On iterated revision in the AGM framework

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Abstract. While AGM belief revision identifies belief states with sets of formulas, proposals for iterated revision are usually based on more complex belief states. In this paper we investigate within the AGM framework several postulates embodying some aspects of iterated revision. Our main results are negative: when added to the AGM postulates, our postulates force revision to be maxichoice (whenever the new piece of information is inconsistent with the current beliefs the resulting belief set is maximal). We also compare our results to revision operators with memory and we investigate some postulates proposed in this framework.

1 Introduction

While AGM belief revision identifies belief states with sets of formulas, proposals for iterated revision are usually based on more complex belief states. Following the work of [7], they are usually represented by total pre-orders on interpretations. In fact in [6], Darwiche and Pearl first stated their postulates (C1-C4) in the classical AGM framework. But it has been shown in [8, 15] that (C2) is inconsistent with AGM, and that under the AGM postulates (C1) implies (C3) and (C4). To remove these contradictions, Darwiche and Pearl rephrased their and the AGM postulates in terms of epistemic states [7]. This has lead to a widely accepted framework for iterated revision, and most of the work on iterated belief revision now uses this more complex framework.

So an interesting question investigated in this paper is which requirements on iteration one can consistently add to the usual AGM framework. We focus on the status of old information, and formulate several postulates embodying that aspect of iterated revision. They all express that old information about A determine in some way the current status of A.

In particular, the first postulate says that if the agent was informed about A before revision (in the sense that either A or $\neg A$ was accepted) then the agent should remain informed about A after revision.

Our second postulate is motivated by the following basic algorithm for the revision of a belief set B by a new piece of information A [11, 19]: first put A in the new belief set, then add as many old beliefs from B as possible. So the second postulate expresses that the corresponding operator is idempotent with respect to B. We also study a family of postulates that generalizes this idea.

We also review other postulates coming from the iterated revision literature, in the classical belief set framework.

Our results are mainly negative: when added to the AGM postulates, our postulates lead to extreme revision operators. In particular the first two postulates force revision to be maxichoice: whenever the new piece of information is inconsistent with the current beliefs then the resulting belief set is maximal.

These "impossibility results" about iterated revision in the usual AGM framework can be seen as a justification for the increase in representational complexity that shows up when one goes from AGM to iterated belief revision frameworks (see e.g. [7, 15, 18, 13, 17, 4]). Instead of "flat" belief sets (alias sets of interpretations), the latter work with epistemic states, that can be represented by pre-orders on interpretations.

The paper is organized as follows. In section 2 we give some definitions and notations. In section 3 we consider the Darwiche and Pearl postulates in the AGM framework. More specifically, we focus on their first postulate. In section 4 we investigate the implications of trying to retain old information as much as possible. In section 5 we explore a family of postulates, saying that re-introducing old pieces of information is harmless. In section 6 we compare our results to revision operators with memory [13, 14] and we investigate the implications of some postulates coming from this work. We conclude in section 7.

2 Preliminaries

We work with a propositional language built from a set of atomic variables, denoted by $p, q, \ldots$ Formulas are denoted by $A, B, C, \ldots$ We identify finite sets of formulas (that we call belief sets) with the conjunction of their elements. A belief set B is *informed about* a formula C if $B \vdash C$ or $B \vdash \neg C$. A belief set B is *maximal* (or complete) if B is informed about every C.

The set of all interpretations is denoted $\mathcal{W}$, and the set of all belief sets is denoted $\mathcal{B}$. For a formula B, $Mod(B)$ denotes the set of models of B, i.e. $Mod(B) = \{\omega \in \mathcal{W} : \omega \models B\}$. For a set of interpretations $M \subseteq \mathcal{W}$, $Form(W)$ denotes the formula (up to logical equivalence) whose set of models is M, i.e. $Form(W) = \{B : \omega \models B \text{ iff } \omega \in M\}$.

A pre-order $\leq$ is a reflexive and transitive relation. $<$ is its strict counterpart: $\omega < \omega'$ if and only if $\omega \leq \omega'$ and $\omega' \not\leq \omega$. And $\simeq$ is defined by $\omega \simeq \omega'$ iff $\omega \leq \omega'$ and $\omega' \leq \omega$. A pre-order is total is for all ω, ω' we have $\omega \leq \omega'$ or $\omega' \leq \omega$. $\min(M, \leq)$ denotes the set $\{\omega \in M | \nexists \omega' \in M : \omega' < \omega\}$.

