Is Promoting Beliefs Useful to Make Them Accepted in Networks of Agents? (extended version including proofs of propositions)

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Abstract

We consider the problem of belief propagation in a network of communicating agents, modeled in the recently introduced Belief Revision Game (BRG) framework. In this setting, each agent expresses her belief through a propositional formula and revises her own belief at each step by considering the beliefs of her acquaintances, using belief change tools. In this paper, we investigate the extent to which BRGs satisfy some monotonicity properties, i.e., whether promoting some desired piece of belief to a given set of agents is actually always useful for making it accepted by all of them. We formally capture such a concept of promotion by a new family of belief change operators. We show that some basic monotonicity properties are not satisfied by BRGs in general, even when the agents merging-based revision policies are fully rational (in the AGM sense). We also identify some classes where they hold.

1 Introduction

We are interested in the issue of monotonicity in a multi-agent system, represented as a Belief Revision Game (BRG) [Schwind *et al.*, 2015]. The BRG setting is a framework for modeling the belief dynamics of a group of agents V, for instance agents involved in a social network. A BRG is a dynamical system where agents have their own belief bases (representing their belief states), and communicate synchronously with their acquaintances. The acquaintance relationship is given through a binary relation A over V, i.e., (V, A) is a graph. Let us introduce a motivating example:

Example 1. *Consider a group of friends Alex, Beth and Chris who are discussing on whether they should trust the quality of the food served in a given restaurant. Alex and Beth know each other, Alex and Chris as well, but Beth and Chris do not know each other. Two meals are considered by them. At the beginning, Alex believes that either the first meal or the second one is healthy, but not both of them; Beth believes that none of the two meals is healthy; whereas Chris believes that at least one of the two meals is healthy.*

At each communication step, each agent revises her beliefs by considering the beliefs of her acquaintances. Several merging-based revision policies have been defined, each of them reflecting how much an agent is ready to question her current beliefs in front of the beliefs of her acquaintances. Then a piece of belief φ is accepted by an agent i of V when there exists a step of the game from which φ holds in the belief bases of i at each subsequent step; and φ is unanimously accepted when it is accepted by every agent of V.

Now, the question is, given a piece of belief φ, is adding "more φ" in a BRG always beneficial to φ? More precisely, whenever a piece of belief φ is unanimously accepted in a BRG, is it always harmless to replace at the beginning some bases by φ, or more generally, by a base that is "closer" to φ, i.e., by a "promotion" of φ? This monotonicity condition is essential when one wants to investigate the potential manipulation of such sytems, in particular the control issue: consider a set of agents from a predefined subset C of V and an additional agent M who can "control" the agents from C, i.e., M can modify the initial beliefs of agents from C, then is it possible for M to make a piece of belief unanimously accepted? Such a control issue is significant for a number of multi-agent problems, including brand crisis management [Dawar and Pillutla, 2000]; in such applications, it is useful to know what information agents from C should convey to their acquaintances in order to avoid the propagation of negative perceptions.

In the next section we provide some preliminaries on BRGs. We then define and investigate in Section 3 the notion of promoting a formula φ in a belief base, and show that many rational change operators (including revision and contraction operators) can be used for the promotion purpose. We then formalize in Section 4 the monotonicity property based on promotion operators and show that this condition is not always satisfied by BRGs. As a consequence, replacing some agents' initial beliefs by φ is not always the best way to make φ unanimously accepted. In Section 5 we show that the monotonicity property holds for BRGs based on the complete graph topology, focusing on revision policies based on a specific merging operator. We discuss some related work in Section 6 before concluding.

The proofs of propositions are given in an appendix.

