we simply replace the belief bases by their weakened counterparts. Hopefully, it can not lead to confusions.

(IC4) *If $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$, then $\triangle_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_1$ is consistent if and only if $\triangle_\mu(\{\varphi_1, \varphi_2\}) \wedge \varphi_2$ is consistent*

(IC5) $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2) \models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$

(IC6) *If $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent, then $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \models \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$*

(IC7) $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$

(IC8) *If $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(\Psi) \models \triangle_{\mu_1}(\Psi)$*

For more explanations on those properties see (Konieczny & Pino Pérez 2002a). So, let us see now what are the properties of BGM operators.

Proposition 2 *BGM operators satisfy properties* **(IC0)**, **(IC1)**, **(IC2)**, **(IC3)**, **(IC7)**, **(IC8)**. *They do not necessarily satisfy properties* **(IC4)**, **(IC5)**, **(IC6)**.

So, as stated in the previous proposition, BGM operators do not fit all properties of IC merging operators. On the other hand, we know for example that the operator $\langle g_{id}, \blacktriangledown_\delta \rangle = \triangle^{d_H,\max}$ satisfies also **(IC4)**, **(IC5)** (Konieczny & Pino Pérez 2002a). So the question is to know if we can ensure more logical properties by making some restrictions on the weakening and/or the choice functions.

A first remark is that **(IC4)** can not be proved to hold for any BGM operator, but it is satisfied for all the particular operators we have defined in this paper.

Proposition 3 *If the weakening function is dilatation or drastic weakening, and if the choice function is a symmetric model-based choice function or the formula-based choice function g^{mc1} or g^{mc2}, then the BGM operator satisfies* **(IC4)**.

The property **(IC5)** can also be recovered for some BGM operators, but **(IC6)** seems hardly recoverable. Those two properties are important for classical merging operators. The BGM operators aim at focusing on sources beliefs interactions, so it seems natural to misses property **(IC6)**. Indeed, whereas classical merging operators aim at giving the result of the merging process in an *ideal* framework, BGM operators seem more adequately reflect the behaviour of *real* multi-source merging process.

Another important logical link to be underlined is the relationship between BGM operators and AGM belief revision operators (Alchourrón, Gärdenfors, & Makinson 1985; Gärdenfors 1988; Katsuno & Mendelzon 1991; Gärdenfors 1992). Belief revision aim is to make the minimum change in a belief base in order to take into account a new information that is more reliable than the current belief base (and that usually contradicts the current belief base). Technically those operators can be described as follows : until the belief base is consistent with the new item of information (seen as an integrity constraint) then weaken the belief base[4]. Stating this way, one can immediately see the parallel with BGM operators since they are describe as follows : until the belief profile is consistent with the constraint then weaken some belief bases. The following result shows more formally that, as explained above, BGM operators can be seen as direct generalization of AGM belief revision operators.

[4]It is the intuitive meaning behind Katsuno and Mendelzon representation theorem in terms of faithful assignments (Katsuno & Mendelzon 1991).

Proposition 4 *Let $\mathcal{N} = \langle g, \blacktriangledown \rangle$ be a BGM operator. Let φ and μ be two belief bases. The operator $\circ$ defined as $\varphi \circ \mu = \mathcal{N}_\mu(\{\varphi\})$ is an AGM belief revision operator (i.e. it satisfies properties (R1-R6) of (Katsuno & Mendelzon 1991)).*

In particular, we have that each BGM using the dilatation weakening function is a generalization of Dalal's revision operator (Dalal 1988).

Finally let us see another cardinality restriction on belief profile.

Proposition 5 *Let $\mathcal{N} = \langle g^{d,h}, \blacktriangledown_\delta \rangle$ be a BGM operator defined from a symmetric model-based choice function and dilatation weakening function. Let φ_1, φ_2 and μ be three belief bases, then the operator $\mathcal{N}_\mu(\{\varphi_1, \varphi_2\})$ is the model-based merging operator $\triangle_\mu^{d_H,\max}(\{\varphi_1, \varphi_2\})$ (Konieczny & Pino Pérez 2002a).*

Note that the previous result holds only when we merge two belief bases.

