Belief Base Rationalization for Propositional Merging *

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Abstract

Existing belief merging operators take advantage of all the models from the bases, including those contradicting the integrity constraints. In this paper, we show that this is not suited to every merging scenario. We study the case when the bases are "rationalized" with respect to the integrity constraints during the merging process. We define in formal terms several independence conditions for merging operators and show how they interact with the standard IC postulates for belief merging. Especially, we give an independence-based axiomatic characterization of a distance-based operator.

1 Introduction

Belief merging operators [Konieczny and Pino Pérez, 2011; Revesz, 1997; Lin, 1996; Liberatore and Schaerf, 1998; Konieczny and Pino Pérez, 2002; Konieczny *et al.*, 2004] aim at defining a base which represents the beliefs of a group of agents given their individual belief bases. Integrity constraints, representing physical laws or norms, are often used in the merging process. There is usually more than a single way to merge a profile of belief bases given some integrity constraints. The rational way to do it is characterized by a set of rationality postulates, the IC postulates [Konieczny and Pino Pérez, 2002], that merging operators should satisfy. Such operators are called IC merging operators.

Existing IC merging operators take advantage of all the models from the bases, including those contradicting the integrity constraints. However, this is not suited to every merging scenario. Especially, when the integrity constraints encode knowledge about the world as physical laws, the exploitation in the merging process of "incorrect" models (i.e., conflicting with the constraints) can be questioned.

For instance, Condotta and al. [2009] recently proposed a framework for merging qualitative spatial or temporal information expressed in propositional logic. Integrity constraints are used for encoding the spatial or temporal laws. Thus, "unfeasible" models (such as a set of three instants t_1, t_2, t_3 of the totally ordered time line $\mathcal{T}$, together with the constraints $t_1 <_\mathcal{T} t_2 <_\mathcal{T} t_3 <_\mathcal{T} t_1$) must be discarded in such a way that they have no impact on the resulting spatial or temporal base.

In order to illustrate the problem on a simple example, let us consider the following situation. There is a room with a bulb and two switches. The bulb is lit when the two switches are in the same position (either both "on" or both "off"), and only in this case. The bulb is currently lit. The switches status are unknown, and the available information about them are contradictory: one source of information states that the first switch is "off" while a second source of information states that both switches are "on". What can be deduced from this? The answer depends on the merging operator under consideration. Considering the IC merging operator based on the Hamming distance and sum as aggregation function ($\Delta^{d_H,\Sigma}$); for this operator, the models of the merged base representing the beliefs of the group are the models of the integrity constraints which are as close as possible to the profile consisting of the two sources of information, where the distance between two models is evaluated as the number of atomic facts on which they differ. There are two possible worlds compatible with the integrity constraints, 00 (both switches are "off") and 11 (both switches are "on".) 00 is at distance 0 from the first source (this world is a model of the corresponding belief base) and at distance 2 from the second one. 11 is at distance 1 from the first source (since only the fact that the second switch is "on" conflicts with the information conveyed by this source) and at distance 0 from the second source. Hence, using sum as an aggregation function, we get that only the second model is kept so that the beliefs of the group is that both switches are "on".

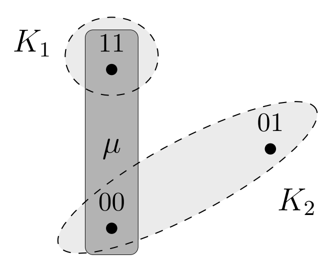

Figure 1: Graphical representation of the integrity constraints μ and the two sources (K_1 and K_2.)

But 11 is at distance 1 from the first source only because

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it is at distance 1 of the model 01 of this source, whereas this model does not correspond to a feasible world given the integrity constraint! Thus it makes sense to disqualify this world. If we do it so that the first source now states that both sources are "off" (the sole possibility compatible with the beliefs from the first source and the integrity constraints) then the merged base obtained using the same belief merging operator states that either both switches are "on" or both switches are "off". This is a more satisfactory result here since there is no reason to give more credit to 11 than to 00 when one source finally states 00 and the other one states 11.

This simple example shows that some IC merging operators allow "impossible worlds" (i.e., those not satisfying the integrity constraints) to play a role in the merging process, namely to have an impact on the resulting merged base. Addressing the cases when this is unexpected calls for a new property for merging operators, which is intended to require them to lead to the same merged base if every input belief base has been "rationalized" with respect to the integrity constraints.

