Recovering Consistency by Forgetting Inconsistency

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Abstract. In this paper, we introduce and study a new paraconsistent inference relation $\models_c$ in the setting of 3-valued paraconsistent logics. Using inconsistency forgetting as a key mechanism for recovering consistency, it guarantees that the deductive closure $Cn_{\models_c}(\Sigma)$ of any belief base Σ is classically consistent and classically closed. This strong feature, not shared by previous inference relations in the same setting, allows to interpret an inconsistent belief base as a set of classical worlds (hence to reason classically from them).

1 Introduction

Reasoning in a non-trivial way from inconsistent pieces of information (the paraconsistency issue) is a fundamental problem in artificial intelligence. Its importance is reflected by the number of approaches developed so far to address it: paraconsistent logics, belief revision, belief merging, reasoning from preferred consistent subsets, knowledge integration, argumentative logics, purification, etc. (see [1–3] for a survey).

The variety of existing approaches can be explained by the fact that paraconsistency can be achieved in various ways, depending on the exact nature of the problem at hand (hence, the available information). Each of them has its own pros and cons, and is more or less suited to different inconsistency handling scenarios. For instance, when Σ represents the (conflicting) beliefs of several agents, a merged base giving the beliefs of the group of agents can be designed by logically weakening some local belief bases (associated to the agents) in order to restore global consistency [4–8].

Compared with the other approaches listed above, paraconsistent logics (taken *stricto sensu*) offer a *basic* way to address the trivialization issue in presence of inconsistency. Indeed, belief revision, belief merging, knowledge integration, reasoning from preferred consistent subsets and purification need some extra-logical information in order to be well-defined and avoid trivializing. Such extra-logical information can be rather poor: A splitting between the belief base and the revision formula in the belief revision setting, a set (or multi-set) organization of the beliefs in a belief merging scenario. They can also be rather sophisticated: Preference relations over beliefs, knowledge gathering actions for purification. In both cases, they are required. In particular, unlike paraconsistent logics, none of those approaches can address in a significant way the case when the available information take the form of a *single* piece (hence encoded as a unique formula in a logical language)[1].

Several (non mutually exclusive) techniques can be used to define an inference relation that avoid trivialization from an inconsistent propositional formula (see [3]). One of them consists in *preventing classically inconsistent belief bases from having no model*, through the consideration of more general notions of interpretations. Several multi-valued logics are related to this line of research (among others, see [10–22]).

In the following, the focus is laid on *three-valued paraconsistent logics*. The additional (epistemic) truth value (called middle element) intuitively means "proved both true and false" and allows to still reasoning meaningfully with variables that are not embedded directly in a contradiction. While a number of paraconsistent inference relations have been defined in this setting, none of them ensures that deductive closures are always classically consistent and classically closed. This is a strong drawback of such approaches since it prevents from interpreting inconsistent belief bases as sets of classical worlds (i.e., 2-interpretations), and consequently it questions the possibility to exploit further the information encoded by an inconsistent belief base using standard inference or decision-making techniques (since such techniques typically require classically consistent information).

In this paper, we fill the gap by introducing and studying a new paraconsistent inference relation $\models_c$ in the setting of three-valued paraconsistent logics. This inference relation elaborates on a valuable paraconsistent inference relation $\models^{\leq}$ introduced by Priest [15]. Basically, the preferred 3-models of a belief base Σ w.r.t. $\models_c$ are the 2-interpretations which are as close as possible to the preferred 3-models of a belief base Σ w.r.t. $\models^{\leq}$. Determining the latter models mainly amounts to forgetting the inconsistent "truth value" in the former interpretations. Interestingly, $\models_c$ guarantees that the deductive closure $Cn_{\models_c}(\Sigma)$ of any belief base Σ is classically consistent and classically closed (what we call the *classical closure property*).

The rest of this paper is organized as follows. In Section 2, we present some background on three-valued paraconsistent logics; especially, we define the logical framework into which our inference relation $\models_c$ takes place. In Section 3, we present the classical closure property and show that three-valued paraconsistent inference relations from the literature do not satisfy it. On this ground, we introduce our relation $\models_c$; we show that it satisfies a number of expected logical properties, including the strong paraconsistency condition (i.e., the deductive closure of a belief base never trivializes), the preservation property (i.e., the deductive closure of a belief base coincides with its classical closure when the

[1] Note that approaches based on consistent subsets take advantage of a specific "comma" connective [9] which is not equivalent to conjunction in the general case; every singleton consisting of an inconsistent formula like $\{a \wedge (\neg a \vee b) \wedge c \wedge \neg c\}$ has $\emptyset$ as its unique consistent subset.

belief base is classically consistent), as well as all the properties of system P [23] but reflexivity. We also investigate some computational aspects of $\models_c$, show that it is not harder than the underlying relation $\models^{\leq}$ from a complexity point of view and explain how to turn any finite belief base Σ into a consistent propositional formula $cl(\Sigma)$ such that $Cn_{\models_c}(\Sigma)$ is equal to the classical closure of $cl(\Sigma)$ (thus, $cl(\Sigma)$ can be viewed as a compilation of Σ as a propositional formula, classically interpreted). Finally, Section 4 concludes the paper. For space reasons, some proofs are omitted. However, they are given in [24], available from the authors.