Comparison between BGM and BNM

In this section we will mainly compare our proposal with Booth's Belief Negotiation Model (BNM) (Booth 2002). Let us first briefly recall Booth's proposal.

Belief profiles in this framework are no more multi-sets but vectors of belief bases, noted $\vec{\Psi}$. Let us note $\vec{\mathcal{E}}$ the set of belief profiles, and let us note $\vec{\Sigma}$ the set of all sequences (vectors) of belief profiles, and $\vec{\sigma}$ one element of this set. When $\vec{X}$ is a vector, we will note $\vec{X}^n$ the nth element of the vector and $\vec{X}^m$ the last element of the vector.

Then a BNM negotiation (choice) function is defined as :

Definition 15 *A* BNM negotiation function *is a function $g^{BNM} : \vec{\Sigma} \rightarrow \vec{\mathcal{E}}$ such that:*

- $g^{BNM}(\vec{\sigma}) \sqsubseteq \vec{\sigma}^m$
- $g^{BNM}(\vec{\sigma}) \neq \emptyset$
- *If $\varphi_i \in g^{BNM}(\vec{\sigma})$, then $\varphi_i \not\equiv \top$*

And a BNM weakening function is defined as :

Definition 16 *A* BNM weakening function *is a function $\blacktriangledown^{BNM} : \vec{\Sigma} \rightarrow \vec{\mathcal{E}}$ such that:*

- $(\vec{\sigma}^m)^i \vdash \blacktriangledown^{BNM}(\vec{\sigma})^i$
- *If $(\vec{\sigma}^m)^i \equiv \blacktriangledown^{BNM}(\vec{\sigma})^i$, then $(\vec{\sigma}^m)^i \equiv \top$*

Finally the solution to a BNM is defined as :

Definition 17 *The solution to a belief profile $\vec{\Psi}$ for a Belief Negotiation Model $\mathcal{N}^{BNM} = \langle g^{BNM}, \blacktriangledown^{BNM} \rangle$ under the integrity constraints μ, noted $\mathcal{N}_\mu^{BNM}(\Psi)$, is given by the function $f^{\mathcal{N}} : \vec{\mathcal{E}} \rightarrow \vec{\Sigma}$ defined as:*

- $f^{\mathcal{N}}(\vec{\Psi}) = \vec{\sigma} = (\vec{\Psi}_0, \ldots, \vec{\Psi}_k)$

with $\vec{\Psi}_0 = \vec{\Psi}$, k *is the smallest integer such that* $\bigwedge \vec{\Psi}_k \wedge \mu$ *is consistent, and for each* $0 \leq j < k$ *we have (* $\vec{\sigma_j}$ *denotes* $(\vec{\Psi}_0, \ldots, \vec{\Psi}_j)$*):*

$$(\vec{\Psi}_{j+1})^i = \begin{cases} \blacktriangledown^{BNM}(\vec{\sigma_j})^i & \text{if } (\vec{\Psi}_j)^i \in g^{BNM}(\vec{\sigma_j}) \\ (\vec{\Psi}_j)^i & \text{otherwise} \end{cases}$$

Finally, the belief base result of the BNM is $\bigwedge \vec{\Psi}_k \wedge \mu$.

We change some notations, in order to show the closeness with our present work. For the original presentation and explanations on the definitions see (Booth 2002; 2001).

The main differences between BNM and BGM are :

i. BNM's definition of belief profile as vectors allows to speak about sources separately. So when there is two identical belief bases in the belief profile, it is possible to weaken only one of this base. It is not possible in the BGM framework.

ii. The BNM negotiation function takes as input the whole negotiation history from the initial belief profile. So it is possible to implement negotiation process such that each source weaken after the previous one (for example, source 1, then source 2, $\ldots$), or such that we prevent a source to weak two times successively. The BGM choice functions are more Markovian, taking only into account the current belief profile.

iii. Similarly, the BNM weakening function also take as input the whole negotiation history. It allows to weaken differently two identical belief bases obtained at different rounds or to weaken differently two identical belief bases of the same belief profile.

iv. According to the previous items ideas, the irrelevance of syntax condition of BGM weakening function, and the anonymity condition of BGM choice function are not required in the BNM framework.

