Constrained Argumentation Frameworks*

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Abstract

We present a generalization of Dung's theory of argumentation enabling to take account for some additional constraints on the admissible sets of arguments, expressed as a propositional formula over the set of arguments. We point out several semantics for such constrained argumentation frameworks, and compare the corresponding inference relations w.r.t. cautiousness. We show that our setting encompasses some previous approaches based on Dung's theory as specific cases. We also investigate the complexity issue for the inference relations in the extended setting. Interestingly, we show that our generalization does not lead to a complexity shift w.r.t. inference for several semantics.

Introduction

Argumentation is a general approach for nonmonotonic reasoning, in which the main issues are the generation of arguments and their use to draw some conclusions based on the way arguments interact (see e.g., (Toulmin 1958; Prakken & Vreeswijk 2002; Bondarenko *et al.* 1997; Kakas & Toni 1999)). Among the many theories of argumentation pointed out so far (see e.g., (Dung 1995; Pollock 1992; Simari & Loui 1992; Elvang-Gøransson, Fox, & Krause 1993a; 1993b; Elvang-Gøransson & Hunter 1995; Vreeswijk 1997; Besnard & Hunter 2001; Amgoud & Cayrol 2002a; 2002b; Cayrol 1995; Dimopoulos, Nebel, & Toni 2002)) Dung's theory (Dung 1995) has received much attention since it encompasses many approaches to nonmonotonic inference and logic programming as specific cases; especially, it has been refined and extended by several authors, including (Baroni, Giacomin, & G.Guida 2000; Baroni & Giacomin 2003; 2004; Cayrol *et al.* 2002; Cayrol & Lagasquie-Schiex 2002).

In Dung's approach, no assumption is made about the nature of an argument and the argument generation issue is not considered: arguments and the way they interact w.r.t. the attack relation are considered as initial data of any argumentation framework, which can thus be viewed as a labeled digraph $AF = \langle A, R \rangle$.

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Despite the simplicity of the setting, several inference relations can be defined within Dung's theory. Usually, inference is defined at the argument level, and an argument is considered derivable when it belongs to one (credulous consequence) (resp. all (skeptical consequence)) extensions of AF under some semantics, where an extension of AF is an admissible set of arguments (i.e., a conflict-free and self-defending set) that is maximal for a given criterion (made precise by the semantics under consideration). Inference can also be easily defined for sets of arguments by asking them to be included into one or all extensions. For the credulous consequence relations, this is not equivalent in the general case to asking each argument to be derivable (see e.g. (Coste-Marquis, Devred, & Marquis 2005c)).

The notion of admissibility for a set of arguments in Dung's theory relies only on the interaction of arguments. Especially, Dung's approach does not offer a way to specify further requirements on the sets of arguments which are expected as extensions, like "extensions must contain argument a when they contain b" or "extensions must not contain one of c or d when they contain a but do not contain b". Actually, when they are not consequences of the interaction of arguments, there is no way to enforce such constraints without revising the given argumentation framework, i.e. producing a new argumentation framework for which the constraint is satisfied. However, revision does not always prove sufficient to ensure that every constraint is satisfied. Furthermore, there are usually many ways to revise an argumentation framework, and the choice of a revision strategy must be guided by the reasons which underly the revision operation. The problem is that such reasons are not always available. For instance, consider $AF = \langle A = \{a, b\}, R = \{(a, b)\} \rangle$ and the constraint "extensions must not contain a". There are several ways to ensure it: adding (a, a) to R, adding a new, fictitious argument c to A and adding (c, c) and (c, a) to R ... If one does not know why "extensions must not contain a", some arbitrary choices must be made, which is not satisfactory: $(a, a) \in R$ has the strong meaning that a is self-conflicting, which is not necessarily believed; similarly, adding (c, c) and (c, a) to R leaves unexplained what c means, hence why c is self-conflicting, why c attacks a, why a does not attack c, and so on. The situation is even worse when the reasons why "extensions must not contain a" have nothing to do with the attack relation (e.g. a is grounded on beliefs that are not plausible); in such a case, there cannot be any meaningful revision strategy.

In this paper, a generalization of Dung's theory of argumentation, obtained by taking advantage of additional information which constrain the sets of arguments which are eligible for admissibility, is presented. Such a constraint takes the form of a propositional formula over the set of symbols used to represent the arguments. This gives rise to new semantics based on the further information conveyed by the constraint – which cannot always be captured by the attack relation.[1] To be more precise, we show how the various semantics considered by Dung (preferred, stable and grounded) can be extended in a natural way so as to take account for such constraints, and how a new semantics, called the weak one, can be obtained. We compare the inference relations induced by those semantics w.r.t. cautiousness. We also show how our setting incorporates some previous approaches based on Dung's theory. We finally investigate the computational issue for the inference relations in the extended setting. Interestingly, we show that our generalization does not lead to a complexity shift w.r.t. inference for several semantics, though the inference relations take place in a strictly more expressive setting than Dung's one.

The rest of this paper is organized as follows. First, we recall the main definitions pertaining to Dung's theory of argumentation. Then, we present our contribution before concluding the article.

Dung's Theory of Argumentation

Let us present some basic definitions at work in Dung's theory of argumentation (Dung 1995). We restrict them to finite argumentation frameworks.

Definition 1 ((finite) argumentation frameworks) *A* (finite) argumentation framework *is a pair $AF = \langle A, R\rangle$ where A is a finite set of so-called arguments and R is a binary relation over A (a subset of $A \times A$), the attack relation.*

Clearly enough, the set of finite argumentation frameworks is a proper subset of the set of Dung's finitary argumentation frameworks, where every argument must be attacked by finitely many arguments.

Formally, we note $AF {\mid\!\sim} S$ where $AF = \langle A, R\rangle$ is an argumentation framework and $S \subseteq A$, to state that S is a consequence of AF under ${\mid\!\sim}$. In the following, an inference relation ${\mid\!\sim}$ is based on a notion of extension, and an inference principle (credulous or skeptical), so that $AF {\mid\!\sim} S$ holds if and only if S is included in all (skeptical) or at least one (credulous) extension(s) of AF, for a given semantics. Formally:

Notation 1 (inference relations) *Let ${\mid\!\sim}^{q,s}$ denote the* inference relation *obtained by considering a semantics s and an inference principle q, either credulous ($q = \exists$) or skeptical ($q = \forall$).*

For instance, $S \subseteq A$ is a consequence of $AF = \langle A, R\rangle$ w.r.t. ${\mid\!\sim}^{\forall,P}$, noted $AF {\mid\!\sim}^{\forall,P} S$, indicates that S is included in every preferred extension of AF.

In order to define a notion of extension, a first important notion is the notion of acceptability: an argument a is acceptable w.r.t. a set of arguments S whenever it is defended by the set, i.e., every argument which attacks a is attacked by an element of S.

Definition 2 (acceptable sets) *Let $AF = \langle A, R\rangle$ be an argumentation framework. An argument $a \in A$ is* acceptable *w.r.t. a subset S of A if and only if for every $b \in A$ s.t. $(b, a) \in R$, there exists $c \in S$ s.t. $(c, b) \in R$. A set of arguments is acceptable w.r.t. S when each of its elements is acceptable w.r.t. S.*

A second important notion is the notion of absence of conflicts. Intuitively, two arguments should not be considered together whenever one of them attacks the other one.

