Complexity Results
for Paraconsistent Inference Relations

Sylvie Coste-Marquis
CRIL/Université d'Artois
rue de l'Université - S.P. 16
62307 Lens Cedex - France
coste@cril.univ-artois.fr

Pierre Marquis
CRIL/Université d'Artois
rue de l'Université - S.P. 16
62307 Lens Cedex - France
marquis@cril.univ-artois.fr

Abstract

In this paper, the complexity of several paraconsistent inference relations, based on multivalued logics, is investigated. Many inference relations pointed out so far by Arieli and Avron, Besnard and Schaub, D'Ottaviano and da Costa, Frisch, Levesque, Priest are considered from the computational side. All these relations can be gathered into two categories: the basic ones stem directly from the notions of models within 3 or 4-valued logics, while the refined ones are based on notions of preferred models for such logics. Completing complexity results by Cadoli and Schaerf (centered on the basic relations), we show that the refined paraconsistent inference relations that have been defined in the framework of multivalued logics are highly intractable. Especially, we prove that the inference problems corresponding to these relations are Π_2^p-complete, even in the simple case where the database is a CNF formula and the query is a propositional symbol.

1 INTRODUCTION

While classical logic is considered as a base line in many AI studies, it is acknowledged that classical entailment cannot be used as such to model commonsense inference. Among its major drawbacks is its inability to handle inconsistent information; indeed, in presence of a contradiction, every formula can be derived (this is the well-known *ex falso quodlibet sequitur* of classical logic).

Such a trivialization problem is very significant both from the theoretical side and from the practical side. Specifically, it is strongly connected to the possibility to handle exceptions (another salient aspect of common-sense reasoning). From the practical side, its importance comes from the fact that many actual, large-sized databases are inconsistent (specifically when they integrate information stemming from multiple sources).

Many logics have been developed so far in order to avoid trivialization in presence of inconsistency. They are called *paraconsistent logics*. Both the complexity of the problem and its significance are reflected by the number of approaches to paraconsistent reasoning that can be found in the literature (see [Besnard and Hunter, 1998; Hunter, 1998] for a survey). Indeed, paraconsistency can be achieved in various ways, mainly:

- by *restricting the proof theory* of classical logic so as to retain only a subset of the classical proofs as admissible (e.g., this is what is done in C_ω logic [da Costa, 1974] and in quasi-classical logic [Besnard and Hunter, 1995; Hunter, 2000]).
- by *focusing on the consistent subsets* of the inconsistent database, eventually restricted to the most preferred ones, when some preferential information can be exploited [Rescher and Manor, 1970; Fagin *et al.*, 1983; Ginsberg, 1986; Baral *et al.*, 1991; Pinkas and Loui, 1992; Benferhat *et al.*, 1993]. Such techniques are closely related to so-called syntax-based approaches to belief revision [Nebel, 1992; Nebel, 1994] and to the framework for supernormal default reasoning with priorities from [Brewka, 1989].
- by associating to each infered formula a justification under the form of a subset of the information used to derive it, and by reasoning on such arguments whenever some mutually inconsistent formulas can be derived. This is the basic idea of *argumentative logics*, see e.g., [Dung, 1995;

Elvang-Goransson and Hunter, 1995; Bondarenko *et al.*, 1997].

- by *merging* the various pieces of belief from the inconsistent database, see e.g., [Lin, 1996; Revesz, 1997; Konieczny and Pino Pérez, 1998; Konieczny, 2000].
- by preventing inconsistent databases from having no model, through the consideration of more general notions of interpretations (or worlds). Several *multivalued logics* are related to this line of research [D'Ottaviano and da Costa, 1970; Belnap, 1977; Frisch, 1987; Levesque, 1989; Priest, 1989; Priest, 1991; Besnard and Schaub, 1997; Besnard and Schaub, 1998; Arieli and Avron, 1998].

Of course, these approaches are not mutually exclusive. For instance, some merging operators and argument selection policies are based on consistent subsets, the set of inference rules used in the proof theory of some multivalued logics is a subset of those used in classical logic, and conversely some logics based on a subclassical proof theory can be given some multivalued semantics.

In this paper, we mainly focus on the *multivalued logic* approach to inconsistency handling. Compared with other approaches to inconsistent-tolerant reasoning, like forms of belief merging and the maxcons approach (i.e., the technique based on the selection of *max*imally *cons*istent subsets of the database) multivalued paraconsistent logics prevent inference from trivializing, even in the restricted case where the database consists of a single inconsistent formula.

In the following, we specifically consider some four-valued logics and three-valued logics proposed so far, including Arieli and Avron's *FOUR* logic [Arieli and Avron, 1998] (based on Belnap's work [Belnap, 1977]), Besnard and Schaub's one [Besnard and Schaub, 1997], D'Ottaviano and da Costa's J_3 logic [D'Ottaviano and da Costa, 1970], and several restrictions of them, especially Priest's LP and LP_m logics [Priest, 1989; Priest, 1991], Frisch's RP logic [Frisch, 1987], Levesque's logic of 3-inference [Levesque, 1989]. We also consider some paraconsistent relations given by Besnard and Schaub [Besnard and Schaub, 1998]. All these relations can be gathered into two categories: the basic ones stem directly from the notions of models within 3 or 4-valued logics, while the refined ones are based on notions of preferred models for such logics.

Many of these inference relations have been investigated in depth from a logical point of view (see e.g., [Arieli and Avron, 1998]), but the study of their computational complexity aspects is far less complete. The notable exception concerns some of the basic relations, the complexity of which has been analyzed by Levesque [Levesque, 1989] and by Cadoli and Schaerf [Cadoli and Schaerf, 1996]. This paper contributes to fill the gap. For each inference relation under consideration, the complexity of the corresponding decision problem is identified, in the general case (i.e., without any restriction over the database Σ or the query γ), and in some restricted cases (specifically, when Σ is a CNF formula and γ is a propositional symbol).