2 Three-valued Paraconsistent Logics

When a belief base is classically inconsistent, every formula is a classical consequence of it (*"ex falso quodlibet sequitur"*). In order to avoid such a trivialization, one can take advantage of any logic in which an (epistemic) truth value "both" ($\top$) denotes that a formula can be proved at the same time "true" (1) and "false" (0). This allows to highlight contradictory pieces of information, but still reasoning "reasonably" about the remaining ones. Thus the third truth value has to be understood as some encoding of the epistemic attitude "proved both true and false", and not as a standard truth value.

Now, there are a number of many-valued paraconsistent logics where such an (epistemic) truth value "both" is considered. In the following, we consider Kleene's strong three-valued logic with middle element designated, restricted to the so-called monotone fragment [18], i.e., the morphology of the language of the logic is reduced to the connectives $\neg$, $\vee$, $\wedge$, only. When restricted to this fragment, this logic is equivalent to a number of other logics pointed out so far in the literature, including LP [14], J_3 [10], $THREE$ [18] and other logics by Levesque [13] and Frisch [12].

Definition 1 (language). *$PROP_{PS}$ is the propositional language generated from a finite set PS of propositional symbols, the unary connective $\neg$ (negation) and the binary connectives $\vee$ (disjunction), and $\wedge$ (conjunction).*

Clearly, this language coincides with a standard language for classical propositional logic.

We will write propositional symbols a, b, ... and formulas from $PROP_{PS}$ will be denoted by lower case Greek letters α, β, ... Belief bases, that will be denoted by upper case Greek letters Σ,... are (conjunctively-interpreted) sets of formulas. In order to alleviate notations, we identify every singleton belief base $\{\alpha\}$ with the formula α in it. $Var(\Sigma)$ denotes the set of propositional symbols occurring in Σ.

A literal is a symbol $x \in PS$ or a negated one $\neg x$. x and $\neg x$ are said to be complementary literals. A proper subset of $PROP_{PS}$ is composed by the CNF formulas, i.e., the (finite) conjunctions of clauses, where a clause is a (finite) disjunction of literals. Another proper subset of $PROP_{PS}$ is composed by the DNF formulas, i.e., the (finite) disjunctions of terms, where a term is a (finite) conjunction of literals.

In the following, we consider a number of inference relations $\vdash$ over $PROP_{PS}$:

Definition 2 (inference relation).

- *An* inference relation $\vdash$ *is a subset of* $2^{PROP_{PS}} \times PROP_{PS}$.
- *For every* Σ *in* $2^{PROP_{PS}}$, $Cn_{\vdash}(\Sigma)$ *denotes the* deductive closure *of a set of formulas* Σ *w.r.t. the inference relation* $\vdash$*, i.e.,* $Cn_{\vdash}(\Sigma) = \{\alpha \in PROP_{PS} \mid \Sigma \vdash \alpha\}$.

We will also need the following notions of interpretations:

Definition 3 (interpretations).

- *A* 3-interpretation ω *over* $PROP_{PS}$ *is a total function from* PS *to* $\{0, 1, \top\}$.
- *A* 2-interpretation ω *over* $PROP_{PS}$ *is a total function from* PS *to* $\{0, 1\}$.

$3 - \Omega$ (resp. $2 - \Omega$) denotes the set of all 3-interpretations (resp. 2-interpretations). 2-interpretations are the classical worlds. Clearly, they are also 3-interpretations. However, the converse does not hold (we have $2 - \Omega \subset 3 - \Omega$).

In the logic under consideration, all the connectives are truth functional ones and the semantics $\omega(\alpha)$ of a formula α from $PROP_{PS}$ in a 3-interpretation ω is defined in the obvious compositional way given the following truth tables.

α	β	$\neg\alpha$	$\alpha \wedge \beta$	$\alpha \vee \beta$
0	0	1	0	0
0	1	1	0	1
0	$\top$	1	0	$\top$
1	0	0	0	1
1	1	0	1	1
1	$\top$	0	$\top$	1
$\top$	0	$\top$	0	$\top$
$\top$	1	$\top$	$\top$	1
$\top$	$\top$	$\top$	$\top$	$\top$

Table 1. Truth tables.