Definition 3 (conflict-free sets) *Let $AF = \langle A, R\rangle$ be an argumentation framework. A subset S of A is* conflict-free *if and only if for every $a, b \in S$, we have $(a, b) \notin R$.*

Requiring the absence of conflicts and the form of autonomy captured by self-acceptability leads to the notion of admissible set:

Definition 4 (admissible sets) *Let $AF = \langle A, R\rangle$ be an argumentation framework. A subset S of A is* admissible *for AF if and only if S is conflict-free and acceptable w.r.t. S.*

The significance of the concept of admissible sets is reflected by the fact that every extension of an argumentation framework under the standard semantics introduced by Dung (i.e., preferred, stable and grounded) is an admissible set, satisfying some form of optimality:

Definition 5 (extensions) *Let $AF = \langle A, R\rangle$ be an argumentation framework and let $S \subseteq A$.*

- *S is* a preferred extension *of AF if and only if it is maximal w.r.t. $\subseteq$ among the set of admissible sets for AF.*
- *S is* a stable extension *of AF if and only if S is conflict-free and $\forall a \in A \setminus S$, $\exists b \in S$ s.t. $(b, a) \in R$.*

A more prudent, semantics is based on the characteristic function $\mathcal{F}_{AF}$ of AF:

Definition 6 (characteristic functions) *The* characteristic function *$\mathcal{F}_{AF}$ of an argumentation framework $AF = \langle A, R\rangle$ is defined as follows:*
$\mathcal{F}_{AF} : 2^A \rightarrow 2^A$
$\mathcal{F}_{AF}(S) = \{a \mid a$ *is acceptable w.r.t.* $S\}$.

Definition 7 (grounded extensions) *Let $AF = \langle A, R\rangle$ be an argumentation framework. The* grounded extension *of AF is the least fixed point of $\mathcal{F}_{AF}$.*

[1] Thus, unlike (Besnard & Doutre 2004), our primary concern is not to encode existing inference relations from argumentation frameworks using constraints, but to exploit constraints as additional inputs.

Dung has shown that every argumentation framework has a (unique) grounded extension and at least one preferred extension, while it may have zero, one or many stable extensions.

These extensions are linked up as follows:

Proposition 1 *Theorem 25 in (Dung 1995)*
Let AF be an argumentation framework. Every preferred (resp. stable) extension of AF contains the grounded extension of AF.

Since the grounded extension of an argumentation framework is unique, we have $\mid\!\sim^{\exists,G} = \mid\!\sim^{\forall,G}$. Hence, we note $\mid\!\sim^{\cdot,G}$ for $\mid\!\sim^{\exists,G} = \mid\!\sim^{\forall,G}$.

Let us illustrate the notions of extensions and the inference relations on a simple example:

Example 1 Let $AF = \langle A = \{a, b, c, d, e, f\}, R = \{(a, b), (a, c), (b, d), (c, e), (e, f), (f, e)\}\rangle$ be the argumentation framework depicted on Figure 1.

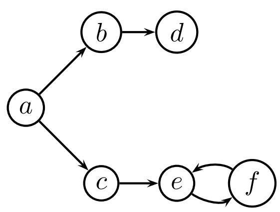

Figure 1: The digraph of AF.

AF has two preferred extensions, $\{a, d, e\}$ and $\{a, d, f\}$, which are stable extensions as well. The grounded extension of AF is $\{a, d\}$. As a consequence, we have (for instance):

- $AF \mid\!\sim^{\cdot,G} \{a, d\}$;
- $AF \mid\!\sim^{\forall,P} \{a, d\}$;
- $AF \mid\!\sim^{\forall,S} \{a, d\}$;
- $AF \mid\!\sim^{\exists,P} \{a, d, e\}$;
- $AF \mid\!\sim^{\exists,S} \{a, d, f\}$.

Constrained Argumentation Frameworks

Definitions

Let us now extend the notion of framework considered by Dung in order to take account for constraints over arguments:

Definition 8 (constrained argumentation frameworks) *Let $PROP_{PS}$ be a propositional language defined in the usual inductive way from a set PS of propositional symbols, the boolean constants $\top$, $\bot$ and the connectives $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$. A* constrained argumentation framework *(CAF) is a triple $CAF = \langle A, R, \mathcal{C}\rangle$ where A is a finite set of arguments and R is a binary relation over A, the attack relation and $\mathcal{C}$ is a propositional formula from $PROP_A$.*

Each subset S of A corresponds to an interpretation over A (i.e. a total function from A to $\{0, 1\}$), given by the completion of A:

Definition 9 (completions) *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework and $S \subseteq A$. S satisfies $\mathcal{C}$ if and only if the completion $\widehat{S} = \{a \mid a \in S\} \cup \{\neg a \mid a \in A \setminus S\}$ of S is a model of $\mathcal{C}$ (denoted by $\widehat{S} \models \mathcal{C}$).*

Since the purpose is to restrict the sets of arguments eligible for extensions to those satisfying $\mathcal{C}$, we need to refine the notion of admissibility:

Definition 10 ($\mathcal{C}$-admissible sets) *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework. A subset S of A is $\mathcal{C}$-*admissible *for CAF if and only if S is admissible for $\langle A, R\rangle$ and satisfies $\mathcal{C}$. We note $\mathcal{A} = \{S \subseteq A \mid S$ is admissible for $\langle A, R\rangle$ and $\widehat{S} \models \mathcal{C}\}$ the set of all $\mathcal{C}$-*admissible *sets for CAF.*

The presence of a constraint $\mathcal{C}$ calls for a notion of *consistency*:

Definition 11 (consistency) *A constrained argumentation framework $CAF = \langle A, R, \mathcal{C}\rangle$ is* consistent *when it has a $\mathcal{C}$-admissible set for CAF.*

Being consistent for a constrained argumentation framework just means that the pieces of information conveyed by its constraint and the pieces of information conveyed by its attack relation are compatible. Of course, this is a highly desirable property (without it, every inference relation trivializes).

Echoing Dung's definitions, we now introduce some definitions of extensions for constrained argumentation frameworks:

Definition 12 (preferred $\mathcal{C}$-extensions) *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework. A $\mathcal{C}$-admissible set $S \subseteq A$ for CAF is* a preferred $\mathcal{C}$-extension *of CAF if and only if $\nexists S' \subseteq A$ s.t. $S \subset S'$ and S' is $\mathcal{C}$-admissible for CAF.*

Definition 13 (stable $\mathcal{C}$-extensions) *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework. A conflict-free subset S of A which satisfies $\mathcal{C}$ is a* stable $\mathcal{C}$-extension *of CAF if and only if $\forall a \in A \setminus S$, $\exists b \in S$ such that $(b, a) \in R$.*

Like for Dung's stable extensions, the definition imposes that a stable $\mathcal{C}$-extension is never empty, except in the trivial case when $A = \emptyset$.

In order to define more prudent inference relations, we need a notion of characteristic function, which is suited to constrained argumentation frameworks:

Definition 14 ($\mathcal{C}$-characteristic functions) *The* $\mathcal{C}$-characteristic function *$\mathcal{F}_{CAF}$ of $CAF = \langle A, R, \mathcal{C}\rangle$ is defined as follows:*
$\mathcal{F}_{CAF} : 2^A \longrightarrow 2^A$
$\mathcal{F}_{CAF}(S) = \{a \mid a$ *is acceptable w.r.t. S and $S \cup \{a\}$ satisfies $\mathcal{C}$*$\}$.

We also need the following notations:

Notation 2 *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework.*

- *$\mathcal{F}_{CAF,\mathcal{A}}$ is the restriction of $\mathcal{F}_{CAF}$ from $\mathcal{A}$ to 2^A, i.e., for any $S \subseteq A$, $\mathcal{F}_{CAF,\mathcal{A}}(S) = \mathcal{F}_{CAF}(S)$ if $S \in \mathcal{A}$ and $\mathcal{F}_{CAF,\mathcal{A}}(S)$ is undefined otherwise.*
- *For any integer i and any $S \in \mathcal{A}$, $\mathcal{F}^0_{CAF,\mathcal{A}}(S) = S$ and $\mathcal{F}^{i+1}_{CAF,\mathcal{A}}(S) = \mathcal{F}_{CAF,\mathcal{A}}(\mathcal{F}^i_{CAF,\mathcal{A}}(S))$, when $\mathcal{F}^i_{CAF,\mathcal{A}}(S) \in \mathcal{A}$, and is undefined otherwise.*

We are now ready to define a notion of *grounded $\mathcal{C}$-extension* of a constrained argumentation framework:

Definition 15 (grounded $\mathcal{C}$-extensions) *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework. If the poset $(\mathcal{A}, \subseteq)$ has a least element and $\mathcal{F}_{CAF,\mathcal{A}}$ is a monotone function from $\mathcal{A}$ to $\mathcal{A}$, then the* grounded $\mathcal{C}$-extension *of CAF is defined as the least fixed point of $\mathcal{F}_{CAF,\mathcal{A}}$. Otherwise, the grounded $\mathcal{C}$-extension of CAF is undefined.*