In the following we define three independence postulates corresponding to some rationalization principles, and we study how they interact with the standard IC operators for belief merging.

The rest of the paper is organized as follows. In Section 2 we give some notations and recall some definitions and results concerning IC merging operators. In Section 3 we define some independence properties in formal terms and present the independence results we have obtained. In Section 4 we give an independence-based axiomatic characterization of the distance-based operator based on the drastic distance and Σ as aggregation function ($\Delta^{d_D,\Sigma}$.) Finally, we conclude in Section 5. A technical report with proof sketches is available at http://www.cril.fr/∼marquis/bbrpm.pdf.

2 Formal Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the usual connectives. $\bot$ (resp. $\top$) is the Boolean constant always false (resp. true.)

An interpretation (or world) is a total function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is denoted $\mathcal{W}$. An interpretation I is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $mod(\phi)$ denotes the set of models of formula ϕ, i.e., $mod(\phi) = \{I \in \mathcal{W} \mid I \models \phi\}$. Let M be a set of interpretations; φ_M denotes a formula from $\mathcal{L}$ whose models are M.

The *integrity constraints* μ are represented by a consistent formula. A *base* K denotes the set of beliefs of an agent, it is a finite set of propositional formulae, interpreted conjunctively, so that K is identified with the conjunction of its elements. A *profile* $\overline{\mathcal{K}} = \langle 1, \ldots, n\rangle$ is a vector of agents involved in the merging process. A *belief profile* $\mathcal{K} = \langle K_1, \ldots, K_n\rangle$ is a vector of bases, each base K_i representing the beliefs of agent i. When it is harmless, one usually does not distinguish the notions of profile and belief profile, i.e., each base is identified with the agent providing it, and the term "profile" is used as a short for "belief profile". In the non-prioritized framework studied in this paper, agents are expected to play equivalent roles in the merging process, so that a profile $\mathcal{K} = \langle K_1, \ldots, K_n\rangle$ is also viewed as a multi-set $\{K_1, \ldots, K_n\}$. A profile is said to be p-consistent if all the bases of the profiles are consistent (this is a standard assumption but it is not made everywhere in this paper.) $\sqcup$ denotes the union on multi-sets and $\equiv$ the equivalence of profiles (two belief profiles are equivalent when there is a bijection between them so that each base from a profile is equivalent to its image in the other profile.) $\mathcal{K}^n$ denotes the multi-set where $\mathcal{K}$ appears n times, i.e., $\mathcal{K}^n = \underbrace{\mathcal{K} \sqcup \ldots \sqcup \mathcal{K}}_{n}$. $\bigwedge \mathcal{K}$ denotes the conjunction of the belief bases of $\mathcal{K}$, i.e., $\bigwedge \mathcal{K} = \bigwedge\{K_i \mid K_i \in \mathcal{K}\}$. Lastly, the notation $I \models \mathcal{K}$ stands for $I \models \bigwedge \mathcal{K}$.

Example 1. *Let us formalize the example drafted in the introduction. We have $\mathcal{P} = \{on\_s_1, on\_s_2\}$ (when on_s_1 (resp. on_s_2) stands for "the first switch (resp. the second one) is on"); $\mathcal{K} = \langle K_1, K_2\rangle$, with $K_1 = \{\neg on\_s_1\}$, $K_2 = \{on\_s_1 \wedge on\_s_2\}$; $\mu = on\_s_1 \leftrightarrow on\_s_2$.*

A merging operator Δ is a mapping associating a formula μ and a profile $\mathcal{K}$ with a new base $\Delta_\mu(\mathcal{K})$. Let us recall the standard logical properties which are expected for belief merging operators [Konieczny and Pino Pérez, 2002]:

Definition 1 (IC merging operator). *A merging operator Δ is an* IC merging operator *iff for every formula μ, μ_1, μ_2, for every p-consistent profile $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and for every consistent belief base K_1, K_2, it satisfies the following postulates:*

(IC0) $\Delta_\mu(\mathcal{K}) \models \mu$;

(IC1) *If μ is consistent, then $\Delta_\mu(\mathcal{K})$ is consistent;*

(IC2) *If $\bigwedge \mathcal{K} \wedge \mu$ is consistent, then $\Delta_\mu(\mathcal{K}) \equiv \bigwedge \mathcal{K} \wedge \mu$;*