From our complexity analysis, the following conclusions can be drawn. Firstly, the inference problem for the basic relations considered in this paper (i.e., Arieli and Avron's $\models^4$, D'Ottaviano and da Costa's $\models^3$ and its restrictions: Priest's $\models_{LP}$, Frisch's $\models_{RP}$ and Levesque's $\models_L$) is coNP-complete in the general case (and even in P in some restricted situations). This is not so high, especially when compared with the complexity of the inference problems associated to other approaches to inconsistency-tolerant inference, like the maxcons ones. Unfortunately, such basic inference relations are typically too cautious. For instance, the disjunctive syllogism is not satisfied by any of them. As a consequence, none of them coincides with classical entailment in the restricted situation where Σ is a (classically) consistent CNF formula. In order to design less cautious inference relations, it is sufficient to focus on some preferred models, e.g., those which are as close as possible to the models of classical logic. This is the key idea underlying the inference relations from the second family considered in this paper, i.e., the refined ones. Secondly, we have established that the inference problem for the refined relations under consideration (i.e., Arieli and Avron's $\models^4_{I_1}$ and $\models^4_{I_2}$ [Arieli and Avron, 1998], Priest's $\models_{LP_m}$ [Priest, 1989; Priest, 1991], Besnard and Schaub $\models_{BS}$ [Besnard and Schaub, 1997]) as well as Besnard and Schaub's $\vdash_s$ and $\vdash^{\pm}_s$ [Besnard and Schaub, 1998] is Π^p_2-complete, even in the restricted case where the database Σ is a CNF formula and the query γ is a propositional symbol. Such an increase in computational complexity seems to be the price to be paid for designing paraconsistent relations that are not too cautious in the framework of multivalued logics.

The rest of this paper is organized as follows. In Section 2, the multivalued logics considered in the paper are described, both from the syntactical side and from the semantical one. On this ground, several paraconsistent inference relations are given in Section 3. The complexity results corresponding to their decision problems are reported in Section 4. Some concluding remarks are provided in Section 5. Proof sketches are reported in an appendix.

2 MULTIVALUED LOGICS

2.1 SYNTACTICAL ASPECTS

In the following, the propositional language $PROP^4_{PS}$, and its restrictions $PROP^3_{PS}$ and $PROP^2_{PS}$ are considered. They are based on a finite set PS of propositional symbols.

Definition 1 ($PROP^4_{PS}$, $PROP^3_{PS}$ **and** $PROP^2_{PS}$)

- $PROP^4_{PS}$ *is the propositional language over PS generated from the constant symbols* $true$, $false$, *both, unknown, and the connectives* $\neg$, $\vee$, $\wedge$, $\supset$, $\oplus$, *and* $\otimes$.
- $PROP^3_{PS}$ *is the propositional language over PS generated from the constant symbols* $true$, $false$, *both, and the connectives* $\neg$, $\vee$, $\wedge$, $\supset$, *and* $\oplus$.
- $PROP^2_{PS}$ *is the propositional language over PS generated from the constant symbols* $true$ *and* $false$, *and the connectives* $\neg$, $\vee$, $\wedge$, *and* $\supset$.

Clearly enough, the fragment $PROP^2_{PS}$ of $PROP^4_{PS}$ coincides with a standard language for classical propositional logic. A restricted fragment of it gathers the formula generated over PS using the connectives $\neg$, $\wedge$, $\vee$, only, which is referred to as $\{\neg, \wedge, \vee\}$ *fragment*. A proper subset of this fragment is composed by the *CNF formulas*, i.e., the (finite) conjunctions of clauses, where a clause is a (finite) disjunction of literals (the symbols from PS, eventually negated).

2.2 SEMANTICAL ASPECTS

Let us now explain how the formulas of $PROP^4_{PS}$ and its subsets can be interpreted. Obviously enough, in multivalued logics, there are more truth values than just 0 (false) and 1 (true):

Definition 2 (Interpretations)
A 4*-interpretation (resp. a* 3*-interpretation, a* 2*-interpretation) over PS is a total function I from PS to* $FOUR = \{0, 1, \top, \bot\}$ *(resp.* $THREE = \{0, 1, \top\}$, $TWO = \{0, 1\}$*).*

Here, $\bot$ intuitively denotes lack of information while $\top$ indicates inconsistency. All the connectives under consideration in this paper are *truth functional* ones. For every 4-interpretation I over PS, we define $I(true) = 1$, $I(false) = 0$, $I(both) = \top$, $I(unknown) = \bot$. The semantics $I(\phi)$ of a formula ϕ from $PROP^4_{PS}$ in I is defined compositionally in the obvious way, given the truth tables reported in Table 1.

The semantics of a formula from $PROP^3_{PS}$ (resp. $PROP^2_{PS}$) in a 3-interpretation (resp. a 2-interpretation) is defined in the obvious compositional way, by considering the reductions of the previous truth tables to $THREE$ (resp. TWO). This explains why *unknown* and the consensus operator $\otimes$ are not considered as connectives in $PROP^3_{PS}$ (just like *unknown*, *both*, $\otimes$, and the gullability operator $\oplus$ are not considered as connectives in $PROP^2_{PS}$), since the set of truth values must be closed under the connectives.

Unlike other multivalued logics, the set of *designated values* considered in $FOUR$ and its restriction $THREE$ is $\{1, \top\}$. This leads to the following notions of *models*:

Definition 3 (Models)
Let I be a 4*-interpretation (resp. a* 3*-interpretation, a* 2*-interpretation) I over PS and* ϕ *be a formula from* $PROP^4_{PS}$ *(resp.* $PROP^3_{PS}$, $PROP^2_{PS}$*). I is a* 4-model *(resp. a 3-model, a 2-model) of* ϕ *iff* $I(\phi) \in \{1, \top\}$.

Example 1 *The* 4*-interpretation I given by* $I(a) = \top$ *and* $I(x) = 1$ *for all* $x \in PS \setminus \{a\}$ *is a* 4*-model of* $\Sigma = a \wedge \neg a \wedge b$ *since* $I(\Sigma) = \top$. *I can also be viewed as a* 3*-model of* Σ. *Note that the fact that* Σ *has no* 2*-models does not prevent it from having some* 3*-models or* 4*-models.*

On this ground, two formulas ϕ and ψ from $PROP^4_{PS}$ (resp. $PROP^3_{PS}$, $PROP^2_{PS}$) are said to be *equivalent*, noted $\phi \equiv^4 \psi$ (resp. $\phi \equiv^3 \psi$, $\phi \equiv^2 \psi$) iff they have the same set of 4-models (resp. 3-models, 2-models).

Once this is stated, two remarks can be done. On the one hand, some of the connectives could easily be defined from others while preserving equivalence; for instance, this is the case for $\vee$ since $\phi \vee \psi \equiv^4 \neg(\neg\phi \wedge \neg\psi)$ (i.e., De Morgan's laws are satisfied in $FOUR$, hence in $THREE$); another example is $\oplus$ which can be defined from *both*, $\vee$ and $\wedge$ (cf. [Arieli and Avron, 1998]). Nevertheless, we keep all of the connectives for the ease of presentation. On the other hand, it would be easy to incorporate some additional connectives in the languages $PROP^4_{PS}$ and its restrictions $PROP^3_{PS}$ and $PROP^2_{PS}$. Among them are:

- $\phi \Leftrightarrow \psi =_{def} (\phi \supset \psi) \wedge (\psi \supset \phi)$;
- $\phi \rightarrow \psi =_{def} (\phi \supset \psi) \wedge (\neg\psi \supset \neg\phi)$;
- $\phi \leftrightarrow \psi =_{def} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$;
- $\phi! =_{def} \phi \wedge \neg\phi$;
- $\Box\phi =_{def} (\neg\phi \supset false) \wedge \neg(\phi \supset \neg\phi)$;
- $\diamond\phi =_{def} \neg\Box\neg\phi$;

Table 1: Truth tables.