It is easy to check that restricting the entries of the previous table to 0 and 1, one recovers the standard semantics for the connectives $\neg$, $\vee$, $\wedge$. Accordingly, a belief base can be considered classically unless it becomes inconsistent (typically via its expansion by a new, yet conflicting, piece of evidence).

In classical logic, notions of model and consequence are defined as:

Definition 4 ($\models_2$). *Let* ω *be a 2-interpretation over* $PROP_{PS}$*. Let* α *be a formula from* $PROP_{PS}$*, and let* Σ *be a set of formulas of* $PROP_{PS}$*:*

- ω *is a* 2-model *of* α *iff* $\omega(\alpha) = 1$.
- ω *is a* 2-model *of* Σ *iff* $\omega(\alpha) = 1$ *for every* $\alpha \in \Sigma$. $2 - mod(\Sigma)$ *denotes the set of* 2*-models of* Σ.

– α *is a* 2-consequence *of* Σ*, noted* $\Sigma \models_2 \alpha$*, iff every* 2*-model of* Σ *is a* 2*-model of* α*.*

A belief base Σ is *classsically consistent* iff it has a 2-model iff $Cn_{\models_2}(\Sigma) \neq PROP_{PS}$. It is well-known that $\models_2$ is not strongly paraconsistent:

Definition 5 (strong paraconsistency). *An inference relation* $\vdash$ *satisfies the* strong paraconsistency property *iff for every* Σ *in* $2^{PROP_{PS}}$, $Cn_{\vdash}(\Sigma) \neq PROP_{PS}$.

When dealing with more than two truth values, one has to make precise the set of designated values, i.e., the set of values that a formula can take to be considered as satisfied. Since we want to define a paraconsistent logic, we choose $\mathcal{D} = \{1, \top\}$: intuitively, a formula is satisfied if it is "at least true" (but it can also be false!). We are now ready to extend the previous notions of model and consequence to the three-valued case:

Definition 6 ($\models_3$). *Let* ω *be a* 3*-interpretation over* $PROP_{PS}$*. Let* α *be a formula from* $PROP_{PS}$*, and let* Σ *be a set of formulas of* $PROP_{PS}$*:*

– ω *is a* 3-model *of* α *iff* $\omega(\alpha) \in \mathcal{D}$*.*
– ω *is a* 3-model *of* Σ *iff* $\omega(\alpha) \in \mathcal{D}$ *for every* $\alpha \in \Sigma$*.* $3 - mod(\Sigma)$ *denotes the set of* 3*-models of* Σ*.*
– α *is a* 3-consequence *of* Σ*, noted* $\Sigma \models_3 \alpha$*, iff every* 3*-model of* Σ *is a* 3*-model of* α*.*

Two formulas α and β are said to be *strongly (*3*-)equivalent* iff for every 3-interpretation ω, we have $\omega(\alpha) = \omega(\beta)$.

Unlike $\models_2$, an interesting feature of the inference relation $\models_3$ is that it is strongly paraconsistent; indeed, every formula from $PROP_{PS}$ has a 3-model (the 3-interpretation $\omega_\top$ such that $\forall x \in PS$, $\omega_\top(x) = \top$). Thus, while we have $a \wedge \neg a \models_2 b$, we do *not* have $a \wedge \neg a \models_3 b$.

A problem is that $\models_3$ is a very weak inference relation. Especially, it is well-known that the disjunctive syllogism is not satisfied by $\models_3$: $a \wedge (\neg a \vee b) \not\models_3 b$. Thus, $\models_3$ does not satisfy the expected preservation property:

Definition 7 (preservation). *An inference relation* $\vdash$ *satisfies the* preservation property *iff for every* Σ *in* $2^{PROP_{PS}}$*, if* Σ *is classically consistent, then* $Cn_{\vdash}(\Sigma) = Cn_{\models_2}(\Sigma)$*.*

In order to circumvent this difficulty, other three-valued paraconsistent inference relations have been proposed. Some of them are based on the following principle: focus on some preferred models of Σ in order to keep as much information as possible. Thus, in LP_m [15], Priest suggests to prefer those 3-models of a belief base Σ which are "as classical as possible". Formally, let us consider the partial preordering $\leq$ over the set of 3-interpretations defined by $\omega \leq \omega'$ if and only if $\{x \in PS \mid \omega(x) = \top\} \subseteq \{x \in PS \mid \omega'(x) = \top\}$; the "most classical" 3-models of a belief base are the 3-models that are minimal w.r.t. $\leq$:

Definition 8 **($\models^{\leq}$).** *Let Σ be a set of formulas of $PROP_{PS}$. Let α be a formula from $PROP_{PS}$. $\Sigma \models^{\leq} \alpha$ iff $\forall \omega \in \min(3-mod(\Sigma), \leq)$, $\omega(\alpha) \in \mathcal{D}$.*

The resulting relation $\models^{\leq}$ is still strongly paraconsistent and it is strictly less cautious than $\models_3$, i.e., we have the inclusion $\models_3 \subset \models^{\leq}$. Unlike $\models_3$, it is non-monotonic; for instance, we have $a \wedge (\neg a \vee b) \models^{\leq} b$ but $a \wedge (\neg a \vee b) \wedge \neg a \not\models^{\leq} b$. Furthermore, $\models^{\leq}$ satisfies the preservation property: the preferred 3-models w.r.t. $\models^{\leq}$ of any classically consistent belief base Σ are exactly its 2-models.