2 Belief Revision Games

Let $\mathcal{L}_{\mathcal{P}}$ be a propositional language built up from a finite set of propositional variables $\mathcal{P}$ and the usual connectives, including $\oplus$, the xor connective. $\bot$ (resp. $\top$) is the Boolean constant always false (resp. true). Formulae are interpreted in a standard way. $Mod(\varphi)$ denotes the set of models of the formula φ, $\models$ denotes logical entailment and $\equiv$ logical equivalence, i.e., $\varphi \models \psi$ iff $Mod(\varphi) \subseteq Mod(\psi)$ and $\varphi \equiv \psi$ iff $Mod(\varphi) = Mod(\psi)$. A *belief base* B denotes the set of beliefs of an agent, it is a finite set of propositional formulae interpreted conjunctively, so that B is identified with the conjunction of its elements. A *profile* $\mathcal{C} = \langle B_1, \dots, B_n\rangle$ is a finite vector of belief bases. A Belief Revision Game (BRG for short) is formalized as follows [Schwind *et al.*, 2015]:

Definition 1 (Belief Revision Game)**.** *A* Belief Revision Game (BRG) *is a tuple* $G = (V, A, \mathcal{L}_{\mathcal{P}}, \mathcal{B}, \mathcal{R})$ *where*

- $V = \{1, \dots, n\}$ *is a finite set of agents;*
- $A \subseteq V \times V$ *is an irreflexive binary relation on* V *which represents the* acquaintances *between the agents;*
- $\mathcal{L}_{\mathcal{P}}$ *is a finite propositional language;*
- $\mathcal{B}$ *is a mapping from* V *to* $\mathcal{L}_{\mathcal{P}}$ *where for each* $i \in V$, $B(i)$ *(noted* B_i*) is the initial belief base of agent* i*;*
- $\mathcal{R} = \{R_1, \dots, R_n\}$*, where each* R_i *is the revision policy of agent* i*, i.e., a mapping from* $\mathcal{L}_{\mathcal{P}} \times {\mathcal{L}_{\mathcal{P}}}^{in(i)}$ *to* $\mathcal{L}_{\mathcal{P}}$ *with* $in(i) = |\{j \mid (j, i) \in A\}|$ *the in-degree of* i*, such that if* $in(i) = 0$*, then* R_i *is the identity function.*

Let $G = (V, A, \mathcal{L}_{\mathcal{P}}, \mathcal{B}, \mathcal{R})$ be a BRG and let us denote $\mathcal{C}_i$ the *context* of i, defined as the profile $\mathcal{C}_i = \langle B_{i_1}, \dots, B_{i_{in(i)}}\rangle$ where $\{i_1, \dots, i_{in(i)}\} = \{i_j \mid (i_j, i) \in A\}$. Then $R_i(B_i, \mathcal{C}_i)$ is the belief base of agent i once revised by taking into account her own current beliefs B_i and her current context $\mathcal{C}_i$.[1]

In a BRG, the beliefs of each agent evolve at each time step according to her revision policy. This induces for each $i \in V$ a *belief sequence* $(B_i^s)_{s\in\mathbb{N}}$ where B_i^s denotes the belief base of agent i after s steps, defined as $B_i^0 = B_i$ and for each $s \geq 0$, $B_i^{s+1} = R_i(B_i^s, \mathcal{C}_i^s)$, where $\mathcal{C}_i^s$ is the context of i at step s. Schwind *et al.* [2015] showed that in any BRG, the belief sequence of each agent i is *cyclic*, i.e., in $(B_i^s)_{s\in\mathbb{N}}$ there exists a finite subsequence $B_i^b, \dots, B_i^e$ such that for every $j > e$, we have $B_j^j \equiv B_i^{b+((j-b) mod (e-b+1))}$; the *belief cycle* $Cyc(i)$ of an agent i corresponds to the series of this subsequence of belief bases $Cyc(i) = B_i^b, B_i^{b+1}, \dots, B_i^e$. As we are interested in determining the pieces of beliefs resulting from the interaction of the agents, we focus on the *outcome* of each agent i in G, denoted $Acc_G(i)$ and defined as:

$$Acc_G(i) = \bigvee\{B_i^s \mid B_i^s \in Cyc(i)\}.$$

We say that a formula φ is *accepted* by i in G if $Acc_G(i) \models \varphi$, φ is *unanimously accepted* in G if φ is accepted by all $i \in V$ in G, and G *converges at step* s if for each $i \in V$, $B_i^{s+1} = B_i^s$.