α	β	$\neg\alpha$	$\alpha \wedge \beta$	$\alpha \vee \beta$	$\alpha \supset \beta$	$\alpha \otimes \beta$	$\alpha \oplus \beta$
0	0	1	0	0	1	0	0
0	1	1	0	1	1	$\bot$	$\top$
0	$\top$	1	0	$\top$	1	0	$\top$
0	$\bot$	1	0	$\bot$	1	$\bot$	0
1	0	0	0	1	0	$\bot$	$\top$
1	1	0	1	1	1	1	1
1	$\top$	0	$\top$	1	$\top$	1	$\top$
1	$\bot$	0	$\bot$	1	$\bot$	$\bot$	1
$\top$	0	$\top$	0	$\top$	0	0	$\top$
$\top$	1	$\top$	$\top$	1	1	1	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	0	1	$\bot$	$\bot$	$\top$
$\bot$	0	$\bot$	0	$\bot$	1	$\bot$	0
$\bot$	1	$\bot$	$\bot$	1	1	$\bot$	1
$\bot$	$\top$	$\bot$	0	1	1	$\bot$	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	1	$\bot$	$\bot$

- $\sim \phi =_{def} \Box\neg\phi$;
- $\odot\phi =_{def} \Box\phi \vee \Box\neg\phi$;
- $\phi \leq \psi =_{def} (\Box\phi \wedge \Box\psi) \vee (\Box\neg\phi \wedge \Box\neg\psi) \vee (\neg\odot\psi \wedge (\psi \vee \neg\psi)) \vee (\neg\odot\phi \wedge ((\phi \wedge \neg\phi) \supset false))$.

These connectives have been taken into account in the languages of some of the logics analyzed in this paper. Thus, the language used in [Arieli and Avron, 1998] to define *FOUR* is the extension of $PROP^4_{PS}$ where $\rightarrow$ and $\leftrightarrow$ are used as additional (binary) connectives. The language of the logic of 3-inference of [Levesque, 1989] (resp. the language of *LP* [Priest, 1989] or equivalently the language of *RP* [Frisch, 1987]) is the propositional language over *PS* generated from the constant $false$ and the connectives $\neg$, $\wedge$, $\vee$ (resp. from the connectives $\neg$, $\wedge$, $\vee$); $\supset$ and $\Leftrightarrow$ are also introduced but as syntactic sugars ($\phi \supset \psi =_{def} \neg\phi \vee \psi$ and $\phi \Leftrightarrow \psi =_{def} (\phi \supset \psi) \wedge (\psi \supset \phi)$). The language of LP_m [Priest, 1989; Priest, 1991] is the propositional language over *PS* generated from the connectives $\neg$, $\wedge$, $\vee$ and ! [1]. The language of J_3 is the propositional language over *PS* generated from the connectives $\neg$, $\wedge$, $\vee$, $\supset$, $\Leftrightarrow$, $\Box$, $\diamond$, $\sim$, $\odot$. The language considered in [Besnard and Schaub, 1997] is the propositional language over *PS* generated from the constants $true$, $false$ and the connectives $\neg$, $\wedge$, $\vee$, $\supset$, $\Leftrightarrow$, $\Box$, $\leq$.

[1] To be more precise, ! is introduced as a notation of the metalanguage in [Priest, 1989], and an element of the object language in [Priest, 1991].

The semantics of all such derived connectives can be easily defined by considering their truth tables. For instance, $\Box\phi$ means that ϕ is *necessarily* true, i.e., for every 4-interpretation I over *PS*, $I(\Box\phi) = 1$ if $I(\phi) = 1$ and $I(\Box\phi) = 0$ otherwise. $\odot\phi$ means that ϕ is classical, i.e., for every 4-interpretation I over *PS*, $I(\odot\phi) = 1$ if $I(\phi) = 1$ or $I(\phi) = 0$ and $I(\odot\phi) = 0$ otherwise. $\phi \leq \psi$ means that the truth value of ϕ must be lower or equal to the truth value of ψ w.r.t. Belnap's knowledge ordering $\leq_k$ defined by the reflexive-transitive closure of $<_k$ s.t. $\bot <_k 0$, $\bot <_k 1$, $0 <_k \top$ and $1 <_k \top$. The main point here is that there is no way to increase the expressivity of $PROP^4_{PS}$ or $PROP^3_{PS}$ through the incorporation of such additional connectives, since none of them can be genuine. Indeed, just like $PROP^2_{PS}$ is functionally complete for *TWO*, $PROP^4_{PS}$ is functionally complete for *FOUR* (Theorem 3.8 from [Arieli and Avron, 1998]) and $PROP^3_{PS}$ is functionally complete for *THREE* (Theorem 5.1 from [Arieli and Avron, 1998]). This contrasts with the language of J_3 [D'Ottaviano and da Costa, 1970] (and its restrictions like *LP* [Priest, 1989], *RP* [Frisch, 1987] and the logic of 3-inference of Levesque [Levesque, 1989]) which does include neither $\oplus$ nor *both*, and despite the presence of other connectives, is not expressive enough to enable the representation of a formula equivalent to *both* (see [Epstein, 1990]). The functional completeness of $PROP^4_{PS}$ (resp. $PROP^3_{PS}$) w.r.t. *FOUR* (resp. *THREE*) explains why the focus has been laid on them; in some sense, all the logics considered in this paper (including classical logic) are restrictions of *FOUR*.

Such an expressive power of $FOUR$ and its restrictions, together with their standard truth functional semantics, is a major feature of these logics. Actually, many usual, intuitive laws of classical logic (including De Morgan's ones, involution of negation, distributivity of $\wedge$ over $\vee$ and vice-versa) still hold in such logics[2].

3 PARACONSISTENT INFERENCE RELATIONS

We are now in position to define several paraconsistent inference relations based on $FOUR$, $THREE$ and their restrictions. We start from the simplest ones to the more sophisticated ones.