(IC3) *If $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(\mathcal{K}_1) \equiv \Delta_{\mu_2}(\mathcal{K}_2)$;*

(IC4) *If $K_1 \models \mu$, $K_2 \models \mu$ and $\Delta_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent, then $\Delta_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent;*

(IC5) $\Delta_\mu(\mathcal{K}_1) \wedge \Delta_\mu(\mathcal{K}_2) \models \Delta_\mu(\mathcal{K}_1 \sqcup \mathcal{K}_2)$;

(IC6) *If $\Delta_\mu(\mathcal{K}_1) \wedge \Delta_\mu(\mathcal{K}_2)$ is consistent, then $\Delta_\mu(\mathcal{K}_1 \sqcup \mathcal{K}_2) \models \Delta_\mu(\mathcal{K}_1) \wedge \Delta_\mu(\mathcal{K}_2)$;*

(IC7) $\Delta_{\mu_1}(\mathcal{K}) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(\mathcal{K})$;

(IC8) *If $\Delta_{\mu_1}(\mathcal{K}) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(\mathcal{K}) \models \Delta_{\mu_1}(\mathcal{K}) \wedge \mu_2$.*

Let us stress that IC merging operators consider that an inconsistent belief base provides no information for the merging process [Konieczny and Pino Pérez, 2002], so they suppose that the input profiles are p-consistent.

Each IC merging operator corresponds to a syncretic assignment [Konieczny and Pino Pérez, 2002]:

Definition 2 (Syncretic assignment). *A* syncretic assignment *is a mapping which associates with every p-consistent profile $\mathcal{K}$ a preorder $\leq_\mathcal{K}$ over worlds*[1] *and such that for every p-consistent profile $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and for every consistent belief base K_1, K_2, $\leq_\mathcal{K}$ satisfies the following conditions:*

[1]For each preorder $\leq_\mathcal{K}$, $\simeq_\mathcal{K}$ denotes the corresponding indifference relation and $<_\mathcal{K}$ the corresponding strict ordering. When $\mathcal{K} = \langle K\rangle$ consists of a single base K, we write $\leq_K$ instead of $\leq_{\langle K\rangle}$ in order to alleviate the notations.

(1) *If* $I \models \mathcal{K}$ *and* $J \models \mathcal{K}$*, then* $I \simeq_{\mathcal{K}} J$*;*

(2) *If* $I \models \mathcal{K}$ *and* $J \not\models \mathcal{K}$*, then* $I <_{\mathcal{K}} J$*;*

(3) *If* $\mathcal{K}_1 \equiv \mathcal{K}_2$*, then* $\leq_{\mathcal{K}_1} = \leq_{\mathcal{K}_2}$*;*

(4) *If* $\forall I \models K_1$*, then* $\exists J \models K_2$ $J \leq_{\langle K_1, K_2 \rangle} I$*;*

(5) *If* $I \leq_{\mathcal{K}_1} J$ *and* $I \leq_{\mathcal{K}_2} J$*, then* $I \leq_{\mathcal{K}_1 \sqcup \mathcal{K}_2} J$*;*

(6) *If* $I <_{\mathcal{K}_1} J$ *and* $I \leq_{\mathcal{K}_2} J$*, then* $I <_{\mathcal{K}_1 \sqcup \mathcal{K}_2} J$*.*

Theorem 1 ([Konieczny and Pino Pérez, 2002])**.** *A merging operator* Δ *is an IC merging operator iff there exists a syncretic assignment associating every p-consistent profile* $\mathcal{K}$ *with a total preorder* $\leq_{\mathcal{K}}$ *such that for every formula* μ*,* $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$*.*

Several families of IC merging operators can be defined, including distance-based merging operators, i.e., those operators characterized by a distance between worlds and an aggregation function f (a mapping which associates with a tuple of non-negative real numbers a non-negative real number) [Konieczny *et al.*, 2004]:

Definition 3 (**Distance-based merging operator**)**.** *Let* d *be a distance between worlds and* f *be an aggregation function. The merging operator* $\Delta^{d,f}$ *is defined for every profile* $\mathcal{K}$ *and every formula* μ *by* $mod(\Delta_\mu^{d,f}(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$ *where the preorder* $\leq_{\mathcal{K}}$ *over worlds induced by* $\mathcal{K}$ *is defined by:*