Definition 1 (AGM belief revision). *An AGM belief revision operator $\star$ is a function that maps a belief set B and a formula A to a belief set $B \star A$ such that :*

(R1) $B \star A \vdash A$
(R2) *If* $B \wedge A \nvdash \bot$, *then* $B \star A \equiv B \wedge A$
(R3) *If* $A \nvdash \bot$, *then* $B \star A \nvdash \bot$
(R4) *If* $B_1 \equiv B_2$ *and* $A_1 \equiv A_2$, *then* $B_1 \star A_1 \equiv B_2 \star A_2$

(R5) $(B \star A) \wedge C \vdash B \star (A \wedge C)$
(R6) *If* $(B \star A) \wedge C \nvdash \bot$, *then* $B \star (A \wedge C) \vdash (B \star A) \wedge C$

The postulates (R1-R4) are often called the *basic* AGM postulates, and the set (R1-R6) the *extended* AGM postulates, indicating that people consider the former to be more fundamental. Notice however that they do not put very hard constraints on $\star$. It is the two last ones (R5) and (R6) that allow to state the below representation theorem, which says that a revision operator corresponds to a family of pre-orders on interpretations. (The theorem is due to Katsuno and Mendelzon, but the idea can be directly traced back to Grove [10].) But first we need the following:

Definition 2 (Faithful assignment). *A function that maps each belief set B to a pre-order $\leq_B$ on interpretations is called a* faithful assignment *if and only if the following holds:*

1. *If $\omega \models B$ and $\omega' \models B$, then $\omega \simeq_B \omega'$*
2. *If $\omega \models B$ and $\omega' \not\models B$, then $\omega <_B \omega'$*
3. *If $B_1 = B_2$, then $\leq_{B_1} = \leq_{B_2}$*

Theorem 1. *A revision operator $\star$ satisfies postulates (R1-R6) if and only if there exists a faithful assignment that maps each belief set B to a total pre-order $\leq_B$ such that:*

$$Mod(B \star A) = \min(Mod(A), \leq_B)$$

We say that the assignment is the faithful assignment *corresponding to* the revision operator.

Let us now introduce a special family of revision operators, called maxichoice revision operators [1, 9].

Definition 3 (maxichoice revision). *A belief revision operator $\star$ is a* maxichoice *revision operator if for every B and A, if $B \vdash \neg A$ then $B \star A$ is maximal.*

Maxichoice revision operators are not very satisfactory, since they are too precise and have a too drastic behaviour. In fact, with those operators, learning any piece of information that conflicts with the current beliefs, however incomplete they are, causes the agent to have beliefs on any formula: for any formula A, either the agent believes that A holds or he believes that $\neg A$ holds. They are considered as an upper-bound for revision operators (the lower-bound being full-meet revision operators [1, 9]).

We will use a characterization of maxichoice operators on the semantical level. First we define:

Definition 4. *A* linear faithful assignment *is a faithful assignment that satisfies*

4. *If $\omega \not\models B$ and $\omega' \not\models B$, then $\omega <_B \omega'$ or $\omega' <_B \omega$*

The following result is is part of the folklore in the literature on revision:

Theorem 2. *A revision operator $\star$ is a maxichoice operator if and only if its corresponding assignment is a linear faithful assignment.*

The proof is straightforward.

3 Darwiche and Pearl postulates in the AGM framework

In [6], Darwiche and Pearl first stated their well-known postulates (C1-C4) in the classical AGM framework.

(C1) If $A \vdash C$, then $(B \star C) \star A \equiv B \star A$
(C2) If $A \vdash \neg C$, then $(B \star C) \star A \equiv B \star A$
(C3) If $B \star A \vdash C$, then $(B \star C) \star A \vdash C$
(C4) If $B \star A \nvdash \neg C$, then $(B \star C) \star A \nvdash \neg C$

But it has been shown in [8, 15] that (C2) is inconsistent with AGM, and that under the AGM postulates (C1) implies (C3) and (C4). To remove these contradictions, Darwiche and Pearl rephrased their and the AGM postulates in terms of epistemic states [7].