3.1 THE BASIC RELATIONS

The basic relations are defined in the obvious way from the notions of 4-models and 3-models:

Definition 4 ($\models^4$-inference and $\models^3$-inference) *Let Σ and γ be formulas from $PROP^4_{PS}$ (resp. $PROP^3_{PS}$). We note $\Sigma \models^4 \gamma$ (resp. $\Sigma \models^3 \gamma$) iff every 4-model (resp. 3-model) of Σ is a 4-model (resp. 3-model) of γ.*

Example 2 *Let $\Sigma = a \wedge \neg a \wedge b$. We have $\Sigma \models^4 a$, $\Sigma \models^4 \neg a$ and $\Sigma \models^4 b$, but we do* **not** *have $\Sigma \models^4 \neg b$, nor $\Sigma \models^4 c$. Similar conclusions can be drawn using $\models^3$ instead of $\models^4$. However, both relations do not coincide: we have $true \models^3 a \vee \neg a$ while we do* **not** *have $true \models^4 a \vee \neg a$ (interpreting a as $\perp$, $a \vee \neg a$ is interpreted as $\perp$ as well).*

As evoked before, both J_3, LP, RP and Levesque's logic of 3-inference can be viewed as restricted cases of $THREE$ (what differs is the underlying language which is a subset of $PROP^3_{PS}$):

Definition 5 ($\models_{J_3}$-, $\models_L$- and $\models_{LP}$-inference) *Let Σ and γ be formulas from the language of J_3 (resp. Levesque's logic of 3-inference, LP) as stated in Section 2. We note $\Sigma \models_{J_3} \gamma$ (resp. $\Sigma \models_L \gamma$, $\Sigma \models_{LP} \gamma$) iff every 3-model of Σ is a 3-model of γ.*

A similar definition can be given for $\models_{RP}$. Since $\models_{RP} = \models_{LP}$ holds, we will mainly focus on $\models_{LP}$ in the following.

[2]Note nevertheless that these logics are typically not self-extensional, which implies that only a weak replacement theorem holds (despite the fact that all connectives are truth functional ones). This can be explained by the fact that two truth values are designated. For instance, while $a \vee \neg a \equiv^3 b \vee \neg b$ holds, we do **not** have $\neg(a \vee \neg a) \equiv^3 \neg(b \vee \neg b)$.

Each of the relations $\models^4$, $\models^3$, $\models_{J_3}$, $\models_L$ and $\models_{LP}$ is paraconsistent [Arieli and Avron, 1998], i.e., whenever $\Sigma \in PROP^2_{PS}$ has no 2-model, it does **not necessarily** [3] mean that every formula γ from $PROP^2_{PS}$ satisfies $\Sigma \models^4 \gamma$, $\Sigma \models^3 \gamma$, $\Sigma \models_{J_3} \gamma$, $\Sigma \models_L \gamma$, or $\Sigma \models_{LP} \gamma$. However, they are **too cautious**. For instance, the disjunctive syllogism inference rule of classical logic is not satisfied by any of them. Thus, we do **not** have

$$p \wedge (\neg p \vee q) \models^4 q.$$

This is due to the fact that every 4-interpretation I s.t. $I(p) = \top$ and $I(q) = 0$ is a 4-model of $p \wedge (\neg p \vee q)$, but not a 4-model of q. Especially, none of the relations above coincides with classical entailment $\models^2$ on the $PROP^2_{PS}$ fragment in the situation where the database Σ has a 2-model - while we would expect it.

3.2 THE REFINED RELATIONS

In order to circumvent such difficulties, these inference relations have been refined. The key idea is to restrict the set of models under consideration by taking advantage of some *preference criteria*, so as to keep as many information as possible. This led to the paraconsistent inference relations $\models^4_{I1}$, $\models^4_{I2}$, $\models_{LP_m}$, $\models_{BS}$.

Let us first focus on $\models^4_{I1}$ and $\models^4_{I2}$ [Arieli and Avron, 1998]. Corresponding to them are two preference criteria. The first one consists in giving more credit to the 4-models of Σ that minimize the amount of inconsistent beliefs in Σ. The second one leads to prefer the 4-models of Σ which are as close as possible to its classical models. Formally, two partial preorderings $\leq_1$ and $\leq_2$ over the set of 4-interpretations over PS can be used to capture these criteria:

- $I \leq_1 J$ iff $\{x \in PS \mid I(x) \in \{\top\}\} \subseteq \{x \in PS \mid J(x) \in \{\top\}\}$;
- $I \leq_2 J$ iff $\{x \in PS \mid I(x) \in \{\top, \perp\}\} \subseteq \{x \in PS \mid J(x) \in \{\top, \perp\}\}$.

The preferred 4-models of Σ w.r.t. the first (resp. second) preference criterion is the set of 4-models of Σ that are minimal w.r.t. $\leq_1$ (resp. $\leq_2$).

Definition 6 ($\models^4_{I1}$-inference and $\models^4_{I2}$-inference) *Let Σ and γ be formulas from $PROP^4_{PS}$. We note $\Sigma \models^4_{I1} \gamma$ (resp. $\Sigma \models^4_{I2} \gamma$) iff every 4-model of Σ that is minimal in its set w.r.t. $\leq_1$ (resp. $\leq_2$) is a 4-model of γ.*

[3]Trivialization is not avoided in every situation; for instance, the set of (nonstandard) consequences of $false$ is the whole representation language, whatever the inference relation among those considered in the paper - as soon as $false$ belongs to the language, of course.

Example 3 *Let* $\Sigma = a \wedge \neg a \wedge b \wedge (\neg b \vee c)$. *We have* $\Sigma \models^4_{I1} c$ *and* $\Sigma \models^4_{I2} c$ *(while we do* **not** *have* $\Sigma \models^4 c$*).*

In [Arieli and Avron, 1998], it is shown that both $\models^4_{I1}$ and $\models^4_{I2}$ are valuable inference relations since they are (four-valued) preferential, i.e., any of them satisfies reflexivity, left logical equivalence, right weakening, or, cautious left monotonicity and cautious cut. Obviously, each of them is a proper superset of $\models^4$ but the two relations do not coincide (see counterexamples in [Arieli and Avron, 1998]).

The preference criteria encoded by $\leq_1$ and $\leq_2$ have also been considered in a three-valued framework [Priest, 1989; Priest, 1991] in order to design the logic LP_m, less cautious than LP. Formally, the partial preordering $\leq_{LP_m}$ over the set of 3-interpretations over PS defined by $I \leq_{LP_m} J$ iff $\{x \in PS \mid I(x) \in \{\top\}\} \subseteq \{x \in PS \mid J(x) \in \{\top\}\}$ is considered. Obviously, the sole difference between $\leq_1$ - equal to $\leq_2$ when $\perp$ is not allowed - and $\leq_{LP_m}$ is the underlying set of nonclassical interpretations.

