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Weighted Merging of Propositional Belief Bases

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Abstract

In standard propositional belief merging, one implicit assumption is that all sources have exactly the same importance. But there are many situations where the sources have different importance/reliability/expertise that have to be taken into account in the merging process. In this work we study the problem of weighted merging operators, which aimed to take these weights into account in a sensible way. We give a syntactical characterization of these operators, and then we state a representation theorem in terms of plausibility preorders on interpretations. We also propose a general method to build weighted distance-based merging operators, and provide some concrete examples, using two different weight functions.

1 Introduction

Belief merging aims at producing a coherent opinion from a set of (typically) conflicting belief bases. Belief merging has been logically characterized (Konieczny and Pino Pérez 2002; Revesz 1997), and a lot of their properties has been studied, such as manipulability (Everaere, Konieczny, and Marquis 2004; Everaere, Konieczny, and Marquis 2007; Mata Díaz and Pino Pérez 2023), truth-tracking performances (Everaere, Konieczny, and Marquis 2010; Everaere, Konieczny, and Marquis 2020), computational complexity (Konieczny, Lang, and Marquis 2004; Haret et al. 2020), etc.

In these works, all the bases carry equal weight, which was crucial for understanding the fundamental framework. However, now that this foundation has been established, it becomes important to consider cases where different weights are assigned to each belief base. This approach is more general, and necessary in numerous applications, allowing for the reflection of varying degrees of credibility among information sources.

Additionally, as advocated by Delgrande, Dubois and Lang (2006), weighted belief merging can be seen as the most general change setting, since we can obtain on one hand classical belief merging as a special case when taking all the weights equal, and on the other hand iterated belief revision, when the last piece of information is given a weight big enough with respect to the previously received ones. Delgrande, Dubois and Lang (2006) discuss these links, but they did not provide any representation theorem. This is what we propose in this work.

Belief merging operators can be use to merge either beliefs or goals. When merging beliefs, then the aim is often to find the correct ("true") state of the world, and quite straightforwardly it is interesting to listen more the most reliable sources, so the weights will encode the reliability (expertise) of the sources. When merging goals to take a collective decision, it may happen that some participants have more importance or more power than others (for instance some participants can contribute more than others), then it is crucial to be able to take these difference for the merging, and the weights will encore the importance (power) of the sources.

In the rest of the paper, after some preliminaries, where we will give the necessary notations and definitions, we introduced weighted IC merging operators, that are characterized by a set of postulates. Then we provide a representation theorem in terms of plausibility preorders on interpretations. We also provide a general class of distance-based weighted merging operators, that are guaranteed to satisfy the postulates. Then we provides some examples of concrete weighted IC merging operators. And we illustrate their behaviour characteristic on some examples.

2 Preliminaries

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional letters. The set of consistent formulas is denoted $\mathcal{L}^*$. An interpretation is a function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is denoted Ω. An interpretation ω is a model of formula if and only if it makes it true in the usual classical truth functional way. $[\![\varphi]\!]$ denotes the set of models of φ, i.e. $[\![\varphi]\!] = \{\omega \in \Omega : \omega \models \varphi\}$. We note $\varphi \equiv \varphi'$ when φ and φ' have exactly the same models. Let M be a set of interpretations, φ_M denotes a formula whose set of models is M. When $M = \{\omega\}$ we will use the notation φ_ω for reading convenience.

An agent is characterized by some beliefs and by a reliability degree, that encodes how important/reliable/expert he is. Thus, an agent a is encoded by a couple of the form $(\varphi\, \alpha)$ where φ is a consistent formula, the beliefs of a, and where α is a strictly positive real number,[1] the degree of reliability associated to agent a.

[1]The set of strictly positive real numbers will be denoted $\mathbb{R}^{*+}$.

The set of agents will be denoted by A. A finite set of agents is called a profile. We use Capital Greek letters to denote profiles. The set of profiles is denoted $\mathcal{E}$.

Let $\mathcal{B}$ and δ be functions given the beliefs and the reliability of an agent respectively, that is $\mathcal{B} : A \to \mathcal{L}$ and $\delta : A \to \mathbb{R}^{*+}$ are the functions such that for every agent $a = (\varphi\ \alpha)$, $\mathcal{B}(a) = \varphi$ and $\delta(a) = \alpha$.

We say that two agents a and a' are equivalent (noted by a $\leftrightarrow$ a') if and only if $\mathcal{B}(a) \equiv \mathcal{B}(a')$ and $\delta(a) = \delta(a')$.

We say that two profiles Ψ and Ψ' are equivalent (noted by $\Psi \leftrightarrow \Psi'$) if and only if there is a bijection g from Ψ to Ψ' such that $a \leftrightarrow g(a)$.

Given two profiles Ψ and Ψ' we can suppose that they are disjoint (if not, we take Ψ'' equivalent to profile Ψ' and disjoint from Ψ). Thus, the union of these two profiles (supposed disjoint) is noted[2] by $\Psi \sqcup \Psi'$. The union between a profile Ψ and a profile $\{a\}$ such that $\mathcal{B}(a) = \varphi$ and $\delta(a) = \alpha$ is noted by $\Psi \sqcup a$ or $\Psi \sqcup (\varphi\ \alpha)$. We define Ψ^n as $\sqcup_{i=1}^{n} \Psi_i$ where for each i, $\Psi_i \leftrightarrow \Psi$ and for $i \neq j$ Ψ_i and Ψ_j are disjoint.

We note the conjunction between the bases of a profile $\mathcal{B}(a_1) \wedge ... \wedge \mathcal{B}(a_n)$ by $\bigwedge \Psi$.

We say that the profile Ψ is consistent if and only if $\bigwedge \Psi$ is consistent, in that case we write $\omega \models \Psi$ instead of $\omega \models \bigwedge \Psi$.

3 Weighted IC Merging

In this section we consider functions Δ mapping a profile Ψ and a consistent formula μ (that represents the integrity constraints) into a formula $\Delta(\Psi, \mu)$, noted $\Delta_\mu(\Psi)$ for short, that is $\Delta : \mathcal{E} \times \mathcal{L}^* \longrightarrow \mathcal{L}^*$. In the following, we give a set of postulates that merging operators must satisfy in order to behave rationally when the agents are related to weights which encode their reliability degrees.

Definition 1. *A merging operator Δ is called a weighted IC merging operator (WIC merging operator for short) if it satisfies the postulates (WIC0-WIC12) below:*

(WIC0) $\Delta_\mu(\Psi) \vdash \mu$

(WIC1) If μ is consistent, then $\Delta_\mu(\Psi)$ is consistent

(WIC2) If Ψ is consistent with μ, then $\Delta_\mu(\Psi) = \bigwedge \Psi \wedge \mu$

(WIC3) If $\Psi_1 \leftrightarrow \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(\Psi_1) \equiv \Delta_{\mu_2}(\Psi_2)$

(WIC4) If $\mathcal{B}(a_1) \vdash \mu$, $\mathcal{B}(a_2) \vdash \mu$ and $\delta(a_1) = \delta(a_2)$, then $\Delta_\mu(a_1 \sqcup a_2) \wedge \mathcal{B}(a_1) \nvdash \bot \Rightarrow \Delta_\mu(a_1 \sqcup a_2) \wedge \mathcal{B}(a_2) \nvdash \bot$

(WIC5) $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2) \vdash \Delta_\mu(\Psi_1 \sqcup \Psi_2)$

(WIC6) If $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$ is consistent, then $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \vdash \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$

(WIC7) $\Delta_{\mu_1}(\Psi) \wedge \mu_2 \vdash \Delta_{\mu_1 \wedge \mu_2}(\Psi)$

(WIC8) If $\Delta_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(\Psi) \vdash \Delta_{\mu_1}(\Psi) \wedge \mu_2$

(WIC9) If $\beta > \alpha$, if $\Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \vdash \varphi$, then $\Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \vdash \varphi$.

(WIC10) If $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \nvdash \bot$ and $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \nvdash \bot$, then $\Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \wedge \varphi \equiv \Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \wedge \varphi$

(WIC11) $\Delta_\mu((\varphi\ \alpha)) \equiv \Delta_\mu((\varphi\ \beta))$

(WIC12) If φ is consistent with μ, then $\exists \alpha, \Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \vdash \varphi$

The postulates (WIC0-WIC8) are an adaptation of (IC0-IC8) from the framework of belief merging (see (Konieczny and Pino Pérez 2002)) to the framework of weighted belief merging. The only difference is for (WIC4) where we need to add an extra condition stating that the two agents must have the same weights. (WIC4) states that if two agents have the same weights, no priority (coming from their names or any other information) can be given to one of them.

The most important postulates here are the ones that enforce a good behaviour with respect to the weights.

(WIC9) states that increasing the weight of a source can only be beneficial for this source. If the result of the merging already implies the beliefs of a source with a given weight, it will continue to do so with a bigger weight for this source.

(WIC10) states that the weight associated to a source has to be seen as a penalty against conflicting formulas, but they have no impact on formulae consistent with the source. So if the result of the merging is consistent with the belief of an agent, it will remain consistent whatever the weight of this agent (this weight will only have an impact on the part of the merging that is not consistent with the beliefs of the agent).

(WIC11) states that the result of the merging of a single source will be the same whatever the weight of this source. This postulate is about a very particular case, without any particular interest by itself, but its implications are important, since it ensures that weights are only used as amplificators of the plausibility relations associated to the beliefs of the sources, without modifying them.

(WIC12) can be seen as a kind of success postulates for the weights. It states that if the weight of a source is sufficiently large, then this source will manage to impose its view for the merging.

Clearly WIC merging operators are safe extensions of classical IC merging operators. More formally, every WIC merging operator Δ induces a merging operator $\bar{\Delta}$ over profiles without weights in a natural way: to any weigthed profile $\Psi = \{(\varphi_1\ 1), \ldots, (\varphi_n\ 1)\}$ we can associate a profile without weights $\bar{\Psi} = \{\varphi_1, \ldots, \varphi_n\}$. Then we define $\bar{\Delta}_\mu(\bar{\Psi}) = \Delta_\mu(\Psi)$, and straightforwardly:

Proposition 1. *For any WIC merging operator Δ, the operator $\bar{\Delta}$ is an IC merging operator.*

Let us now provide a representation theorem for these WIC merging operators in terms of plausibility relations on interpretations.