- $I \leq_{\mathcal{K}} J$ *if and only if* $d(I, \mathcal{K}) \leq d(J, \mathcal{K})$*,*
- $d(I, \mathcal{K}) = f_{K \in \mathcal{K}}(d(I, K))$*,*
- $d(I, K) = \begin{cases} min_{J \models K} d(I, J) & \text{If } K \text{ is consistent}^2, \\ 0 & \text{otherwise.} \end{cases}$

Usual distances are d_D, the drastic distance ($d_D(I, J) = 0$ if $I = J$ and 1 otherwise), and d_H the Hamming distance ($d_D(I, J) = n$ if I and J differ on n variables.) Note that some distance-based operators are not IC merging ones (some conditions must be satisfied by f, see [Konieczny *et al.*, 2004]), but taking advantage of usual aggregation functions as Σ, $Gmax$ ($leximax$), etc. leads to IC merging operators.

Belief merging operators are related to belief revision operators [Alchourrón *et al.*, 1985], defined as follows [Katsuno and Mendelzon, 1991b]:

Definition 4 (**AGM revision operator**)**.** *An AGM revision operator* $\circ$ *is a mapping associating a formula* μ *and a base* K *with a new base* $K \circ \mu$*, such that for every formula* μ, μ_1, μ_2*, for every consistent base* K, K_1, K_2*, it satisfies the following postulates:*

(R1) $K \circ \mu \models \mu$*;*

(R2) *If* $K \wedge \mu$ *is consistent, then* $K \circ \mu \equiv K \wedge \mu$*;*

(R3) *If* μ *is consistent, then* $K \circ \mu$ *is consistent;*

(R4) *If* $K_1 \equiv K_2$ *and* $\mu_1 \equiv \mu_2$*, then* $K_1 \circ \mu_1 \equiv K_2 \circ \mu_2$*;*

(R5) $(K \circ \mu_1) \wedge \mu_2 \models K \circ (\mu_1 \wedge \mu_2)$*;*

(R6) *If* $(K \circ \mu_1) \wedge \mu_2$ *is consistent, then* $K \circ (\mu_1 \wedge \mu_2) \models (K \circ \mu_1) \wedge \mu_2$*.*

Now, if Δ is an IC merging operator, then one can associate with it an AGM revision operator $\circ_\Delta$:

Definition 5 ($\circ_\Delta$)**.** *Let* Δ *be a merging operator. Its corresponding revision operator, denoted* $\circ_\Delta$*, is given by*

$$K \circ_\Delta \mu = \Delta_\mu(\langle K \rangle).$$

Theorem 2 ([Konieczny and Pino Pérez, 2002])**.** *If* Δ *is an IC merging operator (i.e., it satisfies (IC0-IC8)), then* $\circ_\Delta$ *is an AGM revision operator (i.e., it satisfies (R1-R6) [Katsuno and Mendelzon, 1991b].)*

Another family of change operators that is considered in this paper consists of belief update operators [Katsuno and Mendelzon, 1991a]. Update operators perform a model-wise change, whereas belief revision operators make a change at the base level (see [Katsuno and Mendelzon, 1991a] for a discussion.) Such operators are defined as follows:

Definition 6 (**KM update operator**)**.** *An KM update operator* $\diamond$ *is a mapping associating a formula* μ *and a base* K *with a new base* $K \diamond \mu$*, such that for every formula* μ, μ_1, μ_2*, for every consistent base* K, K_1, K_2*, it satisfies the following postulates:*

(U1) $K \diamond \mu \models \mu$*;*

(U2) *If* $K \models \mu$*, then* $K \diamond \mu \equiv K$*;*

(U3) *If* K *is consistent and* μ *is consistent, then* $K \diamond \mu$ *is consistent;*

(U4) *If* $K_1 \equiv K_2$ *and* $\mu_1 \equiv \mu_2$*, then* $K_1 \diamond \mu_1 \equiv K_2 \diamond \mu_2$*;*

(U5) $(K \diamond \mu_1) \wedge \mu_2 \models K \diamond (\mu_1 \wedge \mu_2)$*;*

(U6) *If* $(K \diamond \mu_1) \models \mu_2$ *and* $(K \diamond \mu_2) \models \mu_1$*, then* $K \diamond \mu_1 \equiv K \diamond \mu_2$*;*