Other inference relations have been defined so far for refining the inference relation $\models^{\leq}$ (especially in order to discriminate between the 3-consequences of a belief base Σ which are subject to a contradiction – like a if $\Sigma = a \wedge \neg a \wedge b$ – and those which are contradiction-free – like b if $\Sigma = a \wedge \neg a \wedge b$). Here are the main ones [20]:

Definition 9 (refined inference relations). *Let Σ be a set of formulas of $PROP_{PS}$. Let α be a formula from $PROP_{PS}$.*

- *$\Sigma \models^{\leq}_{arg} \alpha$ iff $\Sigma \models^{\leq} \alpha$ and $\Sigma \not\models^{\leq} \neg\alpha$.*
- *$\Sigma \models^{\leq}_{1} \alpha$ iff $\forall \omega \in \min(3-mod(\Sigma), \leq)$, $\omega(\alpha) = 1$.*
- *$\Sigma \models^{\leq}_{t} \alpha$ iff $\forall \omega \in \min(3-mod(\Sigma), \leq)$, $\omega(\Sigma) \leq_t \omega(\alpha)$ where the so-called "truth ordering" $\leq_t$ is such that $0 \leq_t \top \leq_t 1$.*

Those three relations correspond respectively to three refinement principles:

- considering only argumentative consequences of the belief base.
- selecting those consequences of the belief base that are "conflict-free" (i.e., true but not false).
- selecting as consequences of the belief base formulas that are informally "more true" than the belief base.

All those relations are non-monotonic, strongly paraconsistent and they satisfy the preservation property. Furthermore they are strictly more cautious than $\models^{\leq}$ (see [20] for more details).

3 Recovering Consistency by Forgetting Inconsistency

3.1 The inference relation $\models_c$

Now, a major problem with the inference relations considered in the previous section (except $\models_2$ which is not paraconsistent) is that they do not satisfy the classical closure property:

Definition 10 (classical closure). *An inference relation $\vdash$ satisfies the* classical closure property *iff for every Σ in $2^{PROP_{PS}}$, $\vdash(\Sigma)$ is classically consistent and is closed w.r.t. classical deduction, i.e., $Cn_{\models_2}(Cn_{\vdash}(\Sigma)) = Cn_{\vdash}(\Sigma)$.*

This is obvious for $\models_3$, $\models^{\leq}$, and $\models_t^{\leq}$ since those relations are "reflexive" [20], i.e., for every α in $PROP_{PS}$, we have α is a consequence of α w.r.t. the relation. Thus, take $\Sigma = a \wedge \neg a$; Σ has to belong to its deductive closure w.r.t. any of those three relations, hence it cannot be classically consistent. As to $\models_{\text{arg}}^{\leq}$, consider the classically inconsistent CNF formula $\Sigma = (a \vee b) \wedge (\neg a \vee b) \wedge (a \vee \neg b) \wedge (\neg a \vee \neg b)$. Each of the four clauses in it is a consequence of Σ w.r.t. $\models_{\text{arg}}^{\leq}$: since their conjunction Σ is classically inconsistent, it cannot be the case that $Cn_{\models_{\text{arg}}^{\leq}}(\Sigma)$ is classically consistent and closed w.r.t. classical deduction. Finally, one can prove that $Cn_{\models_t^{\leq}}(\Sigma)$ is always classically consistent but this set is not necessarily closed w.r.t. classical deduction: take $\Sigma = a \wedge \neg a$; we have $\Sigma \not\models_t^{\leq} a \vee \neg a$. Since $a \vee \neg a$ is a classical tautology, the conclusion follows.

Using any of those inference relations thus prevents from interpreting inconsistent belief bases as sets of classical worlds (i.e., 2-interpretations), and consequently it questions the possibility to exploit further the information encoded by an inconsistent belief base using standard inference or decision-making techniques (since such techniques typically require classically consistent information). This motivates the introduction of our inference relation $\models_c$.