Definition 2. *A function $\Psi \to \leq_\Psi$ that maps each profile Ψ to a total preorder over interpretations $\leq_\Psi$ is called a* weighted syncretic assignment *if it satisfies the conditions 1-10 below:*

1. If $\omega \models \Psi$ and $\omega' \models \Psi$, then $\omega \simeq_\Psi \omega'$

2. If $\omega \models \Psi$ and $\omega' \not\models \Psi$, then $\omega \prec_\Psi \omega'$

3. If $\Psi_1 \leftrightarrow \Psi_2$, then $\leq_{\Psi_1} = \leq_{\Psi_2}$

[2] We use this $\sqcup$ notation to insist on the fact that several agents can have exactly the same couple (formula, weight), so a profile can be, equivalently, be considered as a multiset of such couples.

4. For any a, a' with $\delta(a) = \delta(a'), \forall \omega \models \mathcal{B}(a)$, $\exists \omega' \models \mathcal{B}(a')$ such that $\omega'_{a \sqcup a'} \omega$

5. If $\omega_{\Psi_1} \omega'$ and $\omega_{\Psi_2} \omega'$, then $\omega_{\Psi_1 \sqcup \Psi_2} \omega'$

6. If $\omega_{\Psi_1} \omega'$ and $\omega \prec_{\Psi_2} \omega'$, then $\omega \prec_{\Psi_1 \sqcup \Psi_2} \omega'$

7. If $\omega \not\models \varphi$, $\omega' \models \varphi$ and $\beta > \alpha$, then $\omega_{\Psi \sqcup (\varphi\ \beta)} \omega' \implies \omega_{\Psi \sqcup (\varphi\ \alpha)} \omega'$

8. If $\omega, \omega' \models \varphi$ then $\forall \alpha, \beta$, $\omega_{\Psi \sqcup (\varphi\ \alpha)} \omega' \iff \omega_{\Psi \sqcup (\varphi\ \beta)} \omega'$

9. $\omega_{(\varphi\ \alpha)} \omega'$ iff $\omega_{(\varphi\ \beta)} \omega'$

10. If $\omega \models \varphi \wedge \mu$ and $\omega' \models \neg\varphi \wedge \mu$, then $\exists \alpha\ \omega \prec_{\Psi \sqcup (\varphi\ \alpha)} \omega'$

The conditions (1-6) are an adaptation of (1-6) from the framework of belief merging (see (Konieczny and Pino Pérez 2002)) to the framework of weighted belief merging. Only (4) has been weaken to hold only when the agents have the same weights, that is expected with this fairness property: if two agents have the same weights, then no priority should be given to one of them.

Condition 7 says that if an interpretation ω that is not a model of the source is better than an interpretation ω' that is a model of this source, it will remain better if we use a lower weight for this source.

Condition 8 states that the order between interpretations that satisfy a source does not change regardless of the weight associated to that source. So this weight will have an impact only on interpretations that do not satisfy this source.

Condition 9 states that in a singleton profile the order between the interpretations does not change regardless of the weight associated with the source. This condition is mandatory to ensure that the weigths do not interfere with the plausibility relation.

Condition 10 states that any model of a source that is feasible (i.e. that satisfies the constraints μ) can become better than a non-model with a sufficiently large weight.

Observation 1. *It is easy to see that Condition 7 is equivalent to the following condition:*

7' If $\omega \not\models \varphi$, $\omega' \models \varphi$ and $\beta > \alpha$, then $\omega' \prec_{\Psi \sqcup (\varphi\ \alpha)} \omega \implies \omega' \prec_{\Psi \sqcup (\varphi\ \beta)} \omega$

We can also straightforwardly generalize the two main subfamilies of merging operators, namely majority and arbitration operators, to this weighted setting. A majority operator is a WIC merging operator that satisfies **(Maj)**. An arbitration operator is a WIC operator that satisfies **(Arb)**.

(Maj) $\exists n$ s.t. $\Delta_\mu(\Psi_1 \sqcup \Psi_2^n) \vdash \Delta_\mu(\Psi_2)$

(Arb)

$$\left.\begin{array}{l} \Delta_{\mu_1}(a_1) \equiv \Delta_{\mu_2}(a_2) \\ \Delta_{\mu_1 \leftrightarrow \neg\mu_2}(a_1 \sqcup a_2) \equiv (\mu_1 \leftrightarrow \neg\mu_2) \\ \mu_1 \not\vdash \mu_2 \\ \mu_2 \not\vdash \mu_1 \end{array}\right\} \Rightarrow \begin{array}{c} \Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2) \\ \equiv \\ \Delta_{\mu_1}(a_1) \end{array}$$

These conditions have semantic counterparts as we will see in the Representation theorem below:

(Maj-sa) $\omega \prec_{\Psi_2} \omega' \Rightarrow \exists n\ \omega \prec_{\Psi \sqcup \Psi_2^n} \omega'$

(Arb-sa)
$$\left.\begin{array}{l} \omega \prec_{a_1} \omega_1 \\ \omega \prec_{a_2} \omega_2 \\ \omega_1 \simeq_{a_1 \sqcup a_2} \omega_2 \end{array}\right\} \Rightarrow \omega \prec_{a_1 \sqcup a_2} \omega_1$$

Let us now state the representation theorem for WIC merging operators:

Theorem 1. *An operator Δ is an WIC merging operator if and only if there exists a weighted syncretic assignement $\Psi \rightarrow_\Psi$ that maps each profile Ψ to a total preorder on interpretations $_\Psi$ s.t. for every formula μ,*

$$[\![\Delta_\mu(\Psi)]\!] = min([\![\mu]\!], _\Psi)$$

Moreover, a WIC merging operator Δ satisfies (Maj) iff the syncretic assignment satisfies (Maj-sa), and Δ satisfies (Arb) iff the syncretic assignment satisfies (Arb-sa).

Proof. (only if part). Let Δ be a weighted merging operator satisfying postulates (WIC0-WIC12). Let us define a syncretic assignment as follows: for each profile Ψ, we define a total pre-order $_\Psi$ by putting $\forall \omega, \omega' \in \Omega, \omega_\Psi \omega'$ if and only if $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$. First we show that $_\Psi$ is a total pre-order:
Totality: $\forall \omega, \omega' \in \Omega$, from (WIC1), we know that $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi) \neq \emptyset$. From (WIC0), $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi) \vdash \varphi_{\{\omega,\omega'\}}$, so either $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$ or $\omega' \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$. It follows that $\omega_\Psi \omega'$ or $\omega'_\Psi \omega$.
Reflexivity: From (WIC0) and (WIC1) we have that $\Delta_{\varphi_\omega}(\Psi) \equiv \varphi_\omega$. So $\omega_\Psi \omega$.
Transitivity: Assume that $\omega_{1\Psi}\omega_2$ ($\star$) and $\omega_{2\Psi}\omega_3$ ($\star\star$) and suppose towards a contradiction that $\omega_1 \not\leq_\Psi \omega_3$. So by definition and from (WIC0) and (WIC1) $\Delta_{\varphi_{\{\omega_1,\omega_3\}}}(\Psi) \equiv \varphi_{\omega_3}$ ($\star\star\star$). By (WIC7) we know that $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_3\}} \vdash \Delta_{\varphi_{\{\omega_1,\omega_3\}}}(\Psi)$. We consider two cases:
Case 1: $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_3\}}$ is consistent. Then by (WIC7) and (WIC8) we deduce that $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_3\}} \equiv \Delta_{\varphi_{\{\omega_1,\omega_3\}}}(\Psi)$, from that and ($\star\star\star$) we have $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_3\}} \equiv \varphi_{\omega_3}$. Thus we have that $\omega_1 \not\models \Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi)$. But by (WIC1) $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \neq \emptyset$, so by (WIC0), either $[\![\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi)]\!] = \{\omega_2, \omega_3\}$ or $[\![\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi)]\!] = \{\omega_3\}$.

In the first case, by (WIC7) and (WIC8), we know that $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_2\}} \equiv \Delta_{\varphi_{\{\omega_1,\omega_2\}}}(\Psi)$. As $[\![\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi)]\!] = \{\omega_2, \omega_3\}$, $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_2\}} \equiv \{\omega_2\}$ and $\omega_1 \not\models \Delta_{\varphi_{\{\omega_1,\omega_2\}}}(\Psi)$: Contradiction with ($\star$).

In the second case, by (WIC7) and (WIC8), $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_2,\omega_3\}} \equiv \Delta_{\varphi_{\{\omega_2,\omega_3\}}}(\Psi)$. As $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \equiv \varphi_{\omega_3}$, we get that $\omega_2 \not\models \Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi)$ so $\omega_2 \not\models \Delta_{\varphi_{\{\omega_2,\omega_3\}}}(\Psi)$: Contradiction with ($\star\star$).
Case 2: $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_3\}}$ is not consistent. In this case, $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \equiv \varphi_{\omega_2}$. Then $\Delta_{\varphi_{\{\omega_1,\omega_2,\omega_3\}}}(\Psi) \wedge \varphi_{\{\omega_1,\omega_2\}} = \varphi_{\omega_2}$. By (WIC7) and (WIC8) it follows that $\Delta_{\varphi_{\{\omega_1,\omega_2\}}}(\Psi) = \varphi_{\omega_2}$, that is by definition $\omega_2 \prec_\Psi \omega_1$: Contradiction.

As $_\Psi$ is total, reflexive and transitive, it is a total pre-order.

Now we will show that $[\![\Delta_\mu(\Psi)]\!] = min([\![\mu]\!],_\Psi)$.

First we show the inclusion $[\![\Delta_\mu(\Psi)]\!] \subseteq min([\![\mu]\!],_\Psi)$. Assume that $\omega \models \Delta_\mu(\Psi)$ and suppose towards a contradiction that ω is not in $min([\![\mu]\!],_\Psi)$. So we can find a $\omega' \models \mu$ s.t. $\omega' \prec_\Psi \omega$, i.e. $\omega \not\models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$. Since $\Delta_\mu(\Psi) \wedge \varphi_{\{\omega,\omega'\}}$ is consistent, from (WIC7) and (WIC8), we have $\Delta_\mu(\Psi) \wedge \varphi_{\{\omega,\omega'\}} \equiv \Delta_{\mu\wedge\varphi_{\{\omega,\omega'\}}}(\Psi) \equiv \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$. As $\omega \not\models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$ so necessarily $\omega \not\models \Delta_\mu(\Psi)$: Contradiction.