As (C1) is consistent with the AGM postulates, one might wonder what the constraints imposed by this postulate on the revision operators are like. This question has not been investigated as far as we know. The consistency of (C1) with AGM is easily established by noticing that the full meet revision operator satisfies (C1) [15]. But is this the only AGM operator satisfying (C1), or do we face a wider family?

Let us define another particular family of revision operators.

Definition 5. *Let $\leq$ be a total pre-order on interpretations. A revision operator $\star$ is said to be* imposed *by $\leq$ if its corresponding faithful assignment satisfies the following property:*

i. If $\omega \not\models B$ and $\omega' \not\models B$, then ($\omega \leq_B \omega'$ iff $\omega \leq \omega'$).

As far as we know, this family of operators has not been studied yet. Such operators are not satisfactory since the result of a revision does not depend of the belief set, but merely of the new piece of information (see theorem 3). This seems to be counter-intuitive and to go against the basic ideas behind revision. Nevertheless, such operators fulfill all AGM postulates, and the full meet revision operator is a particular case (when $\leq$ is a flat pre-order, i.e. $\omega \simeq \omega', \forall \omega, \omega' \in \mathcal{W}$).

Theorem 3. *Let $\star$ be an AGM revision operator, and let f be any function mapping formulas to formulas such that $f(A) \vdash A$ and if $A_1 \equiv A_2$ then $f(A_1) \equiv f(A_2)$. $\star$ is imposed if and only if for any belief set B and formula A, the following holds:*

(IMP) *If $B \vdash \neg A$ then $B \star A \equiv f(A)$.*

Proof. The only if part is straightforward: define $f(A)$ as $\min(Mod(A), \leq)$.

For the if part we need to build the imposed pre-order $\leq$ from $f(A)$. This can be established by noting that if we take a formula A that has exactly two (distinct) models ω and ω', then by (IMP) for every B such that $A \wedge B \vdash \bot$, we have $B \star A \equiv f(A)$. By (R1) and (R3), $Mod(f(A)) = \{\omega\}$ or $Mod(f(A)) =$

$\{\omega'\}$ or $Mod(f(A)) = \{\omega, \omega'\}$. Since $\star$ is an AGM operator, the faithful assignment gives us, for *every* B inconsistent with A, that $\omega <_B \omega'$ whenever $Mod(f(A)) = \{\omega\}$, $\omega' <_B \omega$ whenever $Mod(f(A)) = \{\omega'\}$, and $\omega \simeq_B \omega'$ whenever $Mod(f(A)) = \{\omega, \omega'\}$. That means that there exists a pre-order $\leq$ defined as $\omega \leq \omega'$ iff $\omega \in Mod(f(Form(\omega, \omega')))$ and such that for all B such that $\omega, \omega' \not\models B$, $\omega \leq_B \omega'$ iff $\omega \leq \omega'$.

This result states that for any revision that is not an expansion the old belief set is not taken into account in the result of the revision.

Now let us return to the case of the (C1) postulate and state the following result:

Theorem 4. *An AGM revision operator satisfies (C1) if and only if it is imposed.*

Proof. The if part is straighforward, since either $B \wedge A$ is consistent and then (C1) is a consequence of (R2), or $B \wedge A$ is not consistent, and then (C1) is a consequence of theorem 3.

For the only if part, suppose that the operator $\star$ satisfies (R1-R6) and (C1). We will show that the operator is imposed and there exists an f such that (IMP) is satisfied. If $\star$ satisfies (C1) then (IMP) holds, since for every A and B such that $A \wedge B$ is not consistent, by (R2) we have that $B \star A \equiv (\neg A \star (A \vee B)) \star A$. Thus by (C1) we get that $(\neg A \star (A \vee B)) \star A = \neg A \star A$, consequently we get $B \star A \equiv \neg A \star A$. Thus f can be defined by stipulating that $f(A) = \neg A \star A$. This means that the result of the revision depends only on the input A.

This result casts serious doubts on the (C1) postulate in the AGM framework.

4 "Keep on being informed about A"

When an agent receives new information she has to modify her current set of beliefs B in order to take it into account. One major requirement of AGM theory is the principle of *minimal change*, that means that when one revises a belief set by a new piece of information, one has to keep *"as much as possible"* of the old belief set.

The following property tries to capture this intuition, by saying that revising by A can not induce a loss of information: if B is informed about C, then learning A can not lead to loose this information.