(U7) *If* K *is a complete base, then* $(K \diamond \mu_1) \wedge (K \diamond \mu_2) \models K \diamond (\mu_1 \vee \mu_2)$*;*

(U8) $(K_1 \vee K_2) \diamond \mu \equiv (K_1 \diamond \mu) \vee (K_2 \diamond \mu)$*;*

(U9) *If* K *is a complete base and* $(K \diamond \mu_1) \wedge \mu_2$ *is consistent, then* $K \diamond (\mu_1 \wedge \mu_2) \models (K \diamond \mu_1) \wedge \mu_2$*.*

3 Independence to Rationalization

Rationalizing a belief base with respect to some integrity constraints consists in modifying it to fit the conceivable worlds according to the integrity constraints. We start with rationalization by expansion and first present the corresponding independence postulate:

Definition 7. ***(Ind)*** *A merging operator* Δ *satisfies (Ind) iff for every formula* μ *and for every profile* $\langle K_1, \ldots, K_n \rangle$*,* $\Delta_\mu(\langle K_1, \ldots, K_n \rangle) \equiv \Delta_\mu(\langle K_1 \wedge \mu, \ldots, K_n \wedge \mu \rangle)$*.*

This postulate states that merging a profile should lead to the same merged base as the one obtained by first removing every model not satisfying the integrity constraints from every base (i.e., expanding every base with the integrity constraints.)

Example 1 shows that some IC merging operators do not satisfy this postulate, (e.g., $\Delta^{d_H, \Sigma}$) :

[2] Usually, distance-based merging operators are applied to p-consistent profiles so K is always consistent. Here, when K is not consistent, we set $d(I, K) = 0$.

Example 1 (continued). $\Delta^{d_H,\Sigma}$ *does not satisfy (Ind). Indeed,* $\Delta^{d_H,\Sigma}_{\mu}(\langle K_1, K_2\rangle) \not\equiv \Delta^{d_H,\Sigma}_{\mu}(\langle K_1 \wedge \mu, K_2 \wedge \mu\rangle)$ *:*

- $\Delta^{d_H,\Sigma}_{\mu}(\langle K_1, K_2\rangle) \equiv on\_s_1 \wedge on\_s_2$*;*
- $\Delta^{d_H,\Sigma}_{\mu}(\langle K_1 \wedge \mu, K_2 \wedge \mu\rangle) \equiv on\_s_1 \leftrightarrow on\_s_2$.

(Ind) is close to the independence of irrelevant alternatives condition (IIA) in social choice theory [Arrow, 1963; Arrow *et al.*, 2002] for aggregation of preference relations. (IIA) states that the (aggregated) preference between two alternatives depends only on the preferences of the individuals on these two alternatives, and not on the preferences with respect to other alternatives. For voting rules (IIA) can be expressed as the fact that two preference profiles which coincide when projected onto a given agenda should always lead to the same winner [Kelly, 1978]. In our belief merging setting, the set of models of the integrity constraints plays the role of the agenda for voting. Accordingly, an (IIA) condition in a belief merging setting can be stated as:

Definition 8. ***(IIA)*** *A merging operator Δ satisfies (IIA) iff for every formula μ and for every profile $\langle K_1, \ldots, K_n\rangle$, $\langle K'_1, \ldots, K'_n\rangle$:*
if $\langle K_1 \wedge \mu, \ldots, K_n \wedge \mu\rangle \equiv \langle K'_1 \wedge \mu, \ldots, K'_n \wedge \mu\rangle$,
then $\Delta_\mu(\langle K_1, \ldots, K_n\rangle) \equiv \Delta_\mu(\langle K'_1, \ldots, K'_n\rangle)$.

Clearly (IIA) is "almost equivalent" to our (Ind) condition. To be more precise:

Proposition 1. *Δ satisfies (IIA) iff Δ satisfies (Ind) and (IC3).*

Now, whenever a belief base is inconsistent with the integrity constraints, its rationalization by expansion leads to an inconsistent base, which is problematic since the IC postulates assume p-consistent belief profiles. In order to fix this problem, we slightly extend the IC postulates:

Definition 9 (EIC merging operator). *A merging operator Δ is an* E(xtended) IC merging operator *iff for every formula μ, for every profile*[3] *$\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and for every consistent belief base K_1, K_2, it satisfies (IC0) - (IC8) and the following additional postulate, for every $n > 0$:*

(Inc) $\Delta_\mu(\langle\bot\rangle^n) \equiv \mu$.