The following proposition states that this definition is correct and explains how the grounded $\mathcal{C}$-extension of a constrained argumentation framewok can be computed when it exists:

Proposition 2 *The notion of grounded $\mathcal{C}$-extension of a constrained argumentation framework is well-founded and the grounded $\mathcal{C}$-extension of any constrained argumentation framework CAF can be computed as $\mathcal{F}^{i_{min}}_{CAF,\mathcal{A}}(M)$ where i_{min} is the least integer i such that $\mathcal{F}^{i+1}_{CAF,\mathcal{A}}(M) = \mathcal{F}^i_{CAF,\mathcal{A}}(M)$, and M is the least element of $(\mathcal{A}, \subseteq)$.*

Proof: Since A is finite, every chain $C = \{A_1, A_2, \ldots\}$ of sets from $\mathcal{A}$ is finite, so it has a greatest element A_k which is also the supremum of C. Since $\mathcal{F}_{CAF,\mathcal{A}}$ is monotone from $\mathcal{A}$ to $\mathcal{A}$, we also have that $sup(\{\mathcal{F}_{CAF,\mathcal{A}}(A_1), \mathcal{F}_{CAF,\mathcal{A}}(A_2), \ldots\}) = \mathcal{F}_{CAF,\mathcal{A}}(A_k)$, showing that $\mathcal{F}_{CAF,\mathcal{A}}$ is a Scott-continuous function. Furthermore, since $(\mathcal{A}, \subseteq)$ has a least element and A is finite, $(\mathcal{A}, \subseteq)$ is a pointed complete partial order (i.e., a partially ordered set with a least element such that each of its directed subsets has a supremum). Hence, from Knaster-Tarski theorem and Scott theorem, the least fixed point of $\mathcal{F}_{CAF,\mathcal{A}}$ exists, and since A is finite, it can be computed as $\mathcal{F}^{i_{min}}_{CAF,\mathcal{A}}(M)$ where i_{min} is the least integer i such that $\mathcal{F}^{i+1}_{CAF,\mathcal{A}}(M) = \mathcal{F}^i_{CAF,\mathcal{A}}(M)$, and M is the least element of the pointed complete partial order $(\mathcal{A}, \subseteq)$. ■

As we will see later on, the grounded $\mathcal{C}$-extension underlies an inference relation which is at least as cautious as the skeptical inference relation based on preferred $\mathcal{C}$-extensions (the situation is similar in Dung's setting when the grounded extension and the preferred extensions are concerned).

Now, when a constrained argumentation framework has no grounded $\mathcal{C}$-extension, it may have a *weak $\mathcal{C}$-extension*:

Definition 16 (weak $\mathcal{C}$-extensions) *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework. If $\mathcal{A}$ has a least element M w.r.t. $\subseteq$, the* weak $\mathcal{C}$-extension *of CAF is defined as $\mathcal{F}^{i_{con}}_{CAF,\mathcal{A}}(M)$ where i_{con} is the least integer i such that $\mathcal{F}^i_{CAF,\mathcal{A}}(M) \in \mathcal{A}$ and if $\mathcal{F}^{i+1}_{CAF,\mathcal{A}}(M) \in \mathcal{A}$, then $\mathcal{F}^{i+1}_{CAF,\mathcal{A}}(M) = \mathcal{F}^i_{CAF,\mathcal{A}}(M)$.*

Since A is finite, the definition of a weak $\mathcal{C}$-extension is well-founded. Note that the existence of a weak $\mathcal{C}$-extension of a constrained argumentation framework CAF is ensured whenever $\emptyset$ satisfies $\mathcal{C}$ (since $\emptyset$ is admissible, and obviously minimal w.r.t. $\subseteq$).

In order to avoid too heavy notations, we use the same notations as before for the inference relations from constrained argumentation frameworks (and we consider in addition the weak semantics W). The context makes precise whether the inference relation at hand concerns constrained frameworks or not. For instance, $CAF \mid\!\sim^{\forall,P} S$ means that the set of arguments S is included into every preferred $\mathcal{C}$-extension of CAF.

Since the grounded $\mathcal{C}$-extension (resp. the weak $\mathcal{C}$-extension) of CAF is unique when it exists, we have $\mid\!\sim^{\exists,G}$ = $\mid\!\sim^{\forall,G}$, and $\mid\!\sim^{\exists,W}$ = $\mid\!\sim^{\forall,W}$ when such an extension exists. Hence, we note $\mid\!\sim^{\cdot,G}$ for $\mid\!\sim^{\exists,G}$ = $\mid\!\sim^{\forall,G}$, and $\mid\!\sim^{\cdot,W}$ for $\mid\!\sim^{\exists,W}$ = $\mid\!\sim^{\forall,W}$ in such a case. Note that the image of CAF by $\mid\!\sim^{\cdot,G}$ (resp. $\mid\!\sim^{\cdot,W}$) is undefined whenever CAF has no grounded $\mathcal{C}$-extension (resp. no weak $\mathcal{C}$-extension).

Let us illustrate the notions of extensions and the inference relations on the running example, slightly extended:

Example 1 (continued) Let us consider the constrained argumentation framework obtained by adding a constraint to the argumentation framework of Example 1. We consider $CAF = \langle A = \{a, b, c, d, e, f\}, R = \{(a, b), (a, c), (b, d), (c, e), (e, f), (f, e)\}, \mathcal{C} = \neg a \vee \neg d \vee \neg e\rangle$. The grounded $\mathcal{C}$-extension of CAF is undefined since $\mathcal{F}_{CAF,\mathcal{A}}$ is not a monotone function from $\mathcal{A}$ to $\mathcal{A}$ ($\mathcal{F}_{CAF,\mathcal{A}}(\{a\}) = \{a, d\}$ is not included in $\mathcal{F}_{CAF,\mathcal{A}}(\{a, e\}) = \{a, e\}$). Contrastingly, since CAF has a least element w.r.t. $\subseteq$ (namely $\emptyset$), its weak extension is defined and equal to $\{a, d\}$. CAF has two preferred $\mathcal{C}$-extensions: $\{a, e\}$, $\{a, d, f\}$, and one stable $\mathcal{C}$-extension: $\{a, d, f\}$. We have (for instance):

- $CAF \mid\!\sim^{\cdot,W} \{a, d\}$;
- $CAF \mid\!\sim^{\forall,P} \{a\}$;
- $CAF \mid\!\sim^{\forall,S} \{a, d, f\}$;
- $CAF \mid\!\sim^{\exists,P} \{a, e\}$;
- $CAF \mid\!\sim^{\exists,S} \{a, d, f\}$.

Properties and cautiousness

Let us now explain how the various notions of $\mathcal{C}$-extensions are connected and how they relate to Dung's extensions. We first give the following easy result:

Proposition 3 *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework.*

- *For each $\mathcal{C}$-admissible set S of CAF, there exists a preferred $\mathcal{C}$-extension E of CAF such that $S \subseteq E$.*

- *If $CAF' = \langle A, R, \mathcal{C}' \rangle$ is a constrained argumentation framework s.t. $\mathcal{C}' \models \mathcal{C}$, then $\mathcal{A}' \subseteq \mathcal{A}$.*

Proof: Point 1. is obvious since A is finite. Point 2. comes from the fact that if a set of arguments S is such that $\widehat{S} \models \mathcal{C}'$ and $\mathcal{C}' \models \mathcal{C}$, then $\widehat{S} \models \mathcal{C}$. ■

We also have that:

Proposition 4 *Let $CAF = \langle A, R, \mathcal{C} \rangle$ be a constrained argumentation framework. For each preferred $\mathcal{C}$-extension E of CAF, there exists a preferred extension E' of $\langle A, R \rangle$ such that $E \subseteq E'$.*

Proof: Let E be a preferred $\mathcal{C}$-extension. Then E is $\mathcal{C}$-admissible for CAF. Hence E is admissible for $\langle A, R \rangle$. Then from Theorem 1 from (Dung 1995), E is included in a preferred extension E' of AF. ■