Intuitively, the preferred 3-models of a belief base Σ w.r.t. $\models_c$ are the 2-interpretations which are as close as possible to the preferred 3-models of a belief base Σ w.r.t. $\models^{\leq}$. Determining the latter models mainly amounts to forgetting the inconsistent "truth value" in the former interpretations. Formally, for any belief base Σ, we define $IncForg(\Sigma)$ as the set of 2-interpretations ω which are as close as possible to a 3-interpretation $\omega' \in \min(3-mod(\Sigma), \leq)$, in the sense that $\forall x \in PS$, if $\omega'(x) \neq \top$, then $\omega'(x) = \omega(x)$. More formally, $IncForg(\Sigma) = \{\omega \in 2-\Omega \mid \exists \omega' \in \min(3-mod(\Sigma), \leq)\ \forall x \in PS$, if $\omega'(x) \neq \top$, then $\omega'(x) = \omega(x)\}$. Computing $IngForg(\Sigma)$ amounts to projecting each preferred 3-models of Σ on the variables classically interpreted in it (hence, forgetting inconsistency) and interpreting the resulting partial interpretations in a classical way. We are now ready to define $\models_c$:

Definition 11 **($\models_c$).** *Let Σ be a set of formulas of $PROP_{PS}$. Let α be a formula from $PROP_{PS}$. $\Sigma \models_c \alpha$ iff $\forall \omega \in IncForg(\Sigma)$, $\omega(\alpha) = 1$.*

Example 1. Let $\Sigma = a \wedge (\neg a \vee b) \wedge c \wedge \neg c$. Assuming that $PS = \{a, b, c\}$, $\min(3-mod(\Sigma), \leq)$ has only one preferred 3-model ω such that $\omega(a) = \omega(b) = 1$ and $\omega(c) = \top$. Accordingly, $IncForg(\Sigma)$ contains two elements ω' and ω'' such that $\omega'(a) = \omega'(b) = \omega''(a) = \omega''(b) = 1$ and $\omega'(c) = 0$ and $\omega''(c) = 1$. As a consequence, we have $\Sigma \models_c a \wedge b$, $\Sigma \not\models_c c$, and $\Sigma \not\models_c \neg c$. This contrasts with $\models^{\leq}$ which is such that $\Sigma \models^{\leq} c \wedge \neg c$.

Clearly enough, $\models_c$ is a non-monotonic inference relation. For instance, we have $a \models_c a$ but $a \wedge \neg a \not\models_c a$.

3.2 Logical properties

We now investigate in more depth the logical properties satisfied by $\models_c$. Interestingly, $\models_c$ compares favourably with the underlying inference relation $\models^{\leq}$ w.r.t.

logical properties: first of all, like $\models^{\leq}$, $\models_c$ also is strongly paraconsistent and satisfies the preservation property. Furthermore, it satisfies the classical closure property:

Proposition 1. *$\models_c$ is strongly paraconsistent and satisfies the preservation property and the classical closure property.*

Proof.
- *Strong paraconsistency:* Direct from the fact that $min(3{-}mod(\Sigma), \leq)$ is not empty whatever the belief base Σ, since this is the case for $3{-}mod(\Sigma)$ and $\leq$ is noetherian since PS is finite.
- *Preservation:* If Σ is classically consistent, then $min(3 - mod(\Sigma), \leq) = 2 - mod(\Sigma)$. Consequently, $IncForg(\Sigma) = 2 - mod(\Sigma)$, conclusion follows.
- *Classical closure:* Since $min(3 - mod(\Sigma), \leq)$ is not empty (see above), this is also the case of $IncForg(\Sigma)$. Hence $\models_c (\Sigma)$ is classically consistent. Since $IncForg(\Sigma) \subseteq 2{-}\Omega$, we obviously have that $\models_c (\Sigma)$ is closed w.r.t. classical deduction: $\models_2 (\models_c (\Sigma)) = \models_c (\Sigma)$.

□

Now, compared with $\models^{\leq}_{\text{arg}}$, $\models^{\leq}_{1}$ and $\models^{\leq}_{t}$, $\models^{\leq}$ exhibits quite a good logical behaviour in the sense that it is a preferential inference relation [20]:

Definition 12 (system P). *An inference relation $\vdash$ is* preferential *iff it satisfies the following properties (system* P*):*

(Ref)	$\alpha \vdash \alpha$	Reflexivity
(LLE)	If α and β are strongly 3-equivalent and $\alpha \vdash \gamma$, then $\beta \vdash \gamma$	Left Logical Equivalence
(RW)	If $\alpha \vdash \beta$ and $\beta \models_3 \gamma$, then $\alpha \vdash \gamma$	Right Weakening
(Or)	If $\alpha \vdash \gamma$ and $\beta \vdash \gamma$, then $\alpha \vee \beta \vdash \gamma$	Or
(Cut)	If $\alpha \wedge \beta \vdash \gamma$ and $\alpha \vdash \beta$, then $\alpha \vdash \gamma$	Cut
(CM)	If $\alpha \vdash \beta$ and $\alpha \vdash \gamma$, then $\alpha \wedge \beta \vdash \gamma$	Cautious Monotony

Following seminal works in non-monotonic logic [25, 26, 23, 27], this set of normative properties that a non-monotonic inference relation should satisfy has been given in [23]. These properties have been primarily stated in the framework of classical logic [23], but they can be extended to multi-valued settings in a straightforward way as above (such an extension has also been considered in [18]).