According to (Inc), merging "trivial" profiles consisting of inconsistent bases must lead to merged bases equivalent to the constraints themselves. This postulate, which is not very demanding, echoes what is achieved by IC merging operators when dealing with trivial profiles consisting only of logically valid bases (indeed, (IC2) ensures that the merged base for such profiles is also equivalent to the constraints.) It is easy to show that (Inc) is satisfied by all distance-based merging operators.

(Inc) as given in Definition 9 is given in a canonical form, in the sense that it tells how a merging operator should behave when merging "trivial" profiles, but it does not explicitly say anything when only some bases of the input profile are inconsistent. Nevertheless, merging operators satisfying (Inc) together with some (IC) postulates (in particular, EIC merging operators applied to any profile) lead to a merged base equivalent to the one obtained by first removing inconsistent bases from the profile. This is formally stated in the following proposition:

[3] Note that, in contrast to the definition of usual IC merging operators (Definition 1), p-consistency of the profiles is not required here.

Proposition 2. *Let Δ be a merging operator satisfying (Inc), (IC0), (IC1), (IC5) and (IC6). Then for every p-consistent profile $\mathcal{K}$, for every formula μ and for every $m > 0$,*

$$\Delta_\mu(\mathcal{K} \sqcup \langle\bot\rangle^m) \equiv \Delta_\mu(\mathcal{K}).$$

Now, in order to derive a representation theorem for EIC operators, one needs the following assignments:

Definition 10 (Extended syncretic assignment). *An* extended syncretic assignment *is a mapping which associates with every profile $\mathcal{K}$ a preorder $\leq_{\mathcal{K}}$ over worlds which satisfies conditions (1) - (6) (cf. Definition 2) and the following additional condition, for every $n > 0$:*

(0) $I \simeq_{\langle\bot\rangle^n} J$.

Then the standard representation theorem for IC merging operators can be extended to EIC operators:

Proposition 3. *A merging operator Δ is an EIC merging operator iff there exists an extended syncretic assignment associating every profile $\mathcal{K}$ with a total preorder $\leq_{\mathcal{K}}$ such that for every formula μ, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$.*

It is easy to verify that a distance-based merging operator which is an IC merging operator is also an EIC merging operator.

Let us go back to the independence issue. Imposing (Ind) for a merging operator could be considered as too demanding since the corresponding rationalization process is rather drastic. Indeed, according to Proposition 2 when merging a profile containing a belief base such that no model of it satisfies the integrity constraints, one can simply remove this base from the profile as an upstream step of the merging process. A more cautious behaviour would be to consider as still relevant the models of every base of the input profile, even when the base is inconsistent with the integrity constraints μ: instead of removing such bases from the profiles, one could "repair" them. For this purpose, one can take advantage of belief change operators in order to derive, for each base inconsistent with μ, the closest base that is fully compatible with μ.

Two kinds of belief change operators appear as valuable candidates in this objective: revision operators [Alchourrón *et al.*, 1985; Katsuno and Mendelzon, 1991b] if one wants to "repair" the bases globally and update operators [Katsuno and Mendelzon, 1991a] if one wants to "repair" the bases locally, in a model-wise fashion. The corresponding independence properties are as follows:[4]

Definition 11. ***(Ind-$\circ$)*** *A merging operator Δ satisfies (Ind-$\circ$) iff one can associate with each agent i an AGM revision operator $\circ_i$ [Katsuno and Mendelzon, 1991b] such that for every formula μ and for every profile $\langle K_1, \ldots, K_n\rangle$,*
$\Delta_\mu(\langle K_1, \ldots, K_n\rangle) \equiv \Delta_\mu(\langle K_1 \circ_1 \mu, \ldots, K_n \circ_n \mu\rangle)$.

[4] Let us recall that each base of a profile corresponds to the beliefs of an agent.

Definition 12. ***(Ind-$\diamond$)*** *A merging operator Δ satisfies (Ind-$\diamond$) iff one can associate with each agent i a KM update operator $\diamond_i$ [Katsuno and Mendelzon, 1991a] such that for every formula μ and for every profile $\langle K_1, \dots, K_n \rangle$,*
$\Delta_\mu(\langle K_1, \dots, K_n \rangle) \equiv \Delta_\mu(\langle K_1 \diamond_1 \mu, \dots, K_n \diamond_n \mu \rangle)$.