(Compl) If $B \vdash C$ then $B \star A \vdash C$ or $B \star A \vdash \neg C$

Unfortunately it can be proved that :

Theorem 5. *If $\star$ satisfies (R1-R6) and (Compl), then $\star$ is a maxichoice revision operator.*

Proof. This can be proved straightforwardly: suppose $B \vdash A$. If $\vdash A$ then the theorem holds. Else we have $B \vdash A \vee C$ and $B \vdash A \vee \neg C$. By (Compl), $B \star \neg A \vdash A \vee C$ or $B \star \neg A \vdash \neg A \wedge \neg C$, and $B \star \neg A \vdash A \vee \neg C$ or $B \star \neg A \vdash \neg A \wedge C$. Among the four cases, the one where $B \star \neg A \vdash (A \vee C) \wedge (A \vee \neg C)$ is impossible because $B \star \neg A \vdash A$ by (R4) and $\nvdash A$. The one where $B \star \neg A \vdash (\neg A \wedge \neg C) \wedge (\neg A \wedge C)$ is impossible because $B \star \neg A \vdash \bot$. It follows that $B \star \neg A \vdash \neg C$ or $B \star \neg A \vdash C$.

It is straightforward to show that every maxichoice revision operator satisfies (Compl). Together with the preceding theorem it follows that (Compl) characterizes maxichoice revision.

Remark 1. Formula (3.17) in [9] is just (COMPL) (modulo a typo). There, proposition (3.19) says that "$B \star A$ is maximal for any sentence A such that $\neg A \in B$", i.e. (3.17) entails maxichoice revision. The proof refers to observation 3.2 of [2], but the latter presupposes already that $\star$ is a maxichoice operator, and establishes that this entails maximality.

So this postulate puts too strong a requirement on classical AGM revision operators.

In the next section we will investigate another requirement also based on the assumption that we can keep as much as possible of the old information.

5 "Re-introducing old information doesn't harm"

Another way of ensuring that one does not forget previous information is to suppose that we can re-introduce the old belief set without changing the current one. It can be seen as some kind of left-idempotency of the revision operator. This idea is very close to the one used for defining revision with memory operators [14, 13, 3].

First we need the following abbreviations.

Definition 6. *Given a set of beliefs B and pieces of information A_i, then for $1 \leq i \leq n$ we define B_i by:*

$$B_i = (...((B \star A_1) \star A_2) \star \ldots) \star A_i$$

Thus $B_0 = B$, $B_1 = B \star A_1$, and $B_2 = (B \star A_1) \star A_2$.

Our abbreviation enables us to concisely formulate the following family of postulates:

(Mem$_i$) $B_i \equiv B \star B_i$, for $i \geq 0$

Hence:

(Mem$_0$) says $B_0 \equiv B \star B_0$, i.e. $B \equiv B \star B$,
(Mem$_1$) says $B_1 \equiv B \star B_1$, i.e. $B \star A_1 \equiv B \star (B \star A_1)$, and
(Mem$_2$) says $B_2 \equiv B \star B_2$, i.e. $(B \star A_1) \star A_2 \equiv B \star ((B \star A_1) \star A_2)$.

$\vdots$

Let us see now what is the relation of the postulates (Mem$_i$) with the AGM postulates.

Theorem 6. *(Mem$_0$) is derivable from the basic AGM postulates.*

The proof only uses the postulate (R2).

Theorem 7. *(Mem$_1$) is derivable from the extended AGM postulates.*

Proof. From (R1) we know that $(B \star A) \wedge A \equiv B \star A$. Now using (R5) and (R6) with $C = B \star A$, we have $B \star (A \wedge (B \star A)) \equiv (B \star A) \wedge (B \star A)$. That is directly $B \star (B \star A) \equiv B \star A$.

Theorem 8. *(Mem$_2$), (Mem$_3$), etc. cannot be derived from the AGM postulates.*

Proof. This can be established e.g. by considering Dalal's revision operator [5], which is known to satisfy the AGM postulates [12] and showing that is does not satisfy the (Mem$_i$) postulates. Indeed, consider $B = \neg p$, $A_1 = \neg q$, $A_2 = p \vee q$. Then $B_2 = (\neg p \star \neg q) \star (p \vee q) = (\neg p \wedge \neg q) \star (p \vee q) = p \oplus q$ where $\oplus$ is the exclusive or. But this is different from $B \star B_2 = \neg p \star ((\neg p \star \neg q) \star (p \vee q))$ $= \neg p \star ((\neg p \wedge \neg q) \star (p \vee q)) = \neg p \star (p \oplus q) = \neg p \wedge q$.