For the other inclusion $[\![\Delta_\mu(\Psi)]\!] \supseteq min([\![\mu]\!],_\Psi)$, suppose that $\omega \in min([\![\mu]\!],_\Psi)$. We want to show that $\omega \models \Delta_\mu(\Psi)$. Since $\omega \in min([\![\mu]\!],_\Psi)$, $\forall\omega' \models \mu, \omega_\Psi\omega'$ and so $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$. As $\Delta_\mu(\Psi) \wedge \varphi_{\{\omega,\omega'\}}$ is consistent, from (WIC7) and (WIC8), we have $\Delta_\mu(\Psi) \wedge \varphi_{\{\omega,\omega'\}} \equiv \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$. But $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi)$ so $\omega \models \Delta_\mu(\Psi)$.

It remains to check the conditions of the syncretic assignment:

1. Suppose that $\omega \models \Psi$ and $\omega' \models \Psi$, then by (WIC2) we have $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi) = \varphi_{\{\omega,\omega'\}}$, so $\omega_\Psi\omega'$ and $\omega'_\Psi\omega$ and then by definition $\omega \simeq_\Psi \omega'$.

2. Suppose that $\omega \models \Psi$ and $\omega' \not\models \Psi$, then by (WIC2) $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi) = \varphi_\omega$, so $\omega_\Psi\omega'$ and $\omega' \not\!\!{}_\Psi\, \omega$, i.e. $\omega \prec_\Psi \omega'$.

3. Let's take $\Psi_1 \leftrightarrow \Psi_2$. We want to show that $_{\Psi_1} =_{\Psi_2}$. From (WIC3), we know that $\Delta(\Psi_1) \equiv \Delta(\Psi_2)$, then $\forall\omega,\omega' \in \Omega, \omega_{\Psi_1}\omega' \Leftrightarrow \omega_{\Psi_2}\omega'$. Thus $_{\Psi_1} =_{\Psi_2}$.

4. We want to show that if $\delta(a') = \delta(a)$, $\forall\omega \models \mathcal{B}(a)$, $\exists\omega' \models \mathcal{B}(a')$ s.t. $\omega'_{a\sqcup a'}\omega$. First we show that $\exists\omega' \models \Delta_{\mathcal{B}(a)\vee\mathcal{B}(a')}(a\sqcup a')\wedge\mathcal{B}(a')$. If not, we have $\Delta_{\mathcal{B}(a)\vee\mathcal{B}(a')}(a\sqcup a') \wedge \mathcal{B}(a') \vdash \bot$. From (WIC0) and (WIC1) we have that $\Delta_{\mathcal{B}(a)\vee\mathcal{B}(a')}(a\sqcup a') \vdash \mathcal{B}(a)$. Now by (WIC4) we get that $\exists\omega'$ s.t. $\omega' \models \Delta_{\mathcal{B}(a)\vee\mathcal{B}(a')}(a\sqcup a')\wedge\mathcal{B}(a')$. We get from (WIC7) and (WIC8) that $\omega' \models \Delta_{\varphi_{\{\omega,\omega'\}}}(a\sqcup a')$. So $\omega'_{a\sqcup a'}\omega$.

5. Suppose that $\omega_{\Psi_1}\omega'$ and $\omega_{\Psi_2}\omega'$ then $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1) \wedge \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_2)$. So from (WIC5), $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1\sqcup\Psi_2)$ and by definition $\omega_{\Psi_1\sqcup\Psi_2}\omega'$.

6. Suppose that $\omega \prec_{\Psi_1} \omega'$ and $\omega_{\Psi_2}\omega'$. We want to show that $\omega \prec_{\Psi_1\sqcup\Psi_2} \omega'$. From assumption, $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1) \wedge \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_2)$ and $\omega' \not\models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1) \wedge \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_2)$. So from (WIC5) and (WIC6), $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1\sqcup\Psi_2) \equiv \varphi_\omega$. That is $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1\sqcup\Psi_2)$ and $\omega' \not\models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1\sqcup\Psi_2)$, thus by definition $\omega \prec_{\Psi_1\sqcup\Psi_2} \omega'$.

7. Suppose that $\omega \not\models \varphi$, $\omega' \models \varphi$, $\omega' \prec_{\Psi\sqcup(\varphi\ \alpha)} \omega$ and $\beta > \alpha$. We have $\omega' \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha))$ and $\omega \not\models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha))$, thus $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha)) \vdash \varphi$. By (WIC9), we deduce that $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta)) \vdash \varphi$. As $\varphi_{\{\omega,\omega'\}}$ is consistent, by (WIC1) $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta))$ is also consistent, thus $\omega' \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta))$. As $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta)) \vdash \varphi$, we deduce that $\omega \not\models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta))$, then $\omega' \prec_{\Psi\sqcup(\varphi\ \beta)} \omega$.

8. We suppose that $\omega,\omega' \models \varphi$, we want to show that, $\forall\alpha,\beta,\omega_{\Psi\sqcup(\varphi\ \alpha)}\omega' \iff \omega_{\Psi\sqcup(\varphi\ \beta)}\omega'$, we prove ($\Rightarrow$) ($\Leftarrow$ is symmetrical). Suppose that $\omega_{\Psi\sqcup(\varphi\ \alpha)}\omega'$, by definition we have $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha))$, so $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha)) \wedge \varphi$, then $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha)) \wedge \varphi \not\vdash \bot$ ($\star$). From (WIC1), we know that $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta))$ is consistent, and with (WIC0) that $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta)) \models \varphi_{\{\omega,\omega'\}}$. Thus $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta))\wedge\varphi \not\vdash \bot$ ($\star\star$). Then from ($\star$), ($\star\star$) and by (WIC10) $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha)) \wedge \varphi \equiv \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta)) \wedge \varphi$. As $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha)) \wedge \varphi$, then $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta)) \wedge \varphi$, thus $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \beta))$, so $\omega_{\Psi\sqcup(\varphi\ \beta)}\omega'$.

9. We prove $\Rightarrow$ ($\Leftarrow$ is symmetrical). Suppose that $\omega_{(\varphi\ \alpha)}\omega'$. We want to show that $\omega_{(\varphi\ \beta)}\omega'$. From assumption and by definition we deduce that $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}((\varphi\ \alpha))$. By (WIC11) we deduce that $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}((\varphi\ \beta))$, then by definition $\omega_{(\varphi\ \beta)}\omega'$.

10. Assume that $\omega \models \varphi\wedge\mu$ and $\omega' \models \neg\varphi\wedge\mu$. We want to show that $\exists\alpha$ s.t. $\omega \prec_{\Psi\sqcup(\varphi\ \alpha)} \omega'$. Suppose towards a contradiction that $\forall\alpha, \omega'_{\Psi\sqcup(\varphi\ \alpha)}\omega$. By (WIC12), as φ is consistent with $\varphi_{\{\omega,\omega'\}}$, we know that $\exists\alpha$ s.t. $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi\sqcup(\varphi\ \alpha)) \vdash \varphi$.
We deduce that $\omega' \models \varphi$: Contradiction.

(If part) Let's consider a syncretic assignment that maps each profile Ψ to a total preorder on interpretations $_\Psi$ and define an operator Δ by putting $[\![\Delta_\mu(\Psi)]\!] = min([\![\mu]\!],_\Psi)$. We want to show that Δ satisfies (WIC0-WIC12).

(WIC0) By definition $[\![\Delta_\mu(\Psi)]\!] \subseteq [\![\mu]\!]$.

(WIC1) If μ is consistent, then $[\![\mu]\!] \neq \emptyset$ and, as we have a finite number of interpretations, we have no infinite descending chains of inequalities, so $min([\![\mu]\!],_\Psi) \neq \emptyset$. Then $\Delta_\mu(\Psi)$ is consistent.

(WIC2) Assume that $\bigwedge\Psi\wedge\mu$ is consistent. We want to show that $min([\![\mu]\!],_\Psi) = [\![\bigwedge\Psi\wedge\mu]\!]$. First note that if $\omega \models \bigwedge\Psi$ then from conditions 1 and 2, $\omega \in min(\Omega,_\Psi)$. So if $\omega \models \Psi\wedge\mu$ then $\omega \in min([\![\mu]\!],_\Psi)$: $min([\![\mu]\!],_\Psi) \supseteq [\![\bigwedge\Psi\wedge\mu]\!]$. For the other inclusion consider $\omega \in min([\![\mu]\!],_\Psi)$. Suppose towards a contradiction that $\omega \not\models \bigwedge\Psi\wedge\mu$. Since $\omega \not\models \bigwedge\Psi$, by condition 2 we have that $\forall\omega' \models \bigwedge\Psi, \omega' \prec_\Psi \omega$. In particular $\forall\omega' \models \bigwedge\Psi\wedge\mu, \omega' \prec_\Psi \omega$. So $\omega \notin min([\![\mu]\!],_\Psi)$. Contradiction.

(WIC3) Assume that $\Psi_1 \leftrightarrow \Psi_2$ and $\mu_1 \equiv \mu_2$, so from condition 3 we have $_{\Psi_1} =_{\Psi_2}$ and $[\![\mu_1]\!] = [\![\mu_2]\!]$, $min([\![\mu_1]\!],_{\Psi_1}) = min([\![\mu_2]\!],_{\Psi_2})$, thus $\Delta_{\mu_1}(\Psi_1) \equiv \Delta_{\mu_1}(\Psi_2)$.

(WIC4) Assume that $\mathcal{B}(a) \vdash \mu, \mathcal{B}(a') \vdash \mu$, $\Delta_\mu(a\sqcup a') \wedge \mathcal{B}(a) \not\models \bot$ and $\delta(a) = \delta(a')$ .We want to show that $\Delta_\mu(a\sqcup a')\wedge\mathcal{B}(a') \not\models \bot$. Consider $\omega \models \Delta_\mu(a\sqcup a')\wedge\mathcal{B}(a)$. Then from condition 4 we have that $\exists\omega' \models \mathcal{B}(a')$ such that $\omega'_{a\sqcup a'}\omega$. Then ω' is minimal in $[\![\mu]\!]$ with respect of $_{a\sqcup a'}$. Then $\omega' \models \Delta_\mu(a\sqcup a')$ and therefore $\Delta_\mu(a\sqcup a') \wedge \mathcal{B}(a') \not\vdash \bot$.