Furthermore, the $\mathcal{C}$-admissible sets of $CAF = \langle A, R, \mathcal{C} \rangle$ can be easily characterized using the $\mathcal{C}$-characteristic function of CAF:

Proposition 5 *Let $CAF = \langle A, R, \mathcal{C} \rangle$ be a constrained argumentation framework and let $S \subseteq A$ be a conflict-free set which satisfies $\mathcal{C}$. S is $\mathcal{C}$-admissible for CAF if and only if $S \subseteq \mathcal{F}_{CAF}(S)$.*

Proof: Let $a \in S$. Since S is $\mathcal{C}$-admissible for CAF, S is admissible for $\langle A, R \rangle$, hence a is acceptable w.r.t. S. Furthermore, $S \cup \{a\} = S$ satisfies $\mathcal{C}$. Subsequently, $a \in \mathcal{F}_{CAF}(S)$. Conversely, let $S \subseteq A$ s.t. $S \subseteq \mathcal{F}_{CAF}(S)$, S is conflict-free and S satisfies $\mathcal{C}$. By definition of $\mathcal{F}_{CAF}$, we have for any $S \subseteq A$ that $\mathcal{F}_{CAF}(S) \subseteq \mathcal{F}_{AF}(S)$. Hence $S \subseteq \mathcal{F}_{AF}(S)$. Since S is conflict-free, we get from (Dung 1995) that S is admissible for $\langle A, R \rangle$. Since S satisfies $\mathcal{C}$, S is $\mathcal{C}$-admissible for CAF. ■

As a consequence, if E is a preferred $\mathcal{C}$-extension of $CAF = \langle A, R, \mathcal{C} \rangle$, then $E = \mathcal{F}_{CAF}(E)(= \mathcal{F}_{CAF,\mathcal{A}}(E)$ since E is $\mathcal{C}$-admissible for CAF when it is a preferred $\mathcal{C}$-extension). The converse does not hold in general (consider Example 1 (cont'd): while $\{a, d\}$ is $\mathcal{C}$-admissible for CAF and a fixed point for $\mathcal{F}_{CAF,\mathcal{A}}$, it is not a preferred $\mathcal{C}$-extension of CAF).

Note that Proposition 3 does not mean that a constrained argumentation framework always has a preferred $\mathcal{C}$-extension. Actually, this is not the case; for instance, in Example 1 (cont'd), if one replaces $\mathcal{C}$ by $\neg a \wedge e$, one obtains a constrained argumentation framework, which does not have any $\mathcal{C}$-admissible set. In particular, the grounded extension $\{a, d\}$ of $\langle A, R \rangle$ is not $\mathcal{C}$-admissible for CAF. This situation contrasts with what happens in Dung's setting (every argumentation framework has an admissible set).

As to the stable $\mathcal{C}$-extensions, we have the following proposition:

Proposition 6 *Let $CAF = \langle A, R, \mathcal{C} \rangle$ be a constrained argumentation framework.*

- *Every stable $\mathcal{C}$-extension of CAF also is a preferred $\mathcal{C}$-extension of CAF. The converse does not hold.*
- *Every stable $\mathcal{C}$-extension of CAF also is a stable (hence preferred) extension of $\langle A, R \rangle$. The converse does not hold.*

Proof:

- If S is a stable $\mathcal{C}$-extension of CAF then for every $a \in A \setminus S$, there exists $b \in S$ s.t. $(b, a) \in R$. Hence $S \cup \{a\}$ is not admissible for $\langle A, R \rangle$ since it is not conflict-free. Accordingly, S is a $\mathcal{C}$-admissible subset for CAF of A which is maximal w.r.t. $\subseteq$, i.e. a preferred $\mathcal{C}$-extension of CAF. Example 1 (cont'd) shows that the converse does not hold.
- Obvious from the definition of a stable $\mathcal{C}$-extension. For the converse, consider again Example 1 (cont'd).

■

Like in Dung's setting when stable extensions are considered, a constrained argumentation framework $CAF = \langle A, R, \mathcal{C} \rangle$ may have zero, one or many stable $\mathcal{C}$-extensions. Note that the existence of a stable extension for $\langle A, R \rangle$ is not sufficient to ensure the existence of a stable $\mathcal{C}$-extension in the general case (consider again Example 1 (cont'd), conjoin $\mathcal{C}$ with $\neg f$ to get a new constraint: while the corresponding argumentation framework $\langle A, R \rangle$ has two stable extensions ($\{a, d, e\}$ and $\{a, d, f\}$), it has no stable $\mathcal{C}$-extensions).

Let us now turn to the grounded $\mathcal{C}$-extension and the weak $\mathcal{C}$-extension of a constrained argumentation framework. Contrariwise to the grounded extension of an argumentation framework, the grounded $\mathcal{C}$-extension of a constrained argumentation framework does not always exist (and this is also the case for the weak $\mathcal{C}$-extension of a constrained argumentation framework). A reason is that $\mathcal{A}$ does not always have a least element w.r.t. $\subseteq$. For instance, in Example 1 (cont'd), conjoining $\mathcal{C}$ with $a \vee f$ to get a new constraint leads to a new constrained argumentation framework which does not have a least $\mathcal{C}$-admissible set.

The grounded $\mathcal{C}$-extension of a constrained argumentation framework CAF is connected to the preferred (and to the stable) $\mathcal{C}$-extensions of CAF:

Proposition 7 *Let $CAF = \langle A, R, \mathcal{C} \rangle$ be a constrained argumentation framework. If the grounded $\mathcal{C}$-extension of CAF exists, it is included in every preferred (hence in every stable) $\mathcal{C}$-extension of CAF.*

Proof: This result easily comes from the fact that every preferred $\mathcal{C}$-extension of a constrained argumentation framework CAF is a fixed point of $\mathcal{F}_{CAF,\mathcal{A}}$, and that the grounded $\mathcal{C}$-extension of CAF is the least fixed point of $\mathcal{F}_{CAF,\mathcal{A}}$. ■

As to the weak $\mathcal{C}$-extension, we have the following easy result:

Proposition 8 *Let $CAF = \langle A, R, \mathcal{C} \rangle$ be a constrained argumentation framework. If the grounded $\mathcal{C}$-extension of CAF exists, then it coincides with the weak $\mathcal{C}$-extension of CAF.*

Proof: Comes immediately from the definition of the weak $\mathcal{C}$-extension and the grounded $\mathcal{C}$-extension of a constrained argumentation framework. ∎

Note that some constrained argumentation frameworks may have a weak $\mathcal{C}$-extension, without having a grounded $\mathcal{C}$-extension (see Example 1 (cont'd)). Note also that it is not the case that the weak $\mathcal{C}$-extension of a constrained argumentation framework CAF is included into every preferred $\mathcal{C}$-extension of CAF (again, see Example 1). Nevertheless the weak $\mathcal{C}$-extension of a constrained argumentation framework CAF is connected to every stable $\mathcal{C}$-extension of CAF:

Proposition 9 *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework. If the weak $\mathcal{C}$-extension of CAF exists, then it is included in every stable $\mathcal{C}$-extension of CAF.*

Proof: The result trivially holds if CAF has no stable $\mathcal{C}$-extensions. Otherwise, we show by induction on $i \leq i_{con}$ that for any stable $\mathcal{C}$-extension S of CAF, the inclusion $\mathcal{F}^i_{CAF,\mathcal{A}}(M) \subseteq S$ holds. The base case is when $i = 0$: $\mathcal{F}^0_{CAF,\mathcal{A}}(M) = M$ is included in S since S is $\mathcal{C}$-admissible for CAF and M is the least $\mathcal{C}$-admissible set for CAF. Now, we assume that the property holds for every i s.t. $i \leq k < i_{con}$ and show that it still holds for $i = k+1$. Assume that there exists $a \in \mathcal{F}^{k+1}_{CAF,\mathcal{A}}(M)$ which does not belong to a stable $\mathcal{C}$-extension S of CAF. Since $a \notin S$, there exists $b \in S$ such that b attacks a. Since a is acceptable w.r.t. $\mathcal{F}^k_{CAF,\mathcal{A}}(M)$, a is defended against b by an element $c \in \mathcal{F}^k_{CAF,\mathcal{A}}(M)$. But the induction hypothesis shows that $c \in S$, hence S is not conflict-free, which is impossible since S is a stable $\mathcal{C}$-extension. ∎

From the previous propositions, the cautiousness picture for the inference relations can be easily drawn. We say that $\mid\!\sim^{q,s}$ is at least as cautious as $\mid\!\sim^{q',s'}$, noted $\mid\!\sim^{q,s} \subseteq \mid\!\sim^{q',s'}$ if and only if for every $CAF = \langle A, R, \mathcal{C}\rangle$ and every $S \subseteq A$, if $CAF \mid\!\sim^{q,s} S$ then $CAF \mid\!\sim^{q',s'} S$.