We can easily find revision operators satisfying these additional postulates :

Theorem 9. *If $\star$ is a maxichoice revision operator then $\star$ satisfies every postulate (Mem$_i$).*

The postulates of this family are ordered by strength, as shows the following result:

Theorem 10. *If $\star$ satisfies postulate (Mem$_{i+1}$) then $\star$ satisfies postulate (Mem$_i$).*

The other way round, (Mem$_i$) does not always imply (Mem$_{i+1}$): this is immediate for $i = 0$.

So is those families of operators, defined from the (Mem$_i$) postulates, are wide ones ? It is not the case. We show that, once again, only maxichoice revision operators satisfy our postulates.

Theorem 11. *If $\star$ satisfies (R1-R6) and (Mem$_2$), then $\star$ is a maxichoice revision operator.*

Proof. Suppose that A is consistent and that $B \vdash \neg A$. We want to show that $B \star A$ is maximal, i.e. for an arbitrary C we have that either $B \star A \vdash C$, or $B \star A \vdash \neg C$.

First, (Mem$_2$) tells us that $(\neg A \vee C) \star B \star A = (\neg A \vee C) \star ((\neg A \vee C) \star B \star A)$, and similarly $(\neg A \vee \neg C) \star B \star A = (\neg A \vee \neg C) \star ((\neg A \vee \neg C) \star B \star A)$. As $B \vdash \neg A$ we have $B = (\neg A \vee C) \star B$ by (R2), and similarly $B = (\neg A \vee \neg C) \star B$. Hence $(\neg A \vee C) \star B \star A = (\neg A \vee C) \star ((\neg A \vee C) \star B \star A) = (\neg A \vee C) \star (B \star A)$, and similarly

$(\neg A \vee \neg C) \star B \star A = (\neg A \vee \neg C) \star ((\neg A \vee \neg C) \star B \star A) = (\neg A \vee \neg C) \star (B \star A)$. Now suppose that not(either $B \star A \vdash C$, or $B \star A \vdash \neg C$), i.e. $B \star A$ is consistent with C, and $B \star A$ consistent with $\neg C$. Then we must have $(\neg A \vee C) \star B \star A = (\neg A \vee C) \star ((\neg A \vee C) \star B \star A) = (\neg A \vee C) \star (B \star A) = (\neg A \vee C) \wedge (B \star A)$, and $(\neg A \vee \neg C) \star B \star A = (\neg A \vee \neg C) \star ((\neg A \vee \neg C) \star B \star A) = (\neg A \vee \neg C) \star (B \star A) = (\neg A \vee \neg C) \wedge (B \star A)$. As $B \star A \vdash A$, we would have that $(\neg A \vee C) \wedge (B \star A) \vdash C$, and $(\neg A \vee \neg C) \wedge (B \star A) \vdash \neg C$. But by AGM $(\neg A \vee C) \star (B \star A)$ must be consistent.

A corollary of the theorems 10 and 11 is that a revision operator satisfies a (Mem_i) postulate if and only if it is a maxichoice revision operator. So each postulate of this family is a characterisation of maxichoice operators.

As explained at the beginning of this section, the idea of this family of postulates seems very close to the one behind the definition of revision with memory operators. In the next section we will investigate more deeply the links between revision with memory operators and the requirements on classical AGM revision operators.

6 The relation with revision with memory operators

Belief revision operators with memory [14, 13] keep trace of the history of beliefs in order to be able to use them whenever further revisions make this possible. They are based on a notion of belief state that is more complex than the flat set of beliefs of the AGM framework.

Basically, if we represent epistemic states Φ by a pre-order on interpretations, noted $\leq_\Phi$, we can extract the associated belief set with the projection operator $Bel(\Phi) = \min(\mathcal{W}, \leq_\Phi)$. The pre-order $\leq_\Phi$ represents the agent's relative confidence in interpretations. For example $\omega <_\Phi \omega'$ means that for the agent in the epistemic state Φ the interpretation ω seems (strictly) more plausible than the interpretation ω'.