(WIC5) Let $\omega \models \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$. Then $\omega \in \min([\![\mu]\!],_{\Psi_1})$ and so $\forall\omega' \models \mu, \omega_{\Psi_1}\omega'$. We have in the same way $\forall\omega' \models \mu, \omega_{\Psi_2}\omega'$. By condition 5, we have that

$\forall \omega' \models \mu, \omega_{\Psi_1 \sqcup \Psi_2} \omega'$. So $\omega \in min([\![\mu]\!], _{\Psi_1 \sqcup \Psi_2})$ and by definition $\omega \models \Delta_\mu(\Psi_1 \sqcup \Psi_2)$.

(WIC6) Assume that $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$ is consistent. We want to show that $\Delta_\mu(\Psi_1 \sqcup \Psi_2) \vdash \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$ holds. Take $\omega \models \Delta_\mu(\Psi_1 \sqcup \Psi_2)$. So $\forall \omega' \models \mu, \omega_{\Psi_1 \sqcup \Psi_2} \omega'$. Suppose towards a contradiction that $\omega \not\models \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$. So $\omega \not\models \Delta_\mu(\Psi_1)$ or $\omega \not\models \Delta_\mu(\Psi_2)$. Suppose that $\omega \not\models \Delta_\mu(\Psi_1)$ (the other case is analogous). As $\Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$ is consistent, then $\exists \omega' \models \Delta_\mu(\Psi_1) \wedge \Delta_\mu(\Psi_2)$, so $\omega' \models \Delta_\mu(\Psi_1)$ and $\omega' \models \Delta_\mu(\Psi_2)$. Then by definition $\omega' \prec_{\Psi_1} \omega$ and $\omega'_{\Psi_2} \omega$, thus by condition 6 $\omega' \prec_{\Psi_1 \sqcup \Psi_2} \omega$ and then $\omega \not\models \Delta_\mu(\Psi_1 \sqcup \Psi_2)$. Contradiction.

(WIC7) Consider that $\Delta_{\mu_1}(\Psi) \wedge \mu_2$ is consistent (if not, the result is straightforward), and let $\omega \models \Delta_{\mu_1}(\Psi) \wedge \mu_2$. So $\forall \omega' \models \mu_1$, $\omega_\Psi \omega'$. So $\forall \omega' \models \mu_1 \wedge \mu_2$, $\omega_\Psi \omega'$, so $\omega \models \Delta_{\mu_1 \wedge \mu_2}(\Psi)$.

(WIC8) Assume that $\Delta_{\mu_1}(\Psi) \wedge \mu_2$ is consistent and let $\omega' \models \Delta_{\mu_1}(\Psi) \wedge \mu_2$. Consider $\omega \models \Delta_{\mu_1 \wedge \mu_2}(\Psi)$ and suppose that $\omega \not\models \Delta_{\mu_1}(\Psi)$. So $\omega' \prec_\Psi \omega$. But $\omega' \models \mu_1 \wedge \mu_2$ then $\omega \notin min([\![\mu_1 \wedge \mu_2]\!], _\Psi)$. Thus $\omega \not\models \Delta_{\mu_1 \wedge \mu_2}(\Psi)$. Contradiction.

(WIC9) Suppose that $\beta > \alpha$ and $\Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \vdash \varphi$. We want to show that $\Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \vdash \varphi$. From assumption we deduce that $\exists \omega \models \Delta_\mu(\Psi \sqcup (\varphi\ \alpha))$ and $\omega \models \varphi$. Suppose towards a contradiction that $\Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \not\vdash \varphi$, then $\exists \omega' \models \Delta_\mu(\Psi \sqcup (\varphi\ \beta))$ and $\omega' \models \neg\varphi$. Since $\omega \models \mu$ by (WIC0), then $\omega'_{\Psi \sqcup (\varphi\ \beta)} \omega$, so by condition 7 we have that $\omega'_{\Psi \sqcup (\varphi\ \alpha)} \omega$ ($\star$). But $\omega \in min([\![\mu]\!], _{\Psi \sqcup (\varphi\ \alpha)})$ and since $\omega' \in [\![\mu]\!]$ by (WIC0), then by ($\star$) $\omega' \in min([\![\mu]\!], _{\Psi \sqcup (\varphi\ \alpha)})$, thus $\omega' \models \Delta_\mu(\Psi \sqcup (\varphi\ \alpha))$, then from assumption $\omega' \models \varphi$. Contradiction.

(WIC10) Assume that $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \not\vdash \bot$ and $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \not\vdash \bot$. We want to show that $\Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \wedge \varphi \equiv \Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \wedge \varphi$. From assumption, we have $\exists \omega \models \varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \alpha))$ and $\exists \omega' \models \varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \beta))$. As a consequence, $\omega, \omega' \models \varphi$, and from condition (8) we have $\forall \alpha, \beta, \omega_{\Psi \sqcup (\varphi\ \alpha)} \omega' \iff \omega_{\Psi \sqcup (\varphi\ \beta)} \omega'$, so $\Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi \sqcup (\varphi\ \alpha)) \equiv \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi \sqcup (\varphi\ \beta))$. We can easily deduce by that this equivalence is preserved for each $\omega \models \varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \alpha))$ and $\omega' \models \varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \beta))$. Thus $\Delta_{\mu \wedge \varphi}(\Psi \sqcup (\varphi\ \alpha)) \equiv \Delta_{\mu \wedge \varphi}(\Psi \sqcup (\varphi\ \beta))$. From the assumption we have that $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \alpha))$ and $\varphi \wedge \Delta_\mu(\Psi \sqcup (\varphi\ \beta))$ are consistent, then from (WIC7) and (WIC8) we have $\Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \wedge \varphi \equiv \Delta_{\mu \wedge \varphi}(\Psi \sqcup (\varphi\ \alpha))$ and $\Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \wedge \varphi \equiv \Delta_{\mu \wedge \varphi}(\Psi \sqcup (\varphi\ \beta))$, thus $\Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \wedge \varphi \equiv \Delta_\mu(\Psi \sqcup (\varphi\ \beta)) \wedge \varphi$.

(WIC11) Direct from condition (9).

(WIC12) Suppose that φ is consistent with μ. We want to show that $\exists \alpha$, s.t. $\Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \vdash \varphi$. Suppose, towards a contradiction that $\forall \alpha$, $\exists \omega' \models \Delta_\mu(\Psi \sqcup (\varphi\ \alpha)) \wedge \neg\varphi$. Consider an increasing and unbounded sequence of positive real numbers $(\alpha_k)_{k \in \mathbb{N}}$. From the previous assumption, for every α_k, there exists an ω_{α_k} such that $\omega_{\alpha_k} \models \Delta_\mu(\Psi \sqcup (\varphi\ \alpha_k)) \wedge \neg\varphi$. As Ω is finite, by the pigeonhole principle, there exists an ω' and a subsequence $(\alpha_{k_n})_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$, $\omega' \models \Delta_\mu(\Psi \sqcup (\varphi\ \alpha_{k_n})) \wedge \neg\varphi$. Thus $\forall \omega \models \mu$, $\omega'_{\Psi \sqcup (\varphi\ \alpha_{k_n})} \omega$. We know that $\varphi \wedge \mu \not\vdash \bot$, so for all $n \in \mathbb{N}$ and $\forall \omega \models \varphi \wedge \mu$, $\omega'_{\Psi \sqcup (\varphi\ \alpha_{k_n})} \omega$ (*). By (WIC0) $\omega' \models \neg\varphi \wedge \mu$. Let's take $\omega \models \varphi \wedge \mu$. By condition 10, we know that $\exists \alpha$ s.t. $\omega \prec_{\Psi \sqcup (\varphi\ \alpha)} \omega'$. Since the sequence $(\alpha_{k_n})_{n \in \mathbb{N}}$ is unbounded there exists $\alpha_{k_{n_0}} > \alpha$ and by condition 7' (equivalent to 7) we have $\omega \prec_{\Psi \sqcup (\varphi\ \alpha_{k_{n_0}})} \omega'$. But this last statement is in contradiction with (*).

Now we prove the second part of the theorem, that is, that for WIC operators (Maj) is equivalent to (Maj-sa) and (Arb) is equivalent to (Arb-sa).

(Maj $\Rightarrow$ Maj-sa) Suppose that $\omega \prec_{\Psi_2} \omega'$. We want to show, assuming (Maj), that $\exists n\ \omega \prec_{\Psi_1 \sqcup \Psi_2^n} \omega'$. From supposition and by definition we have $\omega \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_2)$ and $\omega' \not\models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_2)$. Suppose, towards a contradiction, that $\forall n\ \omega'_{\Psi_1 \sqcup \Psi_2^n} \omega$, then $\omega' \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_1 \sqcup \Psi_2^n)$, thus by **(Maj)** we deduce that $\omega' \models \Delta_{\varphi_{\{\omega,\omega'\}}}(\Psi_2)$. Contradiction.

(Maj-sa $\Rightarrow$ Maj) First note that if $\omega \prec_{\Psi_2} \omega'$ and $\omega \prec_{\Psi_1 \sqcup \Psi_2^n} \omega'$ then, by condition 6, we have $\omega \prec_{\Psi_1 \sqcup \Psi_2^{n+1}} \omega'$. Thus, using induction, it is easy to see that Maj-sa plus condition 6 entail the following statement:

$$\omega \prec_{\Psi_2} \omega' \Rightarrow \exists n_0 \forall n n_0,\ \omega \prec_{\Psi_1 \sqcup \Psi_2^n} \omega'$$

Note that the contrapositive form of this statement is the following one:

$$\forall n_0 \exists n n_0,\ \omega_{\Psi_1 \sqcup \Psi_2^n} \omega' \Rightarrow \omega_{\Psi_2} \omega' \qquad (*)$$

Suppose towards a contradiction that $\forall n, \Delta_\mu(\Psi_1 \sqcup \Psi_2^n) \not\vdash \Delta_\mu(\Psi_2)$. From this we obtain $\forall n, \exists \omega \models \mu, \forall \omega'' \models \mu,\quad \omega_{\Psi_1 \sqcup \Psi_2^n} \omega''$ and $\exists \omega' \models \mu \quad \omega' \prec_{\Psi_2} \omega$. As Ω is finite, by the pigeonhole principle, there exists an ω such that for infinite integers n, we have $\omega_{\Psi_1 \sqcup \Psi_2^n} \omega''$ for any $\omega'' \models \mu$ and such that $\exists \omega' \models \mu, \quad \omega' \prec_{\Psi_2} \omega$. This condition entails the premisses of ($*$), therefore we have $\omega_{\Psi_2} \omega''$ for any $\omega'' \models \mu$ which is in obvious contradiction with the existence of a $\omega' \models \mu, \quad \omega' \prec_{\Psi_2} \omega$.