Definition 7 ($\models_{LP_m}$-inference)
Let Σ *and* γ *be formulas from the language of* LP_m. *We note* $\Sigma \models_{LP_m} \gamma$ *iff every* 3*-model of* Σ *that is minimal in its set w.r.t.* $\leq_{LP_m}$ *is a* 3*-model of* γ.

Another preference criterion is obtained by giving more importance to the syntax of the database. Indeed, when Σ is a finite *set* of formulas, it is possible to give more credit to the 3-models of Σ that maximize (w.r.t. $\subseteq$) the subset of formulas from Σ that are interpreted to 1. In formal terms, the following partial preordering $\leq_{BS}$ over the set of 3-interpretations over PS is considered: $I \leq_{BS} J$ iff $\{\phi \in \Sigma \mid I(\phi) = 1\} \supseteq \{\phi \in \Sigma \mid J(\phi) = 1\}$.

Definition 8 ($\models_{BS}$-inference)
Let Σ *be a finite set of formulas and* γ *be a formula from the language considered in [Besnard and Schaub, 1997]. We note* $\Sigma \models_{BS} \gamma$ *iff every* 3*-model of every formula from* Σ *that is minimal in its set w.r.t.* $\leq_{BS}$ *is a* 3*-model of* γ.

Example 4 *Let* $\Sigma = \{a, \neg a, (a \vee b), (\neg a \vee b)\}$. *We have* $\Sigma \models_{BS} b$ *(while we do* **not** *have* $\Sigma \models_{LP_m} b$ *or* $\Sigma \models^4_{I1} b$ *or* $\Sigma \models^4_{I2} b$*).*

It is easy to observe that both $\models^4_{I1}$, $\models^4_{I2}$, $\models_{LP_m}$ and $\models_{BS}$ are less cautious inference relations than those considered before. Especially, each of them coincides with classical entailment $\models^2$ under the CNF fragment provided that the database Σ has a 2-model and non-tautological queries γ are considered.

To conclude this section, let us also consider some paraconsistent inference relations introduced in [Besnard and Schaub, 1998]. Prima facie, this work is not directly relevant to the multivalued logic approach to inconsistency handling but to the approach based on the selection of consistent subsets. We show nevertheless that it is closely connected to LP_m.

The language considered in [Besnard and Schaub, 1998] is the propositional language over PS generated from the connectives $\neg$, $\wedge$, $\vee$, $\supset$ and $\Leftrightarrow$, where $\supset$ and $\Leftrightarrow$ are classically interpreted. As in [Besnard and Schaub, 1998], we will assume in the following that the database Σ is a *NNF formula* from $PROP^2_{PS}$, i.e., a formula built up from the connectives $\neg$, $\wedge$, $\vee$, only, and for which the scope of every occurrence of $\neg$ is a propositional symbol from PS[4]. Every formula Σ is associated to a default theory $\langle \Sigma^\pm, D_\Sigma \rangle$ where:

- $\Sigma^\pm$ is a formula in the language $PROP^2_{PS^\pm}$ where $PS^\pm = \{x^+ \mid x \in PS\} \cup \{x^- \mid x \in PS\}$; $\Sigma^\pm$ is obtained by replacing in Σ every occurrence of a positive literal x by the positive literal x^+ and every occurrence of a negative literal $\neg x$ by the positive literal x^-.

- $D_\Sigma = \{\delta_x \mid x \in PS\}$ is a set of default rules

$$\delta_x = \frac{: x^+ \Leftrightarrow \neg x^-}{(x \Leftrightarrow x^+) \wedge (\neg x \Leftrightarrow x^-)}$$

In this framework, negation is given a special treatment; a first step consists in making every literal independent of its negation through renaming; then the corresponding dependence relations are reintroduced in a parsimonious way, so that inconsistency is avoided. Based on the extensions of the default theory $\langle \Sigma^\pm, D_\Sigma \rangle$, several paraconsistent consequence relations can be defined. Among them are $\vdash_s$ and $\vdash^\pm_s$, defined as follows.

Definition 9 ($\vdash_s$-inference and $\vdash^\pm_s$-inference)
Let Σ *and* γ *be two formulas from* $PROP^2_{PS}$.

- γ *is a* skeptical unsigned consequence *of* Σ, *noted* $\Sigma \vdash_s \gamma$, *iff* γ *belongs to every extension of* $\langle \Sigma^\pm, D_\Sigma \rangle$.

- γ *is a* skeptical signed consequence *of* Σ, *noted* $\Sigma \vdash^\pm_s \gamma$, *iff* $\gamma^\pm$ *belongs to every extension of* $\langle \Sigma^\pm, D_\Sigma \rangle$.

Example 5 *Let* $\Sigma = (\neg a \vee b) \wedge a \wedge \neg b \wedge c \wedge (\neg c \vee d)$.

[4] The NNF assumption can be relaxed through the notion of polarity (see [Besnard and Schaub, 1998]) but every occurrence of a subformula $\phi \Leftrightarrow \psi$ must be replaced first by $(\phi \supset \psi) \wedge (\psi \supset \phi)$.

- $\Sigma \vdash_s c \wedge d \wedge (a \vee \neg b)$ *but* $\Sigma \not\vdash_s a$, $\Sigma \not\vdash_s \neg b$ *and* $\Sigma \not\vdash_s (\neg a \vee b)$.
- $\Sigma \vdash_s^{\pm} c \wedge d \wedge a \wedge \neg b \wedge (\neg a \vee b)$.

Both inference relations coincide with classical entailment as soon as Σ has a 2-model. The following proposition shows that $\vdash_s^{\pm}$ actually coincides with $\models_{LP_m}$:

Proposition 1 *Let Σ and γ be formulas from $PROP_{PS}^2$*[5]*. We have $\Sigma \vdash_s^{\pm} \gamma$ iff $\Sigma \models_{LP_m} \gamma$.*

4 COMPLEXITY RESULTS

In the following, the complexity of the inference problems for the various paraconsistent inference relations considered above is analyzed. We assume that the reader is familiar with some basic notions of computational complexity (see [Papadimitriou, 1994] for details), especially the complexity classes P, NP, coNP and the complexity classes Σ_2^p and Π_2^p of the polynomial hierarchy.