Thus, an important question is to determine whether going from $\models^{\leq}$ to $\models_c$ leads to lose such valuable logical properties. Fortunately, most important properties still hold but reflexivity:

Proposition 2. *$\models_c$ satisfies all the properties of system* P*, except reflexivity.*

Proof.
- *Reflexivity:* Take $\alpha = a \wedge \neg a$. We have $\alpha \not\models_c \alpha$.
- *Left Logical Equivalence:* Obvious from the fact that (strongly) equivalent formulas have the same 3-models.
- *Right Weakening:* If $\beta \models_3 \gamma$, then $\beta \models_2 \gamma$ due to the inclusion $2 - mod(\beta) \subseteq 3{-}mod(\beta)$. The fact that $\models_c$ satisfies the classical closure property concludes the proof.

– *Or:* We have that $3 - mod(\alpha \vee \beta) = 3 - mod(\alpha) \cup 3 - mod(\beta)$. As a consequence, $min(3 - mod(\alpha \vee \beta), \leq) \subseteq min(3 - mod(\alpha), \leq) \cup min(3 - mod(\beta), \leq)$. Therefore, $IncForg(\alpha \vee \beta) \subseteq IncForg(\alpha) \cup IncForg(\beta)$. Since every $\omega \in IncForg(\alpha) \cup IncForg(\beta)$ is such that $\omega(\gamma) = 1$ when $\alpha \vdash \gamma$ and $\beta \vdash \gamma$, this must be the case for every $\omega \in IncForg(\alpha \vee \beta)$.
– *Cut:* We first prove the following lemma:

  Lemma 1. *Let ω and ω' be two 3-interpretations such that $\forall x \in PS$, if $\omega'(x) \neq \top$, then $\omega'(x) = \omega(x)$. Then for any formula α of $PROP_{PS}$, we have that if $\omega'(\alpha) = 1$ (resp. $\omega'(\alpha) = 0$), then $\omega(\alpha) = 1$ (resp. $\omega(\alpha) = 0$).*

  The proof of this lemma is easy by structural induction on α. Now, by *reductio ad absurdum*, assume that there exists $\omega \in IncForg(\alpha)$ such that $\omega(\gamma) = 0$. Then by definition of $IncForg(\alpha)$, there exists $\omega' \in min(3 - mod(\alpha), \leq)$ such that $\forall x \in PS$, if $\omega'(x) \neq \top$, then $\omega'(x) = \omega(x)$. Since $\omega' \in 3 - mod(\alpha)$, we have that $\omega'(\alpha) \neq 0$. Since $\alpha \models_c \beta$, we have that $\omega(\beta) = 1$. As a consequence of the lemma, we get that $\omega'(\beta) \neq 0$. Hence, we have $\omega'(\alpha \wedge \beta) \neq 0$: $\omega' \in 3 - mod(\alpha \wedge \beta)$. Since $3 - mod(\alpha \wedge \beta) \subseteq 3 - mod(\alpha)$ and $\omega' \in min(3 - mod(\alpha), \leq)$, we must have $\omega' \in min(3 - mod(\alpha \wedge \beta), \leq)$. Hence $\omega \in IncForg(\alpha \wedge \beta)$. Since $\alpha \wedge \beta \models_c \gamma$, we must have $\omega(\gamma) = 1$, contradiction.
– *Cautious Monotony:* We first exploit the previous lemma to show that for any formulas α and β of $PROP_{PS}$, if $\alpha \models_c \beta$, then $\alpha \models^{\leq} \beta$. By *reduction ad absurdum*, assume that there exists $\omega' \in min(3 - mod(\alpha), \leq)$ such that $\omega'(\beta) = 0$. From the lemma, for every 2-interpretation ω that $\forall x \in PS$, if $\omega'(x) \neq \top$, then $\omega'(x) = \omega(x)$, we must have $\omega(\beta) = 0$. Since $\omega' \in min(3 - mod(\alpha), \leq)$, for at least one 2-interpretation $\omega \in IncForg(\alpha)$, we must have $\omega(\alpha) = 0$. This contradicts the fact that $\alpha \models_c \beta$.
  Now, in order to prove the Cautious Monotony property, it is enough to show that whenever $\alpha \models_c \beta$, we have that $min(3 - mod(\alpha \wedge \beta), \leq) = min(3 - mod(\alpha), \leq)$. Let $\omega \in min(3 - mod(\alpha), \leq)$. Since $\alpha \models_c \beta$, we have that $\alpha \models^{\leq} \beta$. Hence, ω is a 3-model of β. Since it is a 3-model of α, it is a 3-model of $\alpha \wedge \beta$. Since $3 - mod(\alpha \wedge \beta) \subseteq 3 - mod(\alpha)$, we have that $\omega \in min(3 - mod(\alpha \wedge \beta), \leq)$. Hence the inclusion $min(3 - mod(\alpha), \leq) \subseteq min(3 - mod(\alpha \wedge \beta), \leq)$ holds. Conversely, assume that there exists $\omega' \in min(3 - mod(\alpha \wedge \beta), \leq) \setminus min(3 - mod(\alpha), \leq)$. Since $3 - mod(\alpha \wedge \beta) \subseteq 3 - mod(\alpha)$, there exists $\omega \in min(3 - mod(\alpha), \leq)$ such that $\omega < \omega'$ (i.e., $\omega \leq \omega'$ and $\omega' \not\leq \omega$). From the previous inclusion, we must have that $\omega \in min(3 - mod(\alpha \wedge \beta), \leq)$. The fact that $\omega < \omega'$ contradicts that $\omega' \in min(3 - mod(\alpha \wedge \beta), \leq)$.