We first focus on constrained argumentation frameworks CAF having a weak $\mathcal{C}$-extension and a stable $\mathcal{C}$-extension. Indeed, if CAF has no weak $\mathcal{C}$-extension, its image by $\mid\!\sim^{\cdot,W}$ is undefined so the relation cannot be compared with any other inference relation w.r.t. cautiousness. If CAF has no stable $\mathcal{C}$-extension, then both $\mid\!\sim^{\forall,S}$ and $\mid\!\sim^{\exists,S}$ trivialize: every set of argument belongs to the image of CAF by $\mid\!\sim^{\forall,S}$, and no set of argument belongs to the image of CAF by $\mid\!\sim^{\exists,S}$. In such a pathological scenario, credulous inference w.r.t. the stable semantics is strictly more cautious than skeptical inference w.r.t. the stable semantics, which is unexpected.

Proposition 10 *The cautiousness relations given in Table 1 hold for any constrained argumentation framework CAF having a weak $\mathcal{C}$-extension and a stable $\mathcal{C}$-extension, but no grounded $\mathcal{C}$-extension.*

Proof: $\subseteq$ in cells (i,i) ($i \in 1\ldots5$) are obvious. $\subseteq$ in cells $(1,2)$ and $(3,4)$ come from the assumption that CAF has a stable $\mathcal{C}$-extension (hence a preferred $\mathcal{C}$-extension from Proposition 6). $\subseteq$ in cells $(1,3)$ and $(4,2)$ come from Proposition 6. $\subseteq$ in cell $(5,3)$ comes from Proposition 9. $\subseteq$ in cell $(5,3)$ comes from $\subseteq$ in cell $(5,3)$ since CAF has a stable $\mathcal{C}$-extension. $\subseteq$ in cell $(5,2)$ comes from $\subseteq$ in cell $(5,4)$ and Proposition 6. The remaining $\subseteq$ (in cells $(1,4)$ and $(3,2)$) come immediately from the previous $\subseteq$ and the transitivity of cautiousness.

All the $\not\subseteq$ in the table come from Example 1 (cont'd), except in cells $(1,5)$ and $(4,3)$. For them, it is sufficient to consider $CAF = \langle\{a,b,c,d\}, \{(a,c),(b,c),(c,d)\}, \neg a \vee \neg b\rangle$. $\emptyset$ is the least element of $\mathcal{A}$ w.r.t. $\subseteq$, hence CAF has a weak $\mathcal{C}$-extension, namely $\emptyset$. It does not have a grounded $\mathcal{C}$-extension since $\mathcal{F}_{CAF,\mathcal{A}}(\emptyset) = \{a,b\} \notin \mathcal{A}$ (hence $\mathcal{F}_{CAF,\mathcal{A}}$ is not from $\mathcal{A}$ to $\mathcal{A}$). $\{a,d\}$ and $\{b,d\}$ are the preferred $\mathcal{C}$-extensions of CAF (and they are the stable ones as well). ∎

	$\mid\!\sim^{\forall,P}$ (1)	$\mid\!\sim^{\exists,P}$ (2)	$\mid\!\sim^{\forall,S}$ (3)	$\mid\!\sim^{\exists,S}$ (4)	$\mid\!\sim^{\cdot,W}$ (5)
$\mid\!\sim^{\forall,P}$ (1)	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\not\subseteq$
$\mid\!\sim^{\exists,P}$ (2)	$\not\subseteq$	$\subseteq$	$\not\subseteq$	$\not\subseteq$	$\not\subseteq$
$\mid\!\sim^{\forall,S}$ (3)	$\not\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\not\subseteq$
$\mid\!\sim^{\exists,S}$ (4)	$\not\subseteq$	$\subseteq$	$\not\subseteq$	$\subseteq$	$\not\subseteq$
$\mid\!\sim^{\cdot,W}$ (5)	$\not\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$

Table 1: Cautiousness links between inference relations from CAFs

When a constrained argumentation framework CAF has a grounded $\mathcal{C}$-extension, we have $\mid\!\sim^{\cdot,G} = \mid\!\sim^{\cdot,W}$, and the only difference is that $\mid\!\sim^{\cdot,W} \subseteq \mid\!\sim^{\forall,P}$ (from Proposition 7). Thus, in this case, the cautiousness picture is similar to the one relating the corresponding inference relations in Dung's setting.

Generality of the approach

Let us now turn to the expressiveness issue. It is easy to prove that the theory of constrained argumentation frameworks generalizes Dung's theory of (finite) argumentation frameworks:[2]

Proposition 11 *Let $AF = \langle A, R\rangle$ be an argumentation framework . Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework where $\mathcal{C}$ is any valid formula over A. Then:*

1. *the preferred extensions of AF are the preferred $\mathcal{C}$-extensions of CAF.*
2. *the stable extensions of AF are the stable $\mathcal{C}$-extensions of CAF.*
3. *the grounded extension of AF is the grounded $\mathcal{C}$-extension of CAF (which coincides with the weak extension of CAF).*

[2]Observe that the converse does not hold; in particular, though every argumentation framework is consistent ($\emptyset$ is an admissible set) this is not the case of every constrained framework ($\widehat{\emptyset}$ does not satisfy every $\mathcal{C}$).

Proof: Points 1. and 2. are easy; point 3. comes from the fact that $(\mathcal{A}, \subseteq)$ is a complete partial order and $\mathcal{F}_{CAF,\mathcal{A}}$ is continuous from $\mathcal{A}$ to $\mathcal{A}$ when $\mathcal{C}$ is valid (Theorem 1 from (Dung 1995)). ■

In the same way, some bipolar argumentation frameworks (Amgoud, Cayrol, & Lagasquie-Schiex 2004; Cayrol & Lagasquie-Schiex 2005a; 2005b; Mardi, Cayrol, & Lagasquie-Schiex 2005) can be efficiently translated into "equivalent" constrained argumentation frameworks:

Definition 17 (bipolar argumentation frameworks) *A* bipolar (finite) argumentation framework *is a triple* $BAF = \langle A, R_{def}, R_{sup} \subseteq A \times A\rangle$. R_{sup} *is a support relation between arguments.*

Several notions of admissibility can be envisioned in this setting, reflecting the various ways one can take advantage of the support relation. Among them, a set of arguments S can be considered as admissible for a bipolar argumentation framework BAF when it is admissible for $\langle A, R_{def}\rangle$ and such that for every $a, b \in A$, if $(a, b) \in R_{sup}$ and $a \in S$, then $b \in S$. This leads immediately to the following notions of extensions:

Definition 18 (set closed for R_{sup}) *Let* $BAF = \langle A, R_{def}, R_{sup} \subseteq A \times A\rangle$ *be a bipolar (finite) argumentation framework. A subset S of A is* closed *for R_{sup} if and only if it contains every argument a such that there exists $b \in S, (b, a) \in R_{sup}$.*

Definition 19 (weakly c-admissible) *Let* $BAF = \langle A, R_{def}, R_{sup} \subseteq A \times A\rangle$ *be a bipolar (finite) argumentation framework. A subset S of A is* weakly c-admissible *for BAF if and only if S is admissible for $\langle A, R_{def}\rangle$ and closed for R_{sup}.*

Definition 20 (weakly c-preferred extensions) *Let* $BAF = \langle A, R_{def}, R_{sup} \subseteq A \times A\rangle$ *be a bipolar (finite) argumentation framework. A* weakly c-preferred extension *of BAF is a subset S of A such that S is maximal w.r.t. $\subseteq$ among the set of weakly c-admissible sets for BAF.*

Definition 21 (weakly c-stable extensions) *Let* $BAF = \langle A, R_{def}, R_{sup} \subseteq A \times A\rangle$ *be a bipolar (finite) argumentation framework. A* weakly c-stable extension *of BAF is a subset S of A such that S is conflict-free, closed for the relation R_{sup} and $\forall a \in A \setminus S, \exists b \in S$ s.t. $(b, a) \in R_{def}$.*

Example 2 Let $BAF = \langle A, R_{def}, R_{sup}\rangle$ with $A = \{a, b, c\}$, $R_{def} = \{(a, b), (b, a)\}$ and $R_{sup} = \{(c, b)\}$. BAF is depicted on Figure 2. $E_1 = \{b, c\}$ is the weakly c-stable extension of BAF. E_1 and $E_2 = \{a\}$ are the weakly c-preferred extensions of BAF.