For each inference relation $\vdash$, the input of the decision problem is a pair $\langle \Sigma, \gamma \rangle$ of formulas from the language of the logic under consideration, and the question is "does $\Sigma \vdash \gamma$ holds?". Three cases are successively considered, from the more general one to the more specific one:

- general case: no restriction is put over Σ and γ;
- $\{\neg, \wedge, \vee\}$ fragment: both Σ and γ are required to be generated using these three connectives only;
- Σ CNF and γ propositional symbol. Abusing words, this means that Σ is a finite set of CNF formulas (and γ a symbol) when $\models_{BS}$ is considered.

4.1 MAIN RESULTS

Our main purpose here is to complete the complexity results reported in [Levesque, 1989] and [Cadoli and Schaerf, 1996], which mainly concern the basic relations considered in the paper ($\models^4$, $\models^3$, $\models_{J_3}$, $\models_{LP}$ and $\models_L$). We have obtained the next proposition:

Proposition 2

1. *Let Σ and γ be two formulas from $PROP_{PS}^4$ (resp. $PROP_{PS}^3$, J_3, LP and Levesque's logic) and $\vdash$ be $\models^4$ (resp. $\models^3$, $\models_{J_3}$, $\models_{LP}$ and $\models_L$). Then $\Sigma \overset{?}{\vdash} \gamma$ is coNP-complete, even if Σ and γ belongs to the $\{\neg, \wedge, \vee\}$ fragment. For Σ in CNF and γ a symbol, $\Sigma \overset{?}{\vdash} \gamma$ is in P.*
2. *Let Σ and γ be two formulas from $PROP_{PS}^4$ (resp. $PROP_{PS}^4$, LP_m, the language given in [Besnard and Schaub, 1997], $PROP_{PS}^2$ and $PROP_{PS}^2$) and $\vdash$ be $\models_{I1}^4$ (resp. $\models_{I2}^4$, $\models_{LP_m}$, $\models_{BS}$, $\vdash_s^{\pm}$ and $\vdash_s$). Then $\Sigma \overset{?}{\vdash} \gamma$ is Π_2^p-complete, even if Σ is in CNF and γ is a symbol.*

Accordingly, the inference problem for the basic relations considered in the paper is just *as hard* as the inference problem for classical entailment in the general case, and *even easier* (unless $\mathsf{P} = \mathsf{NP}$) in the restricted case where Σ is a CNF formula and γ is a propositional symbol. Furthermore, all the complexity results reported in the rightmost column of Table 2 still hold if γ is a CNF formula.

Contrastingly, our results show that the inference problem for the refined relations ($\models_{I1}^4$, $\models_{I2}^4$, $\models_{LP_m}$, $\models_{BS}$) as well as for $\vdash_s^{\pm}$ and $\vdash_s$, is *strictly harder* than the inference problem for classical entailment in the general case (unless the polynomial hierarchy collapses). This is not very surprising due to the preferred models characterization of such relations (intuitively, it is not sufficient to consider all the 4-models or the 3-models of Σ but some computational effort must be spent to select the preferred ones); indeed, a similar computational shift can be observed in classical logic when minimal models are focused on (for various minimality criteria, e.g., those at work in closed world reasoning).

More surprisingly, the Π_2^p-hardness result still holds when Σ is a CNF formula and γ is a propositional symbol, while the corresponding inference problem is in P when no preference over the set of nonclassical interpretations is taken into account. Let us consider $\models^4$ and $\models_{I1}^4$ to give an intuition about it. We know that $\Sigma \models^4 \gamma$ holds iff the formula $\Sigma_{\neg\gamma}$ obtained by replacing in Σ every negated (resp. positive) occurrence of the propositional symbol γ by *true* (resp. *false*) has no 4-model [Cadoli and Schaerf, 1996]; this can be checked in time linear in $|\Sigma_{\neg\gamma}|$ since it is equivalent to check whether $I_{max}(\Sigma_{\neg\gamma}) \notin \{1, \top\}$, where I_{max} is the greatest 4-interpretation over PS w.r.t. $\leq_1$ (i.e., for every symbol $x \in PS$, we have $I_{max}(x) = \top$). The point is that such a characterization result does not hold any longer when the minimal 4-models of Σ w.r.t. $\leq_1$ are selected. On the one hand, there can be exponentially many preferred 4-models, and all of them must be taken into account (one source of com-

[5] Note that $\supset$ and $\Leftrightarrow$ are considered as syntactic sugars here ($\phi \supset \psi =_{def} \neg\phi \vee \psi$ and $\phi \Leftrightarrow \psi =_{def} (\phi \supset \psi) \wedge (\psi \supset \phi)$).

Table 2: Complexity results for paraconsistent inference.

$\Sigma \overset{?}{\vdash} \gamma$	**general case**	$\{\neg, \wedge, \vee\}$ **fragment**	Σ **CNF and** γ **symbol**
$\models^4$	coNP-complete	coNP-complete	in P
$\models^3$	coNP-complete	coNP-complete	in P
$\models_{J_3}$	coNP-complete	coNP-complete	in P
$\models_{LP}$	coNP-complete	coNP-complete	in P
$\models_L$	coNP-complete	coNP-complete	in P
$\models^4_{I1}$	Π^p_2-complete	Π^p_2-complete	Π^p_2-complete
$\models^4_{I2}$	Π^p_2-complete	Π^p_2-complete	Π^p_2-complete
$\models_{LP_m}$	Π^p_2-complete	Π^p_2-complete	Π^p_2-complete
$\models_{BS}$	Π^p_2-complete	Π^p_2-complete	Π^p_2-complete
$\vdash^{\pm}_s$	Π^p_2-complete	Π^p_2-complete	Π^p_2-complete
$\vdash_s$	Π^p_2-complete	Π^p_2-complete	Π^p_2-complete

plexity). On the other hand, determining whether a given 4-model I of Σ is not minimal w.r.t. $\leq_1$ is hard as well, since exponentially many candidates must be taken into account in the worst case to find a certificate (a second source of complexity, independent of the first one).

Additionally, as a direct corollary to the proposition above, we obtained that skeptical inference from a default theory is Π^p_2-hard, even when the default theory $\langle \Sigma, D^{sn}_{\Sigma} = \{\delta^{sn}_x \mid x \in PS\}\rangle$ has a very specific format, namely it is a *supernormal default theory* in which each default is of the form

$$\delta^{sn}_x = \frac{: x^+ \Leftrightarrow \neg x^-}{x^+ \Leftrightarrow \neg x^-}$$

and Σ is a monotone CNF formula (i.e., it contains only positive literals). Accordingly, our complexity results complete known hardness results for skeptical inference from a default theory [Gottlob, 1992; Stillman, 1992].