$\square$

Observe that there would be no way to keep reflexivity while ensuring the classical closure property. Indeed, we have the following easy proposition:

Proposition 3. *No inference relation $\vdash$ satisfies both reflexivity and the classical closure property.*

Proof. Consider $\Sigma = a \wedge \neg a$. If $\Sigma \not\vdash \Sigma$ then it does not satisfy reflexivity. Contrastingly, If $\Sigma \vdash \Sigma$ then it does not satisfy the classical closure property since Σ is classically inconsistent. $\square$

It is also interesting to note that $\models_c$ satisfies other properties which are not shared by $\models^{\leq}$ [20], especially "transitivity" (this is a direct consequence of the fact that it satisfies both the classical closure property and the preservation property):

Proposition 4. *$\models_c$ satisfies transitivity, i.e. for any formulas α, β, γ from $PROP_{PS}$, if $\alpha \models_c \beta$ and $\beta \models_c \gamma$, then $\alpha \models_c \gamma$.*

Finally, it is important to determine whether the relaxation of $\models^{\leq}$ we realised to ensure the classical closure property does not lead to a too weak inference relation $\models_c$. The following inclusions show that this is not the case:

Proposition 5. $\models^{\leq}_1 \quad \subset \quad \models_c \quad \subset \quad \models^{\leq}$.

Thus, all the "conflict-free" consequences α of a belief base Σ w.r.t. $\models^{\leq}$ are preserved by $\models_c$. Furthermore, $\models_c$ does not add consequences that would not be derivable using $\models^{\leq}_1$.

3.3 Computational aspects

In this section, we investigate some computational aspects of $\models_c$. We assume the reader familiar with some basic notions of complexity, especially the complexity classes coNP and Π^p_2 of the polynomial hierarchy PH (see [28] for a survey).

We first consider the complexity of the inference problem for $\models_c$:

Definition 13 **($\models_c$-INFERENCE).** $\models_c$-INFERENCE *is the following decision problem:*

- **Input:** *A finite set Σ of formulas from $PROP_{PS}$ and a formula α in $PROP_{PS}$.*
- **Question:** *Does $\Sigma \models_c \alpha$ hold?*

We have obtained the following result:

Proposition 6. $\models_c$-INFERENCE *is Π^p_2-complete.*

Proof. Membership is easy; one considers the complementary problem: in order to show that $\Sigma \not\models_c \alpha$ holds, we guess a 2-interpretation ω and a 3-interpretation ω' over $Var(\Sigma) \cup Var(\alpha)$; then we check that ω' belongs to $min(3-mod(\Sigma), \leq)$ (one call to an NP oracle since this problem is in coNP); finally, we check in polynomial time that for every $x \in Var(\Sigma) \cup Var(\alpha)$, we have that $\omega(x) = \omega'(x)$ whenever $\omega'(x) \neq \top$, and that $\omega(\alpha) = 1$.

Hardness holds even in the restricted case when Σ is a CNF formula and α is a propositional symbol; we consider the problem of determining, given a CNF formula Σ and a symbol a, whether every element ω of $min(3 - mod(\Sigma), \leq)$ is such that $\omega(a) \neq \top$. This problem has been shown Π^p_2-hard in [22]. The fact that every element ω of $min(3 - mod(\Sigma), \leq)$ is such that $\omega(a) \neq \top$ if and only if $\Sigma \wedge (a \vee b) \wedge (\neg a \vee b) \models_c b$ where $b \in PS \setminus Var(\Sigma)$, completes the proof. $\square$

This proposition shows that $\models_c$ is not harder than the underlying relation $\models^{\leq}$ from a computational complexity point of view; indeed, the inference problem for $\models^{\leq}$ also is Π_2^p-complete [22].