(Arb $\Rightarrow$ Arb-sa) Suppose that $\omega \prec_{a_1} \omega_1$, $\omega \prec_{a_2} \omega_2$ and $\omega_1 \simeq_{a_1 \sqcup a_2} \omega_2$. First if $\omega_1 = \omega_2$ then $\omega \prec_{a_1 \sqcup a_2} \omega_1$ by condition 6. Now if $\omega_1 \neq \omega_2$, then $\Delta_{\varphi_{\{\omega,\omega_1\}}}(a_1) \equiv \Delta_{\varphi_{\{\omega,\omega_2\}}}(a_2) \equiv \varphi_\omega$. Also $\Delta_{\varphi_{\{\omega_1,\omega_2\}}}(a_1 \sqcup a_2) = \varphi_{\{\omega_1,\omega_2\}}$, $\varphi_{\{\omega_1,\omega\}} \wedge \neg\varphi_{\{\omega_1,\omega_2\}}$ and $\neg\varphi_{\{\omega_1,\omega\}} \wedge \varphi_{\{\omega_1,\omega_2\}}$ are consistent. Then by **(Arb)** we have that $\Delta_{\varphi_{\{\omega,\omega_1,\omega_2\}}}(a_1 \sqcup a_2) = \varphi_\omega$. And by (WIC7) and (WIC8) we deduce that $\Delta_{\varphi_{\{\omega,\omega_1,\omega_2\}} \wedge \varphi_{\{\omega,\omega_1\}}}(a_1 \sqcup a_2) \equiv \Delta_{\varphi_{\{\omega,\omega_1\}}}(a_1 \sqcup a_2) \equiv \varphi_\omega$, then $\omega \prec_{a_1 \sqcup a_2} \omega_1$.

(Arb-sa $\Rightarrow$ Arb) Suppose that $\Delta_{\mu_1}(a_1) \equiv \Delta_{\mu_2}(a_2)$, $\Delta_{\mu_1 \leftrightarrow \neg\mu_2}(a_1 \sqcup a_2) \equiv (\mu_1 \leftrightarrow \neg\mu_2)$, $\mu_1 \wedge \neg\mu_2 \not\vdash \bot$ and $\mu_2 \wedge \neg\mu_1 \not\vdash \bot$. We want to show that $\Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2) \equiv \Delta_{\mu_1}(a_1)$.

First, we show that $\Delta_{\mu_1}(a_1) \vdash \Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2)$. Let's take $\omega \vdash \Delta_{\mu_1}(a_1)$, suppose towards a contradiction that $\omega \not\models \Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2)$. Then $\exists \omega_1 \models \mu_1 \vee \mu_2\ \omega_1 \prec_{a_1 \sqcup a_2} \omega$. We consider 3 cases: $\omega_1 \models \mu_1 \wedge \mu_2$, $\omega_1 \models \mu_1 \wedge \neg\mu_2$ and

$\omega_1 \models \neg\mu_1 \wedge \mu_2$.
Case 1: $\omega_1 \models \mu_1 \wedge \mu_2$. As $\omega \models \Delta_{\mu_1}(a_1)$, $\omega_{a_1}\omega_1$. From assumption we have $\Delta_{\mu_1}(a_1) \equiv \Delta_{\mu_2}(a_2)$. Thus $\omega \models \Delta_{\mu_2}(a_2)$ and then $\omega_{a_2}\omega_1$. Then by condition 5 we have $\omega_{a_1 \sqcup a_2}\omega_1$. Contradiction.
Case 2: $\omega_1 \models \mu_1 \wedge \neg\mu_2$ (case 3 is symmetric). As $\omega_1 \not\models \mu_2$, then $\omega_1 \not\models \Delta_{\mu_2}(a_2)$ and from assumption we have $\Delta_{\mu_1}(a_1) \equiv \Delta_{\mu_2}(a_2)$, then we have $\omega_1 \not\models \Delta_{\mu_1}(a_1)$. Then $\omega \prec_{a_1} \omega_1$ and $\omega \prec_{a_2} \omega_1$. From this, by condition 6, we have $\omega \prec_{a_1 \sqcup a_2} \omega_1$, a contradiction.
Now we will show that $\Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2) \vdash \Delta_{\mu_1}(a_1)$. Suppose that $\omega \models \Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2)$ and suppose towards a contradiction that $\omega \not\models \Delta_{\mu_1}(a_1)$. We consider 3 cases:
Case 1: $\omega \models \mu_1 \wedge \mu_2$, then $\exists \omega_1 \models \Delta_{\mu_1}(a_1)$, such that $\omega_1 \prec_{a_1} \omega$. And as $\Delta_{\mu_1}(a_1) \equiv \Delta_{\mu_2}(a_2)$, $\omega_1 \prec_{a_2} \omega$. Thus by condition 6 we have that $\omega_1 \prec_{a_1 \sqcup a_2} \omega$, then $\omega \not\models \Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2)$. Contradiction.
Case 2: $\omega \models \mu_1 \wedge \neg\mu_2$ (case 3, where $\omega \models \neg\mu_1 \wedge \mu_2$ is symmetric). From assumption we know that $\exists \omega_2 \models \neg\mu_1 \wedge \mu_2$. As $\Delta_{\mu_1}(a_1) \equiv \Delta_{\mu_2}(a_2)$, $\exists \omega_1 \models \Delta_{\mu_1}(a_1)$ such that $\omega_1 \prec_{a_1} \omega$ and $\omega_1 \prec_{a_2} \omega_2$. We deduce also from $\Delta_{\mu_1 \leftrightarrow \neg\mu_2}(a_1 \sqcup a_2) \equiv (\mu_1 \leftrightarrow \neg\mu_2)$ that $\omega \simeq_{a_1 \sqcup a_2} \omega_2$, thus by **Arb-sa** $\omega_1 \prec_{a_1 \sqcup a_2} \omega$. Thus $\omega \not\models \Delta_{\mu_1 \vee \mu_2}(a_1 \sqcup a_2)$. Contradiction. □

4 Distance-Based WIC Operators

In this Section we give a general method to construct a weighted merging operator. This method is a generalization of distance-based methods initiated in (Konieczny and Pino Pérez 2002) and extended in (Konieczny, Lang, and Marquis 2004) (see also (Konieczny and Pino Pérez 2011)). The main ingredients will be a (pseudo) distance d between interpretations, an aggregation function f and another function, $\bullet$, allowing us to take into account the weight along with the distance.

Let us first formally define all these ingredients.

Definition 3. *A pseudo-distance over interpretations is a function $d : \Omega \times \Omega \to \mathbb{R}^+$ such that $d(\omega, \omega') = d(\omega', \omega)$ and $d(\omega, \omega') = 0$ iff $\omega = \omega'$.*

Definition 4. *The distance between an interpretation ω and a formula φ is defined by $d(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega')$.*

Definition 5. *A weight function is a function $\bullet : \mathbb{R}^+ \times \mathbb{R}^{*+} \to \mathbb{R}^+$, which satisfies the following properties:*

- ***Increasing:*** *If $d \neq 0$ and $\alpha > \beta$, then $\bullet(d, \alpha) > \bullet(d, \beta)$, and if $d > d'$, then $\bullet(d, \beta) > \bullet(d', \beta)$*
- ***Invariance of 0:*** *$\forall \alpha, \beta, \bullet(0, \alpha) = \bullet(0, \beta) \stackrel{def}{=} \bullet_0$*
- ***Unbounded:*** *$\forall d > 0$, $\forall K > 0, \exists \alpha$ s.t. $\bullet(d, \alpha) > K$*

Definition 6. *An aggregation function is a mapping $f : \bigcup_n R^{+^n} \to (\mathcal{I}, \leq)$, where $\mathcal{I}$ is a totally ordered set*[3]*, which has the following properties:*

- ***Increasing:*** *If $\beta > \gamma$, then $f(\alpha_1, ..., \beta, ..., \alpha_n) > f(\alpha_1, ..., \gamma, ..., \alpha_n)$*
- ***Symmetry:*** *If σ is a permutation over $\{1, ..., n\}$, then $f(\alpha_1, ..., \alpha_n) = f(\alpha_{\sigma(1)}, ..., \alpha_{\sigma(n)})$*
- ***Composition:*** *If $f(\alpha_1, ..., \alpha_n) \geq f(\beta_1, ..., \beta_n)$, then $\forall \gamma \geq 0$, $f(\alpha_1, ..., \alpha_n, \gamma) \geq f(\beta_1, ..., \beta_n, \gamma)$*
- ***Decomposition:*** *$\forall \gamma \geq 0$, if $f(\alpha_1, ..., \alpha_n, \gamma) \geq f(\beta_1, ..., \beta_n, \gamma)$, then $f(\alpha_1, ..., \alpha_n) \geq f(\beta_1, ..., \beta_n)$*
- ***Unbounded:*** *$\forall(\alpha_1, ..., \alpha_n), \forall(\beta_1, ..., \beta_n), \forall\beta, \exists\alpha$ s.t. $f(\alpha_1, ..., \alpha_n, \alpha) > f(\beta_1, ..., \beta_n, \beta)$*

Note that in a lot of works in the literature aggregation functions are functions that maps a profile of real numbers into a real number. We chose a little more complex set for the image of the aggregation function, in order to be able to include slightly more complex functions, like Gmax.