The usual logical notations extend straightforwardly to epistemic states (they in fact denote conditions on the associated belief sets). For example $\Phi \vdash C$, $\Phi \wedge C$ and $\omega \models \Phi$ respectively mean $Bel(\Phi) \vdash C$, $Bel(\Phi) \wedge C$ and $\omega \models Bel(\Phi)$.

Now let us define revision with memory operators. This family of operators is parametrized by a classical AGM operator. It can be seen as a tool to change a classical AGM operator with bad iteration properties into an operator that has good ones.

Definition 7 (Revision with memory). *Suppose that we dispose of a classical AGM operator $\star$. (We will use its corresponding faithful assignment $C \rightarrow \leq_C$.) Then we define the epistemic state (the pre-order) $\Phi \circ C$ that results from the revision with memory of Φ by the new information C as:*

$$\omega \leq_{\Phi \circ C} \omega' \text{ iff } \omega <_C \omega' \text{ or } \omega \simeq_C \omega' \text{ and } \omega \leq_\Phi \omega'$$

This definition means that each incoming piece of information induces some credibility ordering. (The exact ordering induced depends on the classical AGM operator that has been chosen.[1]) And the new epistemic state is built by listening first to this incoming piece of information, and then to the old epistemic state (this is the well known *primacy of update* principle).

In fact, it is shown in [13], that an epistemic state for revision with memory operators can be encoded as the history of the new pieces of information acquired by the agent since its "birth". So we can suppose that the agent starts from an "empty" epistemic state Ξ, that is represented by a flat pre-order[2], and successively accommodates all the pieces of information. So if we suppose that all revision sequences start from Ξ, it can be shown that all revision with memory operators satisfy the (Mem$_i$) postulates, since they all take the history of the revisions into account.

Theorem 12. *A revision operator with memory satisfies (Mem$_i$), $\forall i$.*

In fact, a logical characterization for revision with memory operators has been given in [13]. Most of the postulates are generalizations of AGM postulates in the epistemic states framework, but there are also some specific postulates characterizing revision with memory. We will examine now their status in the classical belief set framework. Those postulates have been written for epistemic states, but we can translate them for belief sets (with some simplifications) as follows :

(Hist1) $(B \star A) \star C \equiv B \star (A \star C)$
(Hist2) If $C \star A \equiv A$, then $(B \star C) \star A \equiv B \star A$
(Hist3) If $C \star A \vdash D$, then $(B \star C) \star A \vdash D$

The first postulate expresses some kind of associativity and aims at expressing the strong influence of the new piece of information. The second one says that if a formula C does not distinguish between the models of A, then learning C before A is without effect on the resulting belief set. The third one says that the consequences of a revision also holds if we first learn another piece of information.

The counterpart of (Hist1), (Hist2) and (Hist3) for epistemic states are respectively named (H7), (H'7) and (H'8) in [13]. It is shown there that in the presence of the other postulates (H1-H6) (that are mainly a generalisation of AGM postulates in the epistemic state framework), (H7) is equivalent to (H'7-H'8).

This equivalence no longer holds in the belief set framework. Let us see now the implications of these three postulates in this framework.

Theorem 13. *There is no operator that satisfies (R1-R6) and (Hist1).*

[1] Note that one of the possibilities is a two level pre-order with the models of the formula at the lowest level, and the counter-models at the top level. That gives the more "classical" operator of the family [18, 16, 20, 3].

[2] that is $\forall \omega, \omega' \; \omega \simeq_{\Xi} \omega'$

Proof. Let $\omega_0, \omega_1, \omega_2, \omega_3$ be 4 distinct interpretations. Now take four formulas A, B, C, D such that $Mod(A) = \{\omega_1, \omega_2\}$, $Mod(B) = \{\omega_0, \omega_1\}$, $Mod(C) = \{\omega_2, \omega_3\}$ and $Mod(D) = \{\omega_1, \omega_3\}$. From (Hist1) we have that $(B \star A) \star C = B \star (A \star C)$, that is from (R2) $(B \wedge A) \star C = B \star (A \wedge C)$. As $Mod(A \wedge C) = \{\omega_2\}$, from (R1) and (R3) it follows that $Mod(B \star (A \wedge C)) = \{\omega_2\}$, hence $Mod((B \wedge A) \star C) = \{\omega_2\}$. On the other side, starting from (Hist1) with $(B \star D) \star C = B \star (D \star C)$, we obtain similarly $Mod(B \star (D \wedge C)) = Mod((B \wedge D) \star C) = \{\omega_3\}$. Now notice that $B \wedge D \equiv B \wedge A$, so (R4) says that $(B \wedge A) \star C \equiv (B \wedge D) \star C$. Contradiction.