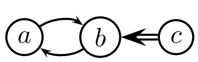

Figure 2: The digraph of BAF.

We have the following translation result:

Proposition 12 *Let* $BAF = \langle A, R_{def}, R_{sup} \subseteq A \times A\rangle$ *be a bipolar (finite) argumentation framework. Let* $CAF = \langle A, R_{def}, \mathcal{C}\rangle$ *be the constrained argumentation framework such that* $\mathcal{C} = \bigwedge_{(a,b)\in R_{sup}} (a \Rightarrow b)$. *Then:*

1. *the weakly c-preferred extensions of BAF are the preferred $\mathcal{C}$-extensions of CAF.*
2. *the weakly c-stable extensions of BAF are the stable $\mathcal{C}$-extensions of CAF.*

Proof: Obvious from that fact that ensuring that a set of arguments S is such that for every $a, b \in A$, if $(a, b) \in R_{sup}$ and $a \in S$, then $b \in S$ amounts exactly to ensuring that $\widehat{S} \models \mathcal{C}$. ■

A notion of weakly c-grounded extension of a bipolar argumentation framework could be also easily defined and computed as the grounded $\mathcal{C}$-extension of the corresponding constrained argumentation framework. The existence of a grounded $\mathcal{C}$-extension in such a case comes from the slightly more general result, as follows:

Proposition 13 *Let* $CAF = \langle A, R, \mathcal{C}\rangle$ *be a constrained argumentation framework. If $\mathcal{C}$ is equivalent to a conjunction of clauses of the form $\neg x \vee y$, then $(\mathcal{A}, \subseteq)$ has a least element and $\mathcal{F}_{CAF,\mathcal{A}}$ is a monotone function from $\mathcal{A}$ to $\mathcal{A}$.*

Proof: First, $\mathcal{C}$ has a least element since $\emptyset$ is admissible for $\langle A, R\rangle$ and it satisfies $\mathcal{C}$. As to monotony, let us consider two subsets S and S' of $\mathcal{A}$ such that $S \subseteq S'$. Let $a \in \mathcal{F}_{CAF,\mathcal{A}}(S)$. By definition, a is acceptable w.r.t. S. Hence, a is also acceptable w.r.t. S'. Furthermore, $S \cup \{a\}$ satisfies $\mathcal{C}$. Assume now that $S' \cup \{a\}$ does not satisfy $\mathcal{C}$. Then there exists an implicate $\neg x \vee y$ of $\mathcal{C}$ which is not satisfied by $S' \cup \{a\}$. Since S' satisfies $\mathcal{C}$, it must be the case that $x = a$ and $y \notin S'$. Therefore, $y \notin S$. As a consequence, $S \cup \{a\}$ does not satisfy $\neg a \vee y$, contradiction. Finally, it remains to show that for any $S \in \mathcal{A}$, we have $\mathcal{F}_{CAF,\mathcal{A}}(S) \in \mathcal{A}$. Since Dung's fundamental lemma (Dung 1995) ensures that $\mathcal{F}_{CAF,\mathcal{A}}(S)$ is admissible for $\langle A, R\rangle$, it remains to show that $\mathcal{F}_{CAF,\mathcal{A}}(S)$ satisfies $\mathcal{C}$. Assume that this is not the case. Then there exists an implicate $\neg x \vee y$ of $\mathcal{C}$ such that $\mathcal{F}_{CAF,\mathcal{A}}(S)$ satisfies x and does not satisfy y. This is equivalent to state that $x \in \mathcal{F}_{CAF,\mathcal{A}}(S)$ and $y \notin \mathcal{F}_{CAF,\mathcal{A}}(S)$. Hence x is acceptable w.r.t. S and $S \cup \{x\}$ satisfies $\mathcal{C}$. As a consequence, $S \cup \{x\}$ satisfies $\neg x \vee y$. So $y \in S$. Now, for any $S \in \mathcal{A}$, we have $S \subseteq \mathcal{F}_{CAF,\mathcal{A}}(S)$; indeed, if a belongs to S, then a is acceptable w.r.t. S since $S \in \mathcal{A}$; besides, $S \cup \{a\} = S$ satisfies $\mathcal{C}$ since $S \in \mathcal{A}$. Finally, since $y \in S$, we must have $y \in \mathcal{F}_{CAF,\mathcal{A}}(S)$, contradiction. ■

We can also show that the prudent semantics and the careful semantics for argumentation frameworks as given in (Coste-Marquis, Devred, & Marquis 2005b; 2005a) can be recovered as the semantics for some correspondings constrained argumentation frameworks (in a nutshell, indirect conflitcs and controversies can be computed and translated into constraints in polynomial time). To be more precise, we need the following definitions:

Definition 22 (controversial arguments) *Let $AF = \langle A, R\rangle$ be an argumentation framework.*

- *Let $a, b \in A$. a* indirectly attacks *b if and only if there exists an odd-length path from a to b in the digraph for AF.*
- *Let $a, b \in A$. a* indirectly defends *b if and only if there exists an even-length path from a to b in the digraph for AF. The length of this path is not zero.*
- *Let $a, b \in A$. a is* controversial *w.r.t. b if and only if a indirectly attacks b and a indirectly defends b.*
- *Let $a, b, c \in A$. (a, b) is* super-controversial *w.r.t. c if and only if a indirectly attacks c and b indirectly defends c.*

Both the prudent semantics and the careful semantics for argumentation frameworks AF aim at restricting the set of admissible sets for AF, so as to prevent any pair of arguments a and b such that a is controversial w.r.t. b from belonging to the same extension, which is not the case for Dung's semantics.

Definition 23 (p-admissible sets) *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework. $S \subseteq A$ is* p(rudent)-admissible *for AF if and only if every $a \in S$ is acceptable w.r.t. S and S is without indirect conflicts, i.e., there is no pair of arguments a and b of S s.t. a indirectly attacks b.*

Definition 24 (c-admissible sets) *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework. $S \subseteq A$ is* c(areful)-admissible *for AF if and only if every $a \in S$ is acceptable w.r.t. S and S is conflict-free and controversial-free for AF, i.e. for every $a, b \in S$ and every $c \in A$, (a, b) is not super-controversial w.r.t. c.*

On this ground, notions of preferred p-extension (resp. preferred c-extension) and of stable p-extension (resp. stable c-extension) can be easily defined:

Definition 25 (preferred p-extensions) *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework. A p-admissible set $S \subseteq A$ for AF is* a preferred p-extension *of AF if and only if $\nexists S' \subseteq A$ s.t. $S \subset S'$ and S' is p-admissible for AF.*

Definition 26 (preferred c-extensions) *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework. A c-admissible set $S \subseteq A$ for AF is* a preferred c-extension *of AF if and only if $\nexists S' \subseteq A$ s.t. $S \subset S'$ and S' is c-admissible for AF.*

Definition 27 (stable p-extensions) *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework. A subset S of A without indirect conflicts S is a* stable p-extension *of AF if and only if S attacks every argument from $A \setminus S$.*

Definition 28 (stable c-extensions) *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework. A conflict-free and controversial-free subset S of A is a* stable c-extension *of AF if and only if S attacks every argument from $A \setminus S$.*

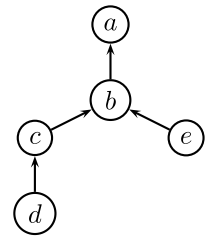

Figure 3: The digraph for AF.