Now we assume we have a pseudo distance d, a weight function $\bullet$ and an aggregation function f. Our next goal will be to define an assignment based on these three functions and to prove that the operator defined via the assigment is a WIC operator. First we define the weighted distance between an interpretation ω and an agent a as follows:

$$d^{\bullet}_{d}(\omega, a) = \bullet(d(\omega, \mathcal{B}(a)), \delta(a))$$

Second, we define the distance between an interpretation ω and a profile Ψ as follows:[4]

$$d^{\bullet}_{d,f}(\omega, \Psi) = f_{a \in \Psi}\, d^{\bullet}_{d}(\omega, a)$$

Third, we define an assignment $\Psi \mapsto^{d^{\bullet}_{d,f}}_{\Psi}$ by putting:

$$\omega^{d^{\bullet}_{d,f}}_{\Psi}\omega' \text{ iff } d^{\bullet}_{d,f}(\omega, \Psi) \leq d^{\bullet}_{d,f}(\omega', \Psi)$$

Finally, we define semantically the operator $\Delta^{d^{\bullet}_{d,f}}_{\mu}$ associated to d, $\bullet$ and f as:

$$[\![\Delta^{d^{\bullet}_{d,f}}_{\mu}(\Psi)]\!] = min([\![\mu]\!], ^{d^{\bullet}_{d,f}}_{\Psi})$$

Theorem 2. *Let d, $\bullet$ and f be a pseudo distance, a weight function and an aggregation function respectively. Then, the operator $\Delta^{d^{\bullet}_{d,f}}$ is a WIC merging operator.*

Before giving the proof of this result we state a lemma with is very useful in order to simplify its proof. For space reasons we don't give the (straightforward) proofs of the lemma.

Lemma 1. *Let f be an aggregation function. Then the following conditions hold:*

1. *If $(\alpha_1, \ldots, \alpha_n)$ and $(\beta_1, \ldots, \beta_n)$ are two vectors of R^{+^n} such that $\forall i \leq n$, $\alpha_i \leq \beta_i$ and there exists $j \leq n$ such that $\alpha_j < \beta_j$, then $f(\alpha_1, \ldots, \alpha_n) < f(\beta_1, \ldots, \beta_n)$.*
2. *$f(\alpha_1, \ldots, \alpha_n) \leq f(\beta_1, \ldots, \beta_n)$ and $f(\alpha'_1, \ldots, \alpha'_k) \leq f(\beta'_1, \ldots, \beta'_k)$ entails $f(\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_k) \leq f(\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_k)$.*
3. *$f(\alpha_1, \ldots, \alpha_n) < f(\beta_1, \ldots, \beta_n)$ and $f(\alpha'_1, \ldots, \alpha'_k) \leq f(\beta'_1, \ldots, \beta'_k)$ entails $f(\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_k) < f(\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_k)$.*

[3] Usually, for example for the sum, $\mathcal{I}$ is simply $\mathcal{R}^+$ and $\leq$ the natural order between real numbers. For Gmax or Gmin, $\mathcal{I}$ is $\bigcup_n R^{+^n}$ and $\leq$ the lexicographic order between vectors of real numbers.

[4] If $\Psi = \{a_1, \ldots, a_n\}$, $f_{a \in \Psi}\, d^{\bullet}_{d}(\omega, a)$ is the short notation of $f(d^{\bullet}_{d}(\omega, a_1), \ldots, d^{\bullet}_{d}(\omega, a_n))$.

Proof of Theorem 2: By definition of $\Delta^{d^\bullet_{d,f}}$ and Theorem 1 it is enough to prove that the mapping $\Psi \mapsto^{d^\bullet_{d,f}}_{\Psi}$ is a weighted syncretic assignment.

1. Suppose that $\omega \models \Psi$ and $\omega' \models \Psi$, we want to show that $\omega \simeq_\Psi \omega'$. We have $\forall a \in \Psi, d(\omega, a) = d(\omega', a) = 0$, so $\forall a \in \Psi, d^\bullet_d(\omega, a) = d^\bullet_d(\omega', a) = \bullet_0$ (invariance of 0), and $f(\{d^\bullet_d(\omega, a) \mid a \in \Psi\}) = f(\{d^\bullet_d(\omega', a) \mid a \in \Psi\})$.

2. Suppose that $\omega \models \Psi$ and $\omega' \not\models \Psi$, we want to show that $\omega \prec^{d^\bullet_{d,f}}_{\Psi} \omega'$. Since, $\bullet$ is increasing in the first variable, we have $\forall a_i \in \Psi, d^\bullet_d(\omega, a_i) \leq d^\bullet_d(\omega', a_i)$, and $\exists a \in \Psi$ s.t. $d^\bullet_d(\omega, a) < d^\bullet_d(\omega', a)$. From this and Lemma 1.1, we obtain easily $\omega \prec^{d^\bullet_{d,f}}_{\Psi} \omega'$.

3. We want to show that if $\Psi_1 \leftrightarrow \Psi_2$, then ${}_{\Psi_1} ={}_{\Psi_2}$. Suppose that $\Psi_1 = \{a_1, ..., a_n\}$ and $\Psi_2 = \{a'_1, ..., a'_n\}$, and for every $i \in \{1, ..., n\}$, $a_i \leftrightarrow a'_i$, i.e. $\forall \omega \in \Omega$, $d(\omega, \varphi_i) = d(\omega, \varphi'_i)$ and $\alpha_i = \alpha'_i$. So $\forall i$, $d^\bullet_d(\omega, a_i) = d^\bullet_d(\omega, a'_i)$, and $f(\{d^\bullet_d(\omega, a_i) \mid a_i \in \Psi_1\}) = f(\{d^\bullet_d(\omega, a'_i) \mid a'_i \in \Psi_2\})$.
As a consequence, ${}_{\Psi_1} ={}_{\Psi_2}$.

4. We want to show that $\forall \omega \models \mathcal{B}(a)$, $\exists \omega' \models \mathcal{B}(a')$ s.t. $\omega' \preceq^{d^\bullet_{d,f}}_{a \sqcup a'} \omega$. We know that $\forall \omega \models \mathcal{B}(a)$, $\bullet(d(\omega, \mathcal{B}(a)), \delta(a)) = \bullet(0, \delta(a)) = \bullet_0$. Let ω be any model of $\mathcal{B}(a)$. $d(\omega, \mathcal{B}(a') = \min_{\omega' \models \mathcal{B}(a')} d(\omega, \omega')$: let us consider $\omega' \models \mathcal{B}(a')$ such that $d(\omega, \omega') = d(\omega, \mathcal{B}(a'))$. We have: $d^\bullet_{d,f}(\omega, a \sqcup a') = f(\bullet(d(\omega, \mathcal{B}(a)), \delta(a)), \bullet(d(\omega, \mathcal{B}(a')), \delta(a'))) = f(\bullet_0, \bullet(d(\omega, \omega'), \delta(a)))$, because $\delta(a) = \delta(a')$ and $d(\omega, \mathcal{B}(a')) = d(\omega, \omega')$. Note that $d^\bullet_{d,f}(\omega', a \sqcup a') = f(\bullet(d(\omega', \mathcal{B}(a)), \delta(a)), \bullet(d(\omega', \mathcal{B}(a')), \delta(a'))) = f(\bullet(d(\omega', \mathcal{B}(a)), \delta(a)), \bullet_0)$. By symmetry, $f(\bullet(d(\omega', \mathcal{B}(a)), \delta(a)), \bullet_0) = f(\bullet_0, \bullet(d(\omega', \mathcal{B}(a)), \delta(a)))$. As $d(\omega', \mathcal{B}(a)) \leq d(\omega, \omega')$, by Lemma 1.1, $f(\bullet_0, \bullet(d(\omega', \mathcal{B}(a)) \;, \delta(a))) \leq f(\bullet_0, \bullet(d(\omega, \omega'), \delta(a)))$, i.e. $d^\bullet_{d,f}(\omega', a \sqcup a') \leq d^\bullet_{d,f}(\omega, a \sqcup a')$. Then by definition $\omega' \preceq^{d^\bullet_{d,f}}_{a \sqcup a'} \omega$.

5. This property follows straightforwardly from Lemma 1.2.

6. This property follows straightforwardly from Lemma 1.3.

7. We prove property 7' which is equivalent to 7. Thus, suppose $\omega \models \varphi$, $\omega' \not\models \varphi$, $\omega \prec^{d^\bullet_{d,f}}_{\Psi \sqcup (\varphi\ \alpha)} \omega'$ and $\beta > \alpha$. We want to show that $\omega \prec^{d^\bullet_{d,f}}_{\Psi \sqcup (\varphi\ \beta)} \omega'$. By definition we have $f(\{d^\bullet_d(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi\ \alpha)\}) < f(\{d^\bullet_d(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi\ \alpha)\})$. Since $\omega \models \varphi$, we have $f(\{d^\bullet_d(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi\ \alpha)\}) = f(\{d^\bullet_d(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi\ \beta)\})$. Then we have $f(\{d^\bullet_d(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi\ \beta)\} < f(\{d^\bullet_d(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi\ \alpha)\})$ $(\star)$. Note that, as $d(\omega', \varphi) \neq 0$ and $\bullet$ is increasing, $\bullet(d(\omega', \varphi), \alpha) < \bullet(d(\omega', \varphi), \beta)$. Thus, by increasing of f, we have $f(\{d^\bullet_d(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi\ \alpha)\}) < f(\{d^\bullet_d(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi\ \beta)\})$. From this and $(*)$ we have $f(\{d^\bullet_d(\omega, a_i) \mid a_i \in \Psi \sqcup (\varphi\ \beta)\}) < f(\{d^\bullet_d(\omega', a_i) \mid a_i \in \Psi \sqcup (\varphi\ \beta)\})$, that is $\omega \prec^{d^\bullet_{d,f}}_{\Psi \sqcup (\varphi\ \beta)} \omega'$.