We now show how to turn any finite belief base Σ (viewed as the conjunction of its elements) into a "classical" consistent propositional formula $cl(\Sigma)$ such that $\models_c (\Sigma)$ is equal to the classical closure of $cl(\Sigma)$. The basic idea is to turn first Σ into a DNF formula which is strongly equivalent. As in classical propositional logic, such a DNF formula can be computed by applying iteratively to Σ the following equivalences, considered as rewrite rules (left-to-right oriented):

- $\neg(\neg\alpha)$ is strongly 3-equivalent to α.
- $\neg(\alpha \vee \beta)$ is strongly 3-equivalent to $(\neg\alpha) \wedge (\neg\beta)$.
- $\neg(\alpha \wedge \beta)$ is strongly 3-equivalent to $(\neg\alpha) \vee (\neg\beta)$.
- $\alpha \wedge (\beta \vee \gamma)$ is strongly 3-equivalent to $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ (and similarly for $(\beta \vee \gamma) \wedge \alpha$).

Of course, the obtained DNF formula can be of exponential size in the size of Σ. It now remains to forget inconsistencies in this DNF formula after isolating terms representing the preferred models (the minimization step); formally, for every term α, let $inc(\alpha)$ be the set of "inconsistencies" occurring in α: $inc(\alpha) = \{x \in PS \mid x \text{ and } \neg x \text{ occur in } \alpha\}$. $cl(\Sigma)$ is the DNF formula obtained by successively:

1. removing in the current DNF every term α such that $inc(\alpha)$ is not minimal w.r.t. set-inclusion in the set $\{inc(\alpha) \mid \alpha \text{ a term in the current DNF}\}$.
2. removing in every term of the resulting DNF formula every literal l when the complementary literal also occurs in the term, then removing every empty term (and finally adding $a \vee \neg a$ if the resulting DNF formula contains no term).

We have that:

Proposition 7. $Cn_{\models_c}(\Sigma) = Cn_{\models_2}(cl(\Sigma))$.

As a matter of illustration, consider again Example 1: let $\Sigma = a \wedge (\neg a \vee b) \wedge c \wedge \neg c$ and $PS = \{a, b, c\}$. Σ is strongly 3-equivalent to the following DNF formula $(a \wedge \neg a \wedge c \wedge \neg c) \vee (a \wedge b \wedge c \wedge \neg c)$. Now, forgetting inconsistency in Σ leads to the DNF formula $cl(\Sigma) = a \wedge b$ (the first term $(a \wedge \neg a \wedge c \wedge \neg c)$ of the previous DNF is removed during the minimization step). We can easily check that $Cn_{\models_c}(\Sigma) = Cn_{\models_2}(cl(\Sigma))$.

Since the computation of $cl(\Sigma)$ can be achieved in time polynomial in the size of Σ when Σ is a DNF and since $cl(\Sigma)$ is a DNF formula, we easily get that:

Proposition 8.

- *Under the restriction where Σ is a DNF formula, $\models_c$-INFERENCE is* coNP*-complete.*
- *Under the restriction where Σ is a DNF formula and α is a CNF formula, $\models_c$-INFERENCE is in* P.

Thus the formula $cl(\Sigma)$ is a classically consistent formula which can be viewed as a compilation of Σ (in the sense that any finite belief base Σ interpreted w.r.t. $\models_c$ is equivalent to the corresponding formula $cl(\Sigma)$ classically interpreted and that the inference problem from $cl(\Sigma)$ is computationally easier than the inference problem from Σ, unless the polynomial hierarchy collapses at the first level).

4 Conclusion

In this paper, we have introduced and studied a new paraconsistent inference relation $\models_c$ in the setting of 3-valued paraconsistent logics. Using inconsistency forgetting as a key mechanism for recovering consistency, it guarantees that the deductive closure $Cn_{\models_c}(\Sigma)$ of any belief base Σ is classically consistent and classically closed. This strong feature, not shared by previous inference relations in the same setting, allows to interpret an inconsistent belief base as a set of classical worlds (hence to reason classically from them).We have investigated the logical properties and the computational complexity of $\models_c$. Among other things, we have shown that $\models_c$ satisfies many interesting properties which are shared by the underlying inference relation $\models^{\leq}$, without any complexity shift compared to it.

We have considered in this paper a basic language for three-valued paraconsistent logic (the monotone fragment). A first perspective for further research is to extend the approach to more complex morphologies. It is also clear that the inconsistency forgetting mechanism at work here could be applied to other many-valued paraconsistent logics, especially four-valued ones. This is another extension of this work that we plan to do.

Acknowledgements

The authors want to thank the anonymous referees for their helpful comments. They have been partly supported by the IUT de Lens.

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