We also derived the following proposition:

Proposition 3 *The inference problem for* $\models^4$, $\models^3$, *and* $\models_{J_3}$ *is* coNP*-complete in the restricted case* γ *is a symbol.*

Note that if Σ is from the $\{\neg, \wedge, \vee\}$ fragment, the complexity falls down to P [Cadoli and Schaerf, 1996]. coNP-hardness is the price to be paid for the improvement of expressive power achieved by the incorporation of the $\supset$ connective in the language in the presence of $false$ (none of $\supset$ or $false$ can be derived using the $\wedge$, $\vee$, $\neg$ connectives in any of the multivalued logics considered in this paper) and the possibility $\supset$ offers to draw some nontrivial inferences in *FOUR* and *THREE* (unlike disjunctive syllogism, the well-known *modus ponens* rule holds in these logics).

4.2 IMPACT ON OTHER RELATIONS

Interestingly, our complexity results can be easily extended to other three-valued or four-valued paraconsistent inference relations, closely related to those analyzed in the paper. Indeed, let us consider the following basic inference relations (Σ and γ are formulas from $PROP^4_{PS}$):

- $\Sigma \models^4_{\supset} \gamma$ iff $\models^4 \Sigma \supset \gamma$;
- $\Sigma \models^4_{inc.} \gamma$ iff $\Sigma \wedge \neg\gamma$ has no 4-model;
- $\Sigma \models^4_{\leq_t} \gamma$ iff for every 4-interpretation I over PS, we have $I(\Sigma) \leq_t I(\gamma)$, where $\leq_t$ is Belnap's truth ordering defined by the reflexive-transitive closure of $<_t$ s.t. $0 <_t \bot$, $0 <_t \top$, $\bot <_t 1$ and $\top <_t 1$;
- $\Sigma \models^4_{true} \gamma$ iff for every 4-model I of Σ, we have $I(\gamma) = 1$.

All these relations correspond to different ways of defining a notion of consequence within a multivalued logic, and they coincide in classical logic.

It is easy to show that $\models^4_{\supset} = \models^4$. In the general case, we have $\models^4_{true} \subset \models^4_{\leq_t} \subset \models^4$. We also have $\models^4_{true} \subset \models^4_{inc.}$. $\models^4_{inc.}$ and $\models^4_{\leq_t}$ (resp. $\models^4$) are incomparable w.r.t. $\subseteq$ in the general case.

We can also define the corresponding three-valued inference relations (just by replacing every 4 by a 3 above). In this situation, $\models^3_{\supset} = \models^3$ and $\models^3_{inc.} = \models^3_{true}$, and we also have $\models^3_{true} \subset \models^3_{\leq_t} \subset \models^3$ in the general case. Taking advantage of our results, we can prove that the decision problems associated to all these relations are coNP-complete, even in the restricted case γ is a propositional symbol.

Based on similar ideas, some refined relations can be defined. Let $*$ be among 1, 2; we let:

- $\Sigma \models^4_{I*,\leq_t} \gamma$ iff for every 4-model I of Σ that is minimal w.r.t. $\leq_*$ in its set, we have $I(\Sigma) \leq_t I(\gamma)$.
- $\Sigma \models^4_{I*,true} \gamma$ iff for every 4-model I of Σ that is minimal w.r.t. $\leq_*$ in its set, we have $I(\gamma) = 1$.

Such refinements lead to more and more cautious relations since in the general case, we have $\models^4_{I*,true} \subset \models^4_{I*,\leq_t} \subset \models^4_{I*}$. Again, the corresponding three-valued inference relations can be obtained by replacing every 4 by a 3 just above (in this case, there is no distinction between $* = 1$ and $* = 2$) and the inclusions above still hold in this case. Unsurprisingly, we can prove that the decision problems associated to all these refined relations are Π^p_2-complete, even in the restricted case Σ is a CNF formula and γ a propositional symbol.

Other paraconsistent inference relations can be tentatively defined by increasing the number of truth values; we then enter the framework of *logical bilattices*, i.e., bilattices on which the sets of designated truth values are prime bifilters (see [Ginsberg, 1988], [Arieli and Avron, 1998]). It has been shown that such an extension of the number of truth values does not add much. Indeed, the corresponding basic inference relations coincide with $\models^4$ (Theorem 6.17 from [Arieli and Avron, 1998]), so our complexity results concerning $\models^4$ immediately apply to them. Furthermore, while extensions to $\models^4_{I1}$ and $\models^4_{I2}$ can be envisioned by preferring models minimizing the set of propositional symbols assigned to an "inconsistent truth value", i.e., those belonging to a given inconsistency set I (see [Arieli and Avron, 1998] for details), it has been shown that this does not lead to inference relations that differ from $\models^4_{I1}$ or $\models^4_{I2}$ (Theorem 6.25 from [Arieli and Avron, 1998]). Since it is possible to determine which one of $\models^4_{I1}$ or $\models^4_{I2}$ is reached independently from the input of the decision problem (Σ and γ), i.e., just by comparing the inconsistency set I with the set of truth values v of the bilattice s.t. neither v nor $\neg v$ belongs to the prime bifilter under consideration, our Π^p_2-completeness results concerning $\models^4_{I1}$ and $\models^4_{I2}$ still apply to such refined relations.

Finally, it is also possible to take advantage of our results to identify the computational complexity of some inference relations proposed so far in the framework of disjunctive logic programming. A basic motivation is to avoid trivial inference from logic programs that do not have any model under usual semantics, like $\mathcal{LP} = \{a \vee b \leftarrow, \neg a \leftarrow, \neg b \leftarrow, c \leftarrow\}$. In a nutshell, the paraconsistent models of an extended disjunctive logic program $\mathcal{LP}$ as defined in [Sakama and Inoue, 1995] are exactly the 4-models of $\mathcal{LP}$ in which every occurrence of $\leftarrow$ is replaced by $\subset$, and every occurrence of *not* l (where *not* is negation as failure, distinct from $\neg$, the "true" negation) is replaced by $(l \supset false)$; this result can be easily established by structural induction on $\mathcal{LP}$. Our complexity results can be extended to show that the corresponding inference problem is coNP-complete, even in the restricted case the query γ is a propositional symbol. Exploiting Proposition 4.14 from [Arieli and Avron, 1998], coNP-completeness still holds when the minimal paraconsistent models of $\mathcal{LP}$ are selected (see [Sakama and Inoue, 1995] for details) and the query belongs to the $\{\wedge, \vee, \neg\}$ fragment. Contrastingly, the complexity of more interesting inference relations based on paraconsistent stable models cannot be identified directly from our results.