Note that in Definitions (Ind-$\circ$) and (Ind-$\diamond$), we do not impose any connection between the *"rationalizing operators"* $\circ_i$ or $\diamond_i$ and the merging operator Δ under consideration. In addition, we do not impose any homogeneity condition, i.e., the agents providing the bases can have different revision/update policies. Hence, these two independence properties are rather permissive.

In spite of it, it turns out that there is no EIC merging operator satisfying (Ind-$\diamond$). Indeed, Proposition 4 shows that for EIC merging operators, the property of independence to rationalization by update is not compatible with the most basic IC properties.

Proposition 4. *There is no merging operator satisfying (IC0), (IC1), (IC2) and (Ind-$\diamond$).*

On the other hand, one can prove that (Ind) and (Ind-$\circ$) are compatible with all IC properties:

Proposition 5. *For any aggregation function f, $\Delta^{d_D,f}$ satisfies (Ind) and (Ind-$\circ$).*

Indeed, since $\Delta^{d_D,\Sigma}$ is an IC merging operator [Konieczny and Pino Pérez, 2002] and since it satisfies (Inc), we get:

Corollary 1. *$\Delta^{d_D,\Sigma}$ is an EIC merging operator satisfying (Ind) and (Ind-$\circ$).*

An interesting issue concerns the set of admissible rationalizing revision operators to be chosen so that (Ind-$\circ$) holds. Actually, this choice is very constrained:

Proposition 6. *Let Δ be an EIC merging operator satisfying (Ind-$\circ$). Then every rationalizing revision operator $\circ_i$ considered in (Ind-$\circ$) is the revision operator $\circ_\Delta$ corresponding to Δ in the sense of Definition 5. Moreover, $\circ_\Delta$ is $\circ^D$, the drastic revision operator, defined as*

$$K \circ^D \mu = \begin{cases} K \wedge \mu & \textit{if } K \wedge \mu \textit{ is consistent,} \\ \mu & \textit{otherwise.} \end{cases}$$

As a noticeable corollary to this proposition, we have that:

Corollary 2. *Let Δ be an EIC merging operator. Δ satisfies (Ind) if and only if Δ satisfies (Ind-$\circ$).*

This last result shows that for EIC merging operators the two notions of independence (Ind) and (Ind-$\circ$) coincide. For this reason, we only focus on (Ind) in the rest of the paper.

4 A Characterization Result

Corollary 1 states that the set of EIC merging operators satisfying (Ind) is not empty, by showing that $\Delta^{d_D,\Sigma}$ belongs to it.

A key question is to determine what are exactly the IC merging operators (not necessarily distance-based ones) satisfying (Ind). In the following, we give a representation theorem which answers this question.

Definition 13 (Filtering assignment)**.** *A* filtering assignment *is an extended syncretic assignment satisfying the following condition, for every belief base K_1, K_2:*

(F) *If $I <_{K_1} J$ and $J <_{K_2} I$, then $I \simeq_{\langle K_1, K_2 \rangle} J$.*

Condition (F) states that if a world is viewed as strictly more plausible than another world for a singleton profile, and the plausibility ordering is reversed for another singleton profile, then these worlds must be considered equally plausible with respect to the joint profiles. Stated otherwise, when condition (F) holds together with conditions (1) and (2) (cf. Definition 2), it is sufficient to compare the plausibility of two distincts worlds I, J with respect to two singleton profiles $\langle K_1 \rangle, \langle K_2 \rangle$ independently in order to determine the relative-plausibility of I and J with respect to the doubleton profile $\langle K_1, K_2 \rangle$.

Observe that condition (F) can be viewed as a stronger version of condition (4) (cf. Definition 2) in the presence of conditions (1) and (2):

Proposition 7. *Every assignment satisfying conditions (1), (2) and (F) satisfies condition (4).*

Indeed, the additional constraint expressed by condition (F) with respect to condition (4) can be illustrated as follows. Consider three pairwise distinct models I, J, L and two belief bases $K_1 \equiv \varphi_{\{I,J\}}$ and $K_2 \equiv \varphi_{\{I,L\}}$. In the presence of conditions (1) and (2), we have $I \simeq_{K_1} J <_{K_1} L$ and $I \simeq_{K_2} L <_{K_2} J$. Targeting an equity behavior, condition (4) alone does not require J and L to be equally plausible with respect to the profile $\langle K_1, K_2 \rangle$: we could have for instance $J <_{\langle K_1, K_2 \rangle} L$. Contrastingly, in such a case, condition (F) implies that $J \simeq_{\langle K_1, K_2 \rangle} L$.