8. This is a straightforward consequence of invariance of 0.

9. Suppose that $\omega_{(\varphi\ \alpha)} \omega'$. We want to show that $\omega_{(\varphi\ \beta)} \omega'$ (that is enough by the symmetrical role of α and β). By definition we have $f(d^\bullet_d(\omega, (\varphi\ \alpha))) \leq f(d^\bullet_d(\omega', (\varphi\ \alpha)))$. We can deduce that $d^\bullet_d(\omega, (\varphi\ \alpha)) \leq d^\bullet_d(\omega', (\varphi\ \alpha))$ (elsewhere, by increasing of f we get a contradiction), i.e. $\bullet(d(\omega, \varphi), \alpha) \leq \bullet(d(\omega', \varphi), \alpha)$. We claim that $d(\omega, \varphi) \leq d(\omega', \varphi)$. Suppose towards a contradiction that $d(\omega, \varphi) > d(\omega', \varphi)$. As $\bullet$ is increasing, we obtain $\bullet(d(\omega, \varphi), \alpha) > \bullet(d(\omega', \varphi), \alpha)$. A contradiction. Then, as $\bullet$ is increasing, we have $\bullet(d(\omega, \varphi), \beta) \leq \bullet(d(\omega', \varphi), \beta)$, *i.e.*, $d^\bullet(\omega, (\varphi\ \beta)) \leq d^\bullet(\omega', (\varphi\ \beta))$. Since f is increasing, $f(d^\bullet_d(\omega, (\varphi\ \beta))) \leq f(d^\bullet_d(\omega', (\varphi\ \beta)))$, *i.e.*, $\omega_{(\varphi\ \beta)} \omega'$.

10. Suppose that $\omega \models \varphi$ and $\omega' \not\models \varphi$. We want to show that $\exists \alpha'$, s.t. $\omega \prec_{\Psi \sqcup (\varphi\ \alpha')} \omega'$. We know that $\forall (\alpha_1, ..., \alpha_n), \forall (\beta_1, ..., \beta_n), \forall \beta, \exists \alpha$ s.t $f(\alpha_1, ..., \alpha_n, \alpha) > f(\beta_1, ..., \beta_n, \beta)$. Let $(\beta_i) = (d^\bullet_d(\omega, a_i) \mid a_i \in \Psi)$, $(\alpha_i) = (d^\bullet_d(\omega', a_i) \mid a_i \in \Psi)$, and $\beta = \bullet_0$. Then $\exists \alpha$ s.t. $f(\alpha_1, ..., \alpha_n, \alpha) > f(\beta_1, ..., \beta_n, \beta)$. From assumption we have $d(\omega, \varphi) = 0$ and $d(\omega', \varphi) > 0$. As $d(\omega, \varphi) = 0$, $\forall \alpha$, $d^\bullet_d(\omega, (\varphi\ \alpha)) = \bullet_0$. As $d(\omega', \varphi) > 0$ and $\bullet$ is unbounded, $\exists \alpha'$ s.t. $\bullet(d(\omega', \varphi), \alpha') > \alpha$. For this α', by increasing of f, we obtain $f(\alpha_1, ..., \alpha_n, \bullet(d(\omega', \varphi), \alpha')) > f(\beta_1, ..., \beta_n, \beta)$. That is, $f(\alpha_1, ..., \alpha_n, \bullet(d(\omega', \varphi), \alpha')) > f(\beta_1, ..., \beta_n, d^\bullet_d(\omega, (\varphi\ \alpha')))$. Therefore $\omega \prec_{\Psi \sqcup (\varphi\ \alpha')} \omega'$.

Let us now give some examples of $\bullet$ and check if the instantiations satisfy these properties. Then, we will give some aggregation operators f, leading to the definition of a set of weighted merging operators by the combination of the instantiation of $\bullet$ and f.

The most natural instances of $\bullet$ in order to take weight into account with the distance is by multiplying the weight by the distance, so our first instantiation will be the usual multiplication $\times$. The second is by taking the power by the weight (we need to shift all distances by 1 in this case, in order to ensure that this function will be strictly increasing for any distance greater than 0).

These functions will be defined as follows:

Definition 7. *Let $\times$ and pow be the functions $\times : R^+ \times R^{*+} \to R^+$ and $pow : R^+ \times R^{*+} \to R^+$, defined by :*

$$\begin{aligned} \times(x, y) &= x \times y \\ pow(x, y) &= (x + 1)^y \end{aligned}$$

Proposition 2. *$\times$ and pow are weighted functions.*

As illustrative examples of aggregation functions, we will take Σ and Gmax.

Definition 8. *Let $\Sigma : \bigcup_n R^{+^n} \to (R^+, \leq)$ an aggregation function, such that: $\Sigma(y_1, \ldots, y_n) = \Sigma_{i=1}^n y_i$*

Definition 9. *Let $\mathsf{Gmax} : \bigcup_n R^{+^n} \to (\bigcup_n R^{+^n}, \leq_{lex})$ an aggregation function, such that:*

$$\mathsf{Gmax}(y_1, \ldots, y_n) = (y^{1^{th}}_{\rho(i_1)}, \ldots, y^{n^{th}}_{\rho(i_n)})$$

where ρ is a permutation of $\{1, \ldots, n\}$ such that the $y_{\rho(i)}$ are in decreasing order and $\leq_{lex}$ is the lexicographic order.

It is not difficult to see that these two functions are aggregation functions in the meaning of Definition 6. More precisely, we have the following proposition:

Proposition 3. *Σ and Gmax are aggregation functions, that is, they satisfy all properties of Definition 6.*

As a straightforward consequence of Theorem 2 and Proposition 3 we have the following result:

Proposition 4. *Whatever the pseudo-distance* d *and the weight function* $\bullet$, $\Delta^{d^\bullet_{d,\Sigma}}$ *and* $\Delta^{d^\bullet_{d,\mathsf{Gmax}}}$ *are weighted IC merging operators.*

Actually, we know more about the behavior of these operators. This will be establish in Propositions 5 and 6.

Proposition 5. *Whatever the pseudo-distance* d *and the weight function* $\bullet$, *any* $\Delta^{d^\bullet_{d,\Sigma}}$ *operator is a majority weighted IC merging operator and is not an arbitration weighted IC merging operator.*

Proof. • **Arb** We know from (Konieczny and Pino Pérez 2002) that $\Delta^{d,\Sigma}$ is not an arbitration operator. So $\Delta^{d^\bullet_{d,\Sigma}}$ cannot satisfy **Arb**.

- **Maj** Suppose that $\omega \prec_{\Psi_2} \omega'$. We want to show that $\exists n, \omega \prec_{\Psi_1 \sqcup \Psi_2^n} \omega'$, then we have to show that $d^\bullet_{d,\Sigma}(\omega, \Psi_1 \sqcup \Psi_2^n) < d^\bullet_{d,\Sigma}(\omega', \Psi \sqcup \Psi_2^n)$, which amounts to showing that $d^\bullet_{d,\Sigma}(\omega, \Psi_1) - d^\bullet_{d,\Sigma}(\omega', \Psi_1) < d^\bullet_{d,\Sigma}(\omega', \Psi_2^n) - d^\bullet_{d,\Sigma}(\omega, \Psi_2^n)$. From assumption we deduce that $d^\bullet_{d,\Sigma}(\omega, \Psi_2) < d^\bullet_{d,\Sigma}(\omega', \Psi_2)$, so $d^\bullet_{d,\Sigma}(\omega', \Psi_2) - d^\bullet_{d,\Sigma}(\omega, \Psi_2) > 0(\star)$. We have two cases: $\omega_{\Psi_1}\omega'$ and $\omega \succ_{\Psi_1} \omega'$.
  Case 1: $\omega_{\Psi_1}\omega'$, by condition 6 we deduce that $\exists n = 1, \omega \prec_{\Psi_1 \sqcup \Psi_2} \omega'$.
  Case 2: $\omega \succ_{\Psi_1} \omega'$, then $d^\bullet_{d,\Sigma}(\omega, \Psi_1) > d^\bullet_{d,\Sigma}(\omega', \Psi_1)$, so $d^\bullet_{d,\Sigma}(\omega, \Psi_1) - d^\bullet_{d,\Sigma}(\omega', \Psi_1) > 0$, as the value of $d^\bullet_{d,\Sigma}(\omega, \Psi_1), d^\bullet_{d,\Sigma}(\omega', \Psi_1), d^\bullet_{d,\Sigma}(\omega', \Psi_2)$ and $d^\bullet_{d,\Sigma}(\omega, \Psi_2)$ are fixed, by $(\star)$, we can easily find n such that $d^\bullet_{d,\Sigma}(\omega, \Psi_1) - d^\bullet_{d,\Sigma}(\omega', \Psi_1) < d^\bullet_{d,\Sigma}(\omega', \Psi_2^n) - d^\bullet_{d,\Sigma}(\omega, \Psi_2^n) = n(d^\bullet_{d,\Sigma}(\omega', \Psi_2) - d^\bullet_{d,\Sigma}(\omega, \Psi_2))$. □

Proposition 6. *Whatever the pseudo-distance* d *and the weight function* $\bullet$, *any* $\Delta^{d^\bullet_{d,\mathsf{Gmax}}}$ *operator is an arbitration weighted IC merging operator and is not a majority weighted IC merging operator.*

Proof. • **Maj** We know from (Konieczny and Pino Pérez 2002) that $\Delta^{d,\mathsf{Gmax}}$ is not a majoritarian operator. So $\Delta^{d^\bullet_{d,\mathsf{Gmax}}}$ cannot satisfy **Maj**.