Note that (Hist2) is stronger that the postulate (C1) proposed by Darwiche and Pearl. As (C1) is consistent with the AGM postulates we will consider a weakening of the (Hist2) postulate, that accounts for the case when $A \not\vdash C$:

(StrictHist2) If $C \star A \equiv A$ and $A \nvdash C$, then $(B \star C) \star A \equiv B \star A$

Theorem 14. *If an operator $\star$ satisfies (R1-R6) and (StrictHist2), then $\star$ is a maxichoice revision operator.*

Proof. We show that if $\star$ satisfies (StrictHist2), then $\star$ is maxichoice. If $\star$ is not maxichoice, then there exists a formula C such that $\leq_C$ is not linear, that means that we can find a formula A and two distinct interpretations ω, ω', with $Mod(A) = \{\omega, \omega'\}$ (with $\omega \neq \omega'$) such that $C \wedge A$ is not consistent[3] and $\omega \simeq_C \omega'$, ie $C \star A = A$. (StrictHist2) then says that for all B $(B \star C) \star A = B \star A$. In particular if we take B such that $Mod(B) = Mod(C) \cup \{\omega\}$, that means that $C \star A = B \star A = A$. But from (R2) we get that $B \star A = B \wedge A$, so $Mod(B \star A) = \{\omega\}$. Contradiction.

So, as a corollary of theorems 14 et 4, every operator satisfying (R1-R6) and (Hist2) must be an imposed maxichoice operator.

Theorem 15. *There is no operator that satisfies (R1-R6) and (Hist3).*

Proof. Let $\omega_0, \omega_1, \omega_2$ be 3 distinct interpretations. Now take four formulas A, B, C, D such that $Mod(A) = \{\omega_1, \omega_2\}$, $Mod(B) = \{\omega_0\}$, $Mod(C) = \{\omega_0, \omega_1\}$, and $Mod(D) = \{\omega_0, \omega_2\}$. As from (R2) $C \star A = C \wedge A$, then $Mod(C \star A) = \{\omega_1\}$, so from (Hist3) and (R3), that means that $Mod((B \star C) \star A) = \{\omega_1\}$. On the other side, starting from $D \star A$, we find similarly that $Mod((B \star D) \star A) = \{\omega_2\}$. Finally, as from (R2) we find easily that $(B \star C) \equiv (B \star D)$, from (R4) we have that $(B \star C) \star A \equiv (B \star D) \star A$. Contradiction.

These three results show, once again, that it is hard to try to formulate iteration postulates in the AGM framework. Whereas those properties are meaningful in the epistemic state framework, two of them, (Hist1) and (Hist3), are not consistent with AGM postulates for belief set revision, and the last one, (StrictHist2), implies the maxichoice property.

[3] When $\star$ is an AGM revision operator and $C \star A \equiv A$, then $C \wedge A \nvdash \bot$ is equivalent to $A \vdash C$.

7 Conclusion

Studies in iterated belief revision have been stated in the epistemic state framework mainly because of the influence of Darwiche and Pearl's proposal [6, 7] and its incompatibility with the AGM belief set framework. But since, few work has been done to see if some properties on iteration can be stated in the classical framework.

We have addressed this issue in this paper by looking at some candidates postulates. In different ways, all of them express that the result of a revision must keep as much as possible of the old information.

Our results are mainly negative. When the proposed postulates are not inconsistent with classical AGM ones, they inexorably lead to the maxichoice property, which is far from satisfactory for a sensible revision operator. So the results obtained in this paper can be seen as "impossibility results" about iteration in the classical AGM framework.

This study is then important to justify the gap, both in terms of knowledge representation and in terms of computational complexity, induced by all the iterated revision approaches that abandon the classical framework and work with more complex objects, viz. epistemic states.

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