Example 3 Let $AF = \langle A, R\rangle$ with $A = \{a, b, c, d, e\}$ and $R = \{(b, a), (e, b), (c, b), (d, c)\}$. The digraph for AF is depicted on Figure 3.

Let $E = \{a, d, e\}$. E is the grounded extension of AF, the unique preferred extension of AF and the unique stable extension of AF.

Since d indirectly attacks a, a and d cannot belong to the same p-extension. The preferred p-extensions of AF are $\{a, e\}$ and $\{d\}$ and AF has no stable p-extension.

(d, e) is super-controversial w.r.t. a. $\{a, e\}$ and $\{d\}$ are the preferred c-extensions of AF, and AF has no stable c-extension.

We have the following translation results:

Proposition 14 *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework Let $CAF = \langle A, R, \mathcal{C}\rangle$ be the constrained argumentation framework such that $\mathcal{C} = \bigwedge_{(a,b)\in A\times A \mid a \text{ indirectly attacks } b}(a \Rightarrow \neg b)$. Then:*

1. *the p-preferred extensions of AF are the preferred $\mathcal{C}$-extensions of CAF.*
2. *the p-stable extensions of AF are the stable $\mathcal{C}$-extensions of CAF.*

Proof: Direct from the fact that $\mathcal{C}$ prevents arguments a and b from belonging to the same $\mathcal{C}$-admissible set for CAF exactly when a indirectly attacks b. ■

Proposition 15 *Let $AF = \langle A, R\rangle$ be a (finite) argumentation framework Let $CAF = \langle A, R, \mathcal{C}\rangle$ be the constrained argumentation framework such that $\mathcal{C} = \bigwedge_{(a,b)\in A\times A \mid \exists c\in A, (a,b) \text{ is super-controversial w.r.t. } c}(a \Rightarrow \neg b)$. Then:*

1. *the c-preferred extensions of AF are the preferred $\mathcal{C}$-extensions of CAF.*
2. *the c-stable extensions of AF are the stable $\mathcal{C}$-extensions of CAF.*

Proof: Direct from the fact that $\mathcal{C}$ prevents arguments a and b from belonging to the same $\mathcal{C}$-admissible set for CAF exactly when (a, b) is super-controversial w.r.t. some argument c. ■

Computational aspects

Finally, we have investigated the complexity of inference, and related problems, in the theory of constrained argumentation frameworks.[3] A first important complexity issue when dealing with constrained argumentation frameworks is the consistency one. Indeed, inference from an inconsistent constrained argumentation framework CAF trivializes: $\mid\!\sim^{\cdot,G}$ and $\mid\!\sim^{\cdot,W}$ are undefined, and since CAF has no preferred $\mathcal{C}$-extensions, $CAF \mid\!\sim^{\forall,s} S$ holds for every $S \subseteq A$ when s is P or S, while $CAF \mid\!\sim^{\exists,s} S$ holds for no $S \subseteq A$ when s is P or S. While this problem is obvious when argumentation frameworks are considered (since argumentation frameworks are consistent!), it is computationally hard in the general case. Furthermore, the complexity does not come solely from the satisfiability problem for $\mathcal{C}$:

Proposition 16 *Deciding whether $CAF = \langle A, R, \mathcal{C}\rangle$ is consistent is* NP*-complete, even if $\mathcal{C}$ is a positive CNF formula or a negative CNF formula.*[4]

Proof: Membership is easy. Hardness is obtained by reduction from 3-CNF-SAT. To every $3-CNF$ formula Σ over $\{x_1, \ldots, x_n\}$, we associate in polynomial time a constrained argumentation frameworks CAF where $A = \{x_1, \ldots, x_n, x'_1, \ldots, x'_n\}$, $R = \{(x_i, x'_i), (x'_i, x_i) \mid i = 1 \ldots n\}$ and $\mathcal{C}$ is the positive (resp. negative formula) obtained by replacing in Σ every negative literal $\neg x_i$ by x'_i (resp. every positive literal x_i by $\neg x'_i$). Σ is satisfiable if and only if CAF is consistent. ■

Consistency is necessary and sufficient to ensure that $\mid\!\sim^{\forall,P}$ and $\mid\!\sim^{\exists,P}$ do not trivialize. However, it is not sufficient to ensure that $\mid\!\sim^{\cdot,G}$ and $\mid\!\sim^{\cdot,W}$ are well-defined and that $\mid\!\sim^{\forall,S}$ and $\mid\!\sim^{\exists,S}$ do not trivialize. As to non-trivialization, we have derived the following results:

Proposition 17 *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework.*

1. *Determining whether $\mathcal{A}$ has a least element is* coNP*-hard and in Θ_2^p.*
2. *Determining whether $\mathcal{A}$ has a least element and $\mathcal{F}_{CAF,\mathcal{A}}$ is a monotone function from $\mathcal{A}$ to $\mathcal{A}$ is* coNP*-hard and in Θ_2^p.*
3. *Determining whether CAF has a stable $\mathcal{C}$-extension is* NP*-complete (even if CAF is known as consistent).*

Proof: For points 1. and 2., in (Besnard & Doutre 2004) (Proposition 6), it is shown how an argumentation framework $AF = \langle A, R\rangle$ can be associated in polynomial time to a propositional formula $\mathcal{C}_{AF}$ over A such that $S \subseteq A$ is admissible for $\langle A, R\rangle$ if and only if S satisfies $\mathcal{C}_{AF}$. Subsequently, $S \subseteq A$ is $\mathcal{C}$-admissible for CAF if and only if S satisfies $\mathcal{C}_{AF} \wedge \mathcal{C}$. Hence, $\mathcal{A}$ has a least element if and only if $\mathcal{C}_{AF} \wedge \mathcal{C}$ has a least model w.r.t. $\subseteq$. This holds if and only if the closure $CWA(\mathcal{C}_{AF} \wedge \mathcal{C})$ of $\mathcal{C}_{AF} \wedge \mathcal{C}$ w.r.t. the closed world assumption has a model; furthermore, the least model M of $\mathcal{C}_{AF} \wedge \mathcal{C}$ is the unique model of $CWA(\mathcal{C}_{AF} \wedge \mathcal{C})$, which can be computed as $CWA(\mathcal{C}_{AF} \wedge \mathcal{C}) = \mathcal{C}_{AF} \wedge \mathcal{C} \wedge \bigwedge_{a \in A \mid \mathcal{C}_{AF} \wedge \mathcal{C} \not\models a} \neg a$ (see Lemma 5 in (Eiter & Gottlob 1992)). Accordingly, determining whether $\mathcal{A}$ has a least element amounts to determining whether $CWA(\mathcal{C}_{AF} \wedge \mathcal{C})$ is consistent, which is in Θ_2^p (and not in coBH_2 unless the polynomial hierarchy collapses) (Eiter & Gottlob 1992). For point 2., adding the further requirement that $\mathcal{F}_{CAF,\mathcal{A}}$ is a monotone function from $\mathcal{A}$ to $\mathcal{A}$ does not lead to a complexity shift since determining whether $\mathcal{F}_{CAF,\mathcal{A}}$ is a monotone function from $\mathcal{A}$ to $\mathcal{A}$ can be easily shown in coNP. For point 1., coNP-hardness comes from the fact that determining whether the closure of a propositional formula Σ over A is consistent is coNP-hard (Eiter & Gottlob 1992). For point 2., we exhibit a reduction from UNSAT. Let Σ be a propositional formula over $\{x_1, \ldots, x_n\}$. We can associate to Σ in polynomial time the constrained argumentation framework $\langle\{x_1, \ldots, x_n, y_1, y_2\}, \emptyset, \mathcal{C} = (\Sigma \wedge (y_1 \vee y_2) \wedge (\neg y_1 \vee \neg y_2)) \vee (\bigwedge_{i=1}^n \neg x_i \wedge \neg y_1 \wedge \neg y_2)\rangle$. If Σ is unsatisfiable then $\mathcal{A} = \{\emptyset\}$; since $\mathcal{F}_{CAF,\mathcal{A}}(\emptyset) = \emptyset$, $\mathcal{F}_{CAF,\mathcal{A}}$ is a monotone function from $\mathcal{A}$ to $\mathcal{A}$. If Σ is satisfiable, then there exists $S \subseteq \{x_1, \ldots, x_n\}$ s.t. S satisfies Σ. Since $R = \emptyset$, every subset of $\{x_1, \ldots, x_n, y_1, y_2\}$ is admissible for $\langle A, R\rangle$ and every argument of $\{x_1, \ldots, x_n, y_1, y_2\}$ is acceptable w.r.t. it. By construction, $S \cup \{y_1\}$ and $S \cup \{y_2\}$ satisfies $\mathcal{C}$ but for every $S' \subseteq \{x_1, \ldots, x_n, y_1, y_2\}$ s.t. $S \cup \{y_1, y_2\} \subseteq S'$, S' does not satisfy $\mathcal{C}$. Hence $\mathcal{F}_{CAF,\mathcal{A}}(S) \notin \mathcal{A}$. For point 3., membership is easy (guess $S \subseteq A$ and check in polynomial time that it is a stable $\mathcal{C}$-extension of CAF) and hardness comes from (Dimopoulos & Torres 1996; Dunne & Bench-Capon 2002) in the restricted case when $\mathcal{C}$ is valid. ■