- **Arb** Suppose that $\omega \prec_{a_1} \omega'$, $\omega \prec_{a_2} \omega''$ and $\omega' \simeq_{a_1 \sqcup a_2} \omega''$. We want to show that $\omega \prec_{a_1 \sqcup a_2} \omega'$. By contradiction, suppose that $\omega \geq_{a_1 \sqcup a_2} \omega'$. Then $\mathsf{Gmax}(d^\bullet_d(\omega, a_1), d^\bullet_d(\omega, a_2)) \geq_{lex} \mathsf{Gmax}(d^\bullet_d(\omega', a_1), d^\bullet_d(\omega', a_2))$ $(\star)$. As $\omega \prec_{a_1} \omega'$, we know that $d^\bullet_d(\omega, a_1) < d^\bullet_d(\omega', a_1)$. So, because of $(\star)$, it must be the case that $d^\bullet_d(\omega, a_2) \geq d^\bullet_d(\omega', a_2)$ and $d^\bullet_d(\omega', a_2) \geq d^\bullet_d(\omega', a_1)$. As $\omega' \simeq_{a_1 \sqcup a_2} \omega''$, we know that $d^\bullet_d(\omega'', a_2) = d^\bullet_d(\omega', a_1)$ or $d^\bullet_d(\omega'', a_2) = d^\bullet_d(\omega', a_2)$. So $d^\bullet_d(\omega, a_2) \geq d^\bullet_d(\omega'', a_2)$ and $\omega \succeq_{a_2} \omega''$: contradiction. □

One very interesting point is that not all standard aggregation functions still works in this weighted setting. In particular Gmin operators, that are IC merging operators in the classical case, are not WIC merging operators:

Definition 10. *Let* $\mathsf{Gmin} : \bigcup_n R^{+^n} \to (\bigcup_n R^{+^n}, \leq_{lex})$ *an aggregation function, such that:*

$$\mathsf{Gmin}(y_1, \ldots, y_n) = (y^{1^{th}}_{\rho(i_1)}, \ldots, y^{n^{th}}_{\rho(i_n)})$$

where ρ *is a permutation of* $\{1, \ldots, n\}$ *such that the* $y_{\rho(i)}$ *are in increasing order and* $\leq_{lex}$ *is the lexicographic order.*

Proposition 7. *For all pseudo-distance* d *and all weighted function* $\bullet$, $\Delta^{d^\bullet_{d,\mathsf{Gmin}}}$ *is not a weighted merging operator.*

Proof. Consider $a_1 = (\varphi_1, 1)$, where $[\![\varphi_1]\!] = \{00, 01\}$; $a_2 = (\varphi_2, 2)$, where $[\![\varphi_2]\!] = \{01\}$, suppose $[\![\mu]\!] = \{00, 01\}$. Let d and $\bullet$ be respectively any pseudo-distance and any weighted function. We note $d_1 = d(00, \varphi_2) > 0$.

	a_1	a_2	a_2	$\Delta^{d^\bullet_{d,\mathsf{Gmin}}}$
00	0	d_1	d_1	$(0, \bullet(d_1, 2), \bullet(d_1, 2))$
01	0	0	0	**(0,0,0)**

With $\Psi = \{a_1, a_2, a_2\}$, $[\![\Delta^{d^\bullet_{d,\mathsf{Gmin}}}(\Psi)]\!] = \{01\}$. Consider $[\![\varphi]\!] = \{00\}$, and note $d_2 = d(01, \varphi) > 0$.

	a_1	a_2	a_2	$(\varphi\ \alpha)$	$\Delta^{d^\bullet_{d,\mathsf{Gmin}}}$
00	0	d_1	d_1	0	$(0, 0, \bullet(d_1, 2), \bullet(d_1, 2))$
01	0	0	0	d_2	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \bullet(\mathbf{d_2}, \alpha))$

Whatever the weight α associated with φ, we obtain $[\![\Delta^{d^\bullet_{d,\mathsf{Gmin}}}(\Psi \sqcup (\varphi, \alpha))]\!] = \{01\}$. So it is not the case that $\Delta^{d^\bullet_{d,\mathsf{Gmin}}}(\Psi \sqcup (\varphi, \alpha)) \vdash \alpha$ and **(WIC12)** is not satisfied. □

When looking to postulates **(WICP12)** and **(Maj)** one can see that they look similar, but they encode truly different behaviours:

Proposition 8. *The postulates* **(WICP12)** *and* **(Maj)** *are independent.*

Proof. $\Delta^{d^\bullet_{d,\mathsf{Gmin}}}$ is a majoritarian merging operator which does not satisfy **(WICP12)**. $\Delta^{d^\bullet_{d,\mathsf{Gmax}}}$ satisfies **(WICP12)** and is not a majoritarian merging operator. Finally, $\Delta^{d^\bullet_{d,\Sigma}}$ is a majoritarian merging operator and satisfies **(WICP12)**. □

5 Examples

In this section we will provide some examples in order to illustrate the impact of the weights on the behaviour of the operators.

Example 1. *Four doctors* (a_1, a_2, a_3 *and* a_4) *are unable to come to an agreement on which of the candidates should be recruited as resident doctors based on their evaluation of their respective skills. They have asked the director of the hospital for assistance in making their decision. The doctors will make their decisions based on three criteria* C_1, C_2 *and* C_3 *(respectively. mobility, experience and perfect academic background).*

Doctor a_1 *believes that the best candidate is one who meets only the third criterion* (C_3)*, while Doctors* a_2 *and* a_3 *believe that the best choice is someone who satisfies either the first or second criterion. However, Doctor* a_4 *believes that the candidate should meet all criteria.*

The director of the hospital assigns weights to each doctor based on an his estimation of the reliability of the doctors for this task, so he assigns a weight of 1 *to Doctor* a_1, 4

	$(\varphi_1\ 1)$	$(\varphi_2\ 4)$	$(\varphi_3\ 5)$	$(\varphi_4\ 6)$	$\Delta^{d^{\times}_{d_H},\Sigma}$	$\Delta^{d^{pow}_{d_H},\Sigma}$	$\Delta^{d^{\times}_{d_H},\mathsf{Gmax}}$
000	1	1	1	3	28	4146	(18, 5,4,1)
001	0	2	2	2	30	1054	(12,10,8,0)
010	2	0	0	2	**14**	734	(12, 2,0,0)
011	1	1	1	1	16	**114**	**(6, 5,4,1)**
100	2	0	0	2	**14**	734	(12, 2,0,0)
101	1	1	1	1	16	**114**	**(6, 5,4,1)**
110	3	1	1	1	18	116	(6, 5,4,3)
111	2	2	2	0	20	328	(10, 8,2,0)

Table 1: The Behaviour of WIC Operators

to Doctor a_2, 5 to Doctor a_3 and 6 to Doctor a_4. So we have $(\varphi_i\ \alpha_i)$ for each agent a_i, s.t. $[\![\varphi_1]\!] = \{001\}$, α_1 = 1; $[\![\varphi_2]\!] = \{010, 100\}$, α_2 = 4; $[\![\varphi_3]\!] = \{010, 100\}$, α_3 = 5; $[\![\varphi_4]\!] = \{111\}$, α_4 = 6.

Table 1 summarizes the computations and allow comparisons between the operators. We see that with the same weight function $\times$, the $\Delta^{d^{\times}_{d_H},\Sigma}$ operator and the $\Delta^{d^{\times}_{d_H},\mathsf{Gmax}}$ operators provide distinct results, the $\Delta^{d^{\times}_{d_H},\Sigma}$ operator, with a majority merging behavior, choose the best option for a_2 and a_3, whereas $\Delta^{d^{\times}_{d_H},\mathsf{Gmax}}$ tends towards a more consensual result, as expected for an arbitration operator. We see also that changing the weight function have an impact on the result since when we compare $\Delta^{d^{\times}_{d_H},\Sigma}$ and $\Delta^{d^{pow}_{d_H},\Sigma}$ we see that the results are different, since the *pow* function gives a higher importance to the weights, so preventing the results to be too far from agents with high weights.

Let us now look more specifically to Gmax operators, to see how varying the weight of agents affects the behavior of these arbitration operators.

Example 2. *Let's take $[\![\varphi_1]\!] = [\![\varphi_2]\!] = [\![\varphi_3]\!] = \{000\}$ and $[\![\varphi_4]\!] = \{111\}$ and let's take $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 3$. Computations for $\Delta^{d^{\times}_{d_H},\mathsf{Gmax}}$ are given in Table 2.*

	$(\varphi_1\ 1)$	$(\varphi_2\ 1)$	$(\varphi_3\ 1)$	$(\varphi_4\ 3)$	$\Delta^{d^{\times}_{d_H},\mathsf{Gmax}}$
000	0	0	0	3	(9,0,0,0)
001	1	1	1	2	(6,1,1,1)
010	1	1	1	2	(6,1,1,1)
011	2	2	2	1	**(3,2,2,2)**
100	1	1	1	2	(6,1,1,1)
101	2	2	2	1	**(3,2,2,2)**
110	2	2	2	1	**(3,2,2,2)**
111	3	3	3	0	(3,3,3,0)

Table 2: $\Delta^{d^{\times}_{d_H},\mathsf{Gmax}}$

We can see the impact of the weight versus the number of repetitions for Gmax and $\times$, more exactly we want to compare the effect of applying a weight of 1 to a formula φ_1 repeated n times versus applying a weight of n to the formula φ_4. None of the models of φ_1 and φ_4 are present in the result of merging, but the result is closer to φ_4 than to φ_1, even when φ_1 is repeated more times. And if we increase the weight associated to φ_4 from 3 to 4 (the value can vary depending on the number of propositional variables and the number of model bases), then the result will imply φ_4 (as ensured by (WIC12) there is always a weight where this happens).

6 Conclusion

In this paper we proposed a characterization of weighted propositional belief merging operators, i.e. operators that merge the beliefs (or goals) of several sources expressed in propositional logic when some sources are more important/reliable/expert than others. Some previous works in the literature already defined instances of such operators, but this is, as far as we know, the first time that a family of such operators is logically characterized.

Note that there are also in the literature other related works that use weights for the merging (Lin 1996; Delgrande, Dubois, and Lang 2006), or where the information is intrinsically prioritized, as for instance the works on merging of possibilistic merging bases (Benferhat et al. 1999; Benferhat and Kaci 2003; Qi, Liu, and Bell 2006; Qi et al. 2006; Benferhat, Lagrue, and Rossit 2009; Benferhat, Lagrue, and Rossit 2014). But none of these works proposes a representation theorem. And, even if one can try to do some technical translation between these settings, it is intrinsically different to weight propositional belief bases to reflect either their importance, reliability, expertise, than to merge a set of bases of equal weights with quantified uncertainty expressed by a weighted logic. So the most general setting would be to allow to do a weighted merging of weighted logic. Nevertheless, in this work, we stick to the basic propositional logic setting: we study the merging of weighted bases expressed in propositional logic.

A lot remains to be done, in particular defining interesting sub-classes of operators, where for instance the weight function is more constrained. An interesting future work is a more systematic study of weighted arbitration operators, since these notions seem quite antinomic: arbitration operators are a translation in merging of egalitarist aggregation approaches, where the aim is to satisfy (as much as possible) everyone, whereas weights' aim is to distort the process in favor of higher weights. Exploring this kind of "egalitarist with weights" approaches could be useful not only for belief merging, but for several social choice problems.

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