The main difference between Booth's proposal and our is that Booth's one take the sources as candidates to weakening, whereas we restrict ourselves to *"points of view"*. That means that in Booth's if one source has to weaken, it can be the case that another source with exactly the same beliefs do not have to weaken too. Our proposal add more anonymity by saying that only beliefs decide who has to weaken, not the identity of one source. Similarly, the choice functions are more Markovian in our framework than in Booth's one. We think that those hypothesis are more realistic (and necessary) on a belief merging point of view. Whereas Booth's framework allows to model more generalized negotiation frameworks, where one can decide for example that each source has to weaken one after the other. So, despite the closeness of the models, and the objective fact that our proposal is a particular case of Booth's one (i.e. each of our operators can be defined in Booth's framework), the intended applications of those two frameworks are quite different. And the particular properties achieved by adding those restrictions shows that this framework forms a consistent family of merging operators. It explains why it worths to focus on the model we defined.

A last difference is that, in this paper, we are interested on the result of the process (as a belief base), whereas BNM framework aims at studying the resulting profile, in connection with a notion of *"social contraction"*. See (Booth 2002) for a study of logical properties for social contraction.

An additional contribution of this work is to give examples of purely propositional logic BNM operators. In (Booth 2002), Booth propose two examples of BNM, both working on ordinal conditional functions (OCF) (Spohn 1987), but none on propositional belief base. So this work can be seen as an investigation of what kind of operators this definition can give on propositional belief bases (through adding additional requirements).

Conclusion

We have proposed in this paper a new family of belief merging operators, that we call Belief Game Model (BGM) operators. The hypothesis for those operators is that all the sources are *a priori* reliable, or that we know that some sources are less reliable than the others, but without knowing which ones. This hypothesis lead to choose a majority approach, justified by Condorcet's Jury Theorem. The idea behind Belief Game Model is simple : Until a coherent set of sources is reached, at each round a contest is organized to find out the weakest sources, then those sources has to concede (weaken their point of view). This idea leads to numerous new interesting operators and opens new perspectives for belief merging. Some existing operators are also recovered as particular cases.

Non-surprisingly, the operators defined do not satisfy all logical properties proposed for IC merging operators. The reason is that those logical properties aim at give constraints on the result of the merging in an *ideal* framework, whereas BGM operators aim at describing more accurately what can happen in a *real* multi-source environment. So usual IC merging operators can be seen as a *normative* approach to merging. They show the way to a purely logical result. Conversely, BGM operators adopt a *descriptive* approach to merging, taking into account the interaction between the sources. They try to simulate more adequately what can happen in a group-decision process. So they are maybe more realistic.

This paper mainly aims to introduce BGM operators, but it provides several open questions that are left for further research.

The first one is about the definition of BGM operators and the computation of the result. We give an iterative definition of BGM operators, that leads to an iterative computation of the result. The question is to know if we can find a non-iterative equivalent definition. We know that some simple operators can be defined non-iteratively. But the question is to know if all operators or a non-trivial subclass of them are also definable non-iteratively.

Another open question is about the logical characterization of this family. In this paper we study the logical properties of this family with respect to the general definition of IC merging operators. The question is to know if we can find a set of logical properties that characterizes BGM operators.

Finally, we have recently studied the strategy-proofness of usual propositional merging operators, showing that most of them are not strategy-proof (Everaere, Konieczny, & Marquis 2004). And we have exhibit several restrictions on which strategy-proofness can be achieved. So an interesting question is to compare the strategy-proofness of BGM operators with the one of classical merging operators.

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