Now, the following proposition shows that through a filtering assignment, all the worlds are ranked over at most two plausibility levels for any singleton profile $\langle K \rangle$:

Proposition 8. *Every assignment satisfying conditions (1), (2), (6) and (F) maps every singleton belief profile $\langle K \rangle$ to a unique total preorder $\leq_K$ over worlds such that $I <_K J$ iff $I \models K$ and $J \not\models K$.*

Proposition 8 is a key result to prove the following stronger result on filtering assignments. Let us denote $|I(\mathcal{K})| = |\{K_i \in \mathcal{K} \mid I \models K_i\}|$, i.e., the number of belief bases in $\mathcal{K}$ for which I is a model. The following proposition holds:

Proposition 9. *Let $\leq_\mathcal{K}$ be the preorder over worlds associated with a profile $\mathcal{K}$ by a filtering syncretic assignment. We have*

$$I <_\mathcal{K} J \textit{ iff } |I(\mathcal{K})| > |J(\mathcal{K})|.$$

An important consequence of Proposition 9 is the following representation theorem for EIC merging operators satisfying (Ind):

Proposition 10. *An EIC merging operator Δ satisfies (Ind) iff there exists a filtering syncretic assignment associating every profile $\mathcal{K}$ with a total preorder $\leq_\mathcal{K}$ such that for every formula μ, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_\mathcal{K})$.*

Another consequence of Proposition 9 is given by the following corollary:

Corollary 3. *There is only one filtering syncretic assignment.*

Finally, as a direct consequence of Corollary 3, Corollary 1 and Proposition 10, we get that:

Corollary 4. $\Delta^{d_D,\Sigma}$ *is the only EIC merging operator satisfying (Ind).*

This result gives a full axiomatic characterization of the IC distance-based operator $\Delta^{d_D,\Sigma}$ in terms of independence.

5 Conclusion

In this paper, we have studied the case where belief bases are "rationalized" with respect to the integrity constraints during the merging process. Such a rationalization is expected in scenarios for which some IC merging operators can lead to unexpected merged bases because they give too much credit to unfeasible worlds. This is especially true when the integrity constraints encode strong constraints such as physical laws. In particular when the propositional formulae are obtained via a translation from representations coming from a more expressive framework (such as qualitative temporal of spatial settings), the integrity constraints must be used to rule out unfeasible worlds (meaningless worlds created during the translation process) [Condotta *et al.*, 2009].

We have defined in formal terms several independence conditions for merging operators and studied how they interact with the standard IC postulates for belief merging. Especially:

- Since rationalization by expansion may lead to inconsistent bases, we have extended the IC postulates with a new axiom (Inc) which constraints the behavior of merging operators applied to profiles consisting of inconsistent bases; we gave a representation theorem for the augmented set of postulates, called EIC, where the p-consistency condition on the profiles is relaxed.
- We have shown that independence to rationalization by update is impossible for EIC operators, since this independence property conflicts with some basic IC postulates. We have also shown that independence to rationalization by revision is equivalent to independence to rationalization by expansion for EIC operators.
- Finally, we have proven that there is a unique EIC operator satisfying the independence property to rationalization by expansion (or equivalently, by revision), namely the distance-based operator $\Delta^{d_D,\Sigma}$, where the drastic distance d_D and sum as an aggregation function are used. As far as we know, this is the first IC merging operator for which a full axiomatic characterization is given.

In this paper we have shown that there is a unique EIC operator satisfying (Ind). It is possible to associate any EIC operator Δ and a rationalization method R with a "rationalized" one Δ^R, by first making an explicit rationalization using R, followed by the merging using Δ. The results of this paper show that these two operators Δ and Δ^R are distinct. Studying the behaviours and properties of such operators Δ^R is an interesting issue. Pointing out other (non EIC) operators satisfying (Ind) and investigating the relationships between (Ind), (Ind-$\circ$), and (Ind-$\diamond$) for non EIC operators are other perspectives for further research.

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