5 CONCLUSION

In this paper, the complexity of several paraconsistent inference relations based on multivalued logics has been investigated. A number of inference relations pointed out by Arieli and Avron, D'Ottaviano and da Costa, Priest, Besnard and Schaub, Levesque have been considered from the computational side, in the general case and when some restrictions are put on the database and the query. Our main contribution has been to show that the refined paraconsistent inference relations that have been defined in the framework of multivalued logics are at the second level of the polynomial hierarchy, even in some quite restricted cases. This is where the complexity of many inconsistent-tolerant inference relations proposed so far in the literature lies (especially, those based on the selection of maximally consistent subsets of the database) as well as many other forms of inference considered by the AI community (see e.g., [Cadoli and Schaerf, 1993]). Another contribution has been to show close relationships between some of the relations considered in the paper. Specifically, we proved that $\models_{LP_m}$ coincides with $\vdash^{\pm}_s$ on their common language. On the one hand, this gives a multivalued semantics to $\vdash^{\pm}_s$. On the other hand, this equivalence gives a default logic characterization to LP_m inference. This is particularly helpful from the practical side since the default theory $\langle \Sigma^{\pm}, D_\Sigma \rangle$ can be replaced by the supernormal default theory $\langle \Sigma^{\pm}, D^{sn}_\Sigma \rangle$, without questioning the set of (skeptical) signed consequences (see Theorems A.1 and A.2 from [Besnard and Schaub, 1998] for details). Furthermore, several implementations of this simple fragment of default logic – like Theorist [Poole, 1988] – exist. Finally, we have also presented some complexity results for related inference relations, both basic and refined ones.

There are at least two ways to interpret the complexity results we gave. A pessimistic interpretation is that none of the inference relations considered in the paper is really satisfying, since either it is very cautious or it is highly intractable. A more optimistic interpretation is reached when intractability is viewed as a feature and not as a drawback. Observing that $PROP^4_{PS}$ and $PROP^3_{PS}$ are strictly more expressive than $PROP^2_{PS}$ and the CNF fragment of it, our analysis shows that, without any extra computational cost, we can take advantage of quite expressive languages (like full $PROP^4_{PS}$ instead of the language of LP) to encode the database.

This work calls for several perspectives. One of them consists in investigating restrictions on the database that would lead to more tractable inference relations. Especially, it would be interesting to consider the case Σ is a Horn CNF formula (since the satisfiability problem for such formulas is tractable in classical logic). Another way to circumvent the intractability would be to restrict the number of preferred symbols (or formulas) when interpreted classically; a preliminary work relevant to such a resource-bounded approach to paraconsistent inference is [Marquis and Porquet, 2001].

Acknowledgements

Many thanks to the anonymous reviewers and to Jérôme Lang for their comments and suggestions.

The authors have been partly supported by the IUT de Lens, the Université d'Artois, the Région Nord/Pas-de-Calais under the TACT-TIC project, and by the European Community FEDER Program.

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Appendix: proof sketches

Proposition 1: First, we exploit the fact that $\Sigma \vdash_s^{\pm} \gamma$ iff $\gamma^{\pm}$ belongs to every extension of the supernormal default theory $T = \langle \Sigma^{\pm}, \{\frac{: x^+ \Leftrightarrow \neg x^-}{x^+ \Leftrightarrow \neg x^-} \mid x \in PS\}\rangle$ (see Theorems A.1 and A.2 from [Besnard and Schaub, 1998]). Then we take advantage of the fact that $\Sigma^{\pm}$ is a monotone (positive) formula to show that every preferred model I of T is s.t. for every $x \in PS$, we do **not** have $I(x^+) = I(x^-) = 0$. Accordingly, we can associate to each such preferred model I of T over $PS^{\pm}$ in a bijective way the 3-model $3(I)$ of Σ over PS s.t. $3(I)(x) = I(x^+)$ and $3(I)(\neg x) = I(x^-)$. It remains to show that $3()$ is a morphism (i.e., the preference ordering between interpretations is preserved) to conclude the proof. ■

Proposition 2:

- Membership: The membership-to-P results are easy consequences from [Cadoli and Schaerf, 1996]. As to $\vdash_s$ and $\vdash_s^{\pm}$, the membership to Π_2^p comes directly from Theorem 5.2 from [Gottlob, 1992] (see also [Stillman, 1992]). The membership-to-coNP results are not very difficult: just consider the complementary problems and solve them by guessing an interpretation and verifying that it is a model of Σ but not a model of γ. A similar approach can be followed for the remaining results; the unique difference is that the guessed interpretation must be shown preferred using a call to an NP oracle; this is easy to do by considering the complementary problem (if an interpretation is not preferred then another interpretation can be guessed and checked as strictly more preferred than the first one in deterministic polynomial time).

- Hardness: coNP-hardness results are easy consequences from [Cadoli and Schaerf, 1996]. For Π_2^p-hardness results, we exploit the fact that both $\models_{I1}^4$, $\models_{I2}^4$ and $\vdash_s^{\pm}$ coincide with $\models_{LP_m}$ whenever Σ is a CNF formula and γ a symbol. We proceed through a number of lemmata to prove that determining whether $\Sigma \not\models_{LP_m} \gamma$ is Σ_2^p-hard whenever Σ is a CNF formula and γ is a symbol. The key ones are:

  1. Given a finite set Σ of clauses from $PROP_{PS}^2$, we show that determining whether $x \in Var(\Sigma)$[6] belongs to at least one var-conflict of Σ is Σ_2^p-hard. A var-conflict is a minimal (w.r.t. $\subseteq$) set of variables occurring in a conflict of Σ (i.e., a minimal (w.r.t. $\subseteq$) subset of Σ that has no 2-model). To achieve this Σ_2^p-hardness proof, we reduce the complementary of WIDTIO inference problem [Winslett, 1990] to the latter problem, taking advantage of Lemma 6.2 and Theorem 8.2 from [Eiter and Gottlob, 1992].
  2. Then, we show that $x \in Var(\Sigma)$ belongs to at least one var-conflict of Σ iff Σ has a 3-model I minimal w.r.t. $\leq_{LP_m}$ s.t. $I(x) = \top$.

  Finally, we reduce the inference problem in LP_m to the one in the approach of Besnard and Schaub [Besnard and Schaub, 1997] and to the skeptical unsigned inference problem [Besnard and Schaub, 1998].

■

Proposition 3: The membership results follow from the previous proposition. Hardness is obtained by reduction from UNSAT, the problem of determining whether a CNF formula from $PROP_{PS}^2$ has no 2-model, and is achieved by taking advantage of the $\supset$ connective.

[6] $Var(\Sigma)$ is the subset of propositional symbols occurring in Σ.