Proposition 18 *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework and $S \subseteq A$. The complexity of determining whether $CAF \mid\!\sim^{q,s} S$ holds (and is well-defined when $s = G$ or $s = W$) is as reported in Table 2.*

$\mid\!\sim^{\forall,P}$	Π_2^p-complete
$\mid\!\sim^{\exists,P}$	NP-complete
$\mid\!\sim^{\forall,S}$	coNP-complete
$\mid\!\sim^{\exists,S}$	NP-complete
$\mid\!\sim^{\cdot,G}$	coNP-hard and in Δ_2^p
$\mid\!\sim^{\cdot,W}$	coNP-hard and in Δ_2^p

Table 2: Complexity of inference from constrained argumentation frameworks

Proof: Let us first focus on the first four rows of Table 2. All hardness results come directly from Proposition 11 and results from (Dimopoulos & Torres 1996; Dunne

[3] We assume the reader acquainted with basic notions of complexity theory, especially the polynomial hierarchy. See e.g. (Papadimitriou 1994) otherwise.

[4] A positive (resp. negative) formula is a formula in negation normal form in which only positive (resp. negative) literals occur. Such formulas are always satisfiable when propositional constants are not included in the morphology of the propositional language. Positive (resp. negative) CNF formulas form a proper subset of the reverse Horn CNF formulas (resp. Horn CNF formulas).

& Bench-Capon 2002) and hold even in the restricted case the constrained argumentation framework CAF is known as consistent. As to membership, we have shown in (Coste-Marquis, Devred, & Marquis 2005c) that considering *sets* of arguments S as queries (instead of arguments) does not lead to a complexity shift for the inference issue in Dung's setting; similar membership proofs can be derived here, taking advantage of the fact that deciding whether S satisfies $\mathcal{C}$ can be easily done in time polynomial in the input size (i.e. $|CAF| + |S|$). Let us now consider the last two rows of Table 18. Since deciding whether S satisfies $\mathcal{C}$ is easy for any $S \subseteq A$, $\mathcal{F}_{CAF,\mathcal{A}}(S)$ can be computed in time polynomial in $|CAF|$; subsequently, computing the grounded $\mathcal{C}$-extension of CAF and computing the weak $\mathcal{C}$-extension of CAF is just as hard as computing the least model M of $\mathcal{C}$ (which requires to determining whether it exists). Indeed, once M is available, it is enough to apply iteratively $\mathcal{F}_{CAF,\mathcal{A}}$ to it, until either (1) reaching a fixed point or (2) obtaining a set which does not satisfy $\mathcal{C}$. Since $\mathcal{F}_{CAF,\mathcal{A}}(S)$ can be computed in time polynomial in $|CAF|$, deciding whether S satisfies $\mathcal{C}$ is easy and the number of iterations is bounded by $|A|$, the process can be achieved in time polynomial in $|CAF|$ when M is given. In case (1), the fixed point is the grounded $\mathcal{C}$-extension of CAF (which coincides with its weak $\mathcal{C}$-extension), while in case (2), CAF has no grounded $\mathcal{C}$-extension but the set obtained after the last iteration is the weak $\mathcal{C}$-extension of CAF. Finally, since M is the unique model of $CWA(\mathcal{C})$, computing it can be easily achieved using $\mathcal{O}(|A|)$ calls to an NP oracle (this is immediate from the definition of $CWA(\mathcal{C})$). ■

It is noticeable to observe that the four inference relations based on the preferred or the stable semantics in the constrained setting are just as hard as the corresponding relations in Dung's setting. It is unlikely to be the case for the inference relations based on the grounded (or the weak) $\mathcal{C}$-extension since the grounded extension of an argumentation framework can be computed in polynomial time.

From the computational point of view, it is interesting to note that the translation approach proposed by Creignou (Creignou 1995) (in the context of graph theory) and by Besnard and Doutre (Besnard & Doutre 2004) for encoding extensions as logical interpretations can be applied as well to constrained frameworks. Such a translation approach shows how to reduce the inference issue from argumentation frameworks to logic-based inference. The idea consists in associating to any constrained argumentation framework CAF a propositional formula the models (resp. the maximal models) of which encode exactly the stable $\mathcal{C}$-extensions (resp. the preferred $\mathcal{C}$-extensions) of CAF:

Proposition 19 *Let $CAF = \langle A, R, \mathcal{C}\rangle$ be a constrained argumentation framework.*

- *$S \subseteq A$ is a stable $\mathcal{C}$-extension of CAF if and only if S satisfies the formula*

$$(\bigwedge_{a \in A} (a \Leftrightarrow \bigwedge_{b \mid (b,a) \in R} \neg b)) \wedge \mathcal{C}.$$

- *$S \subseteq A$ is a preferred $\mathcal{C}$-extension of CAF if and only if $\widehat{S}$ is a maximal model of the formula*

$$(\bigwedge_{a \in A} ((a \Rightarrow \bigwedge_{b \mid (b,a) \in R} \neg b)$$
$$\wedge (a \Rightarrow \bigwedge_{b \mid (b,a) \in R} (\bigvee_{c \mid (c,b) \in R} c)))) \wedge \mathcal{C}.$$

Proof:

- Direct from Proposition 5.1 from (Creignou 1995) since a stable $\mathcal{C}$-extension of CAF is a stable extension of CAF which satisfies $\mathcal{C}$, and the converse also holds.
- Direct from Proposition 6 from (Besnard & Doutre 2004) since a $\mathcal{C}$-admissible set of CAF is an admissible set of the corresponding AF, satisfying $\mathcal{C}$, and the preferred $\mathcal{C}$-extensions of CAF are the maximal $\mathcal{C}$-admissible sets of it.

■

Such polytime translations allow for taking advantage of many results from automated reasoning (including SAT solvers) so as to decide our inference relations.

Conclusion and Perspectives

We have presented a generalization of Dung's theory of argumentation, which takes account for additional constraints on admissible sets. We have pointed out several semantics for such constrained argumentation frameworks, and compared the corresponding inference relations w.r.t. cautiousness. While it encompasses some previous approaches based on Dung's theory as specific cases, we have shown that our generalization does not lead to a complexity shift w.r.t. inference for several semantics.

This paper calls for several perspectives. One of them consists in identifying additional restrictions on constraints $\mathcal{C}$ for which the existence of the weak $\mathcal{C}$-extension (or of the grounded $\mathcal{C}$-extension) would be ensured. Another one consists in pointing out restrictions on constrained argumentation frameworks for which deciding the inference relations would be less complex (and possibly tractable).

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