The formalization of a BRG allows each agent i to consider any revision policy $R_i \in \mathcal{R}$. However, one can take advantage of theoretical tools from Belief Change Theory (see e.g. [Alchourrón *et al.*, 1985]), more precisely, belief revision and merging operators for defining revision policies. A merging operator Δ associates any formula μ (the *integrity constraint*) and any profile $\mathcal{C}$ with a new formula $\Delta_\mu(\mathcal{C})$ (the *merged base*). A merging operator Δ aims at defining the merged base as the beliefs of a group of agents represented by the profile, under some integrity constraint. Standard properties (denoted **(IC0)–(IC8)**) are expected for merging operators, and such operators are called *IC merging operators* (see [Konieczny and Pino Pérez, 2002] for details on these properties).

IC merging operators include some *distance-based* operators, i.e., operators that are based on the selection of some models of the integrity constraint, the "closest" ones to the given profile. These operators are characterized by a distance d between interpretations and an aggregation function f [Konieczny *et al.*, 2004]. They associate with every formula μ and every profile $\mathcal{C}$ a belief base $\Delta_\mu^{d,f}(\mathcal{C})$ satisfying $Mod(\Delta_\mu^{d,f}(\mathcal{C})) = \min(Mod(\mu), \leq_{\mathcal{C}}^{d,f})$, where $\leq_{\mathcal{C}}^{d,f}$ is the total preorder over interpretations induced by $\mathcal{C}$ defined by $\omega \leq_{\mathcal{C}}^{d,f} \omega'$ if and only if $d^f(\omega, \mathcal{C}) \leq d^f(\omega', \mathcal{C})$, where $d^f(\omega, \mathcal{C}) = f_{B\in\mathcal{C}}\{d(\omega, B)\}$ and $d(\omega, B) = \min_{\omega'\models B} d(\omega, \omega')$. Usual distances are d_D, the drastic distance ($d_D(\omega, \omega') = 0$ if $\omega = \omega'$ and 1 otherwise), and d_H the Hamming distance ($d_H(\omega, \omega') = n$ if ω and ω' differ on n variables). Using aggregation functions such as Σ and GMax lead to IC merging operators. For instance, GMax operators consider for each profile $\mathcal{C}$ the total preorder over interpretations $\leq_{\mathcal{C}}^{d,\mathsf{GMax}}$ defined by $\omega \leq_{\mathcal{C}}^{d,\mathsf{GMax}} \omega'$ if and only if $d^{\mathsf{GMax}}(\omega, \mathcal{C}) \leq^{lex} d^{\mathsf{GMax}}(\omega', \mathcal{C})$ (where $\leq^{lex}$ is the lexicographic ordering induced by the natural order) and $d^{\mathsf{GMax}}(\omega, \mathcal{C})$ is the vector of numbers $d_1, \dots, d_n$ obtained by sorting in a descending order the vector $\langle d(\omega, B_i) \mid B_i \in \mathcal{C}\rangle$. Lastly, *belief revision operators* can be seen as belief merging operators applied to singleton profiles: indeed, if Δ is an IC merging operator then the revision operator $\circ_\Delta$ *induced by* Δ defined for all bases B_1, B_2 as $B_1 \circ_\Delta B_2 = \Delta_{B_2}(\langle B_1\rangle)$ satisfies the standard KM revision postulates [Katsuno and Mendelzon, 1992].

Let us go back to BRGs. Six classes of revision policies have been proposed in [Schwind *et al.*, 2015]. Each of them, denoted R_Δ^k ($k \in \{1, \dots, 6\}$) is parameterized by an IC merging operator Δ. Each class is defined as follows,[2] at each step s and for any agent i such that $\mathcal{C}_i \neq \emptyset$:

[1] Abusing notations, $R_i(B_i, \mathcal{C}_i)$ stands for $R_i(B_i, B_{i_1}, \dots, B_{i_{in(i)}})$ where $\mathcal{C}_i = \langle B_{i_1}, \dots, B_{i_{in(i)}}\rangle$.

[2] When using a merging operator without integrity constraints we just note $\Delta(\mathcal{C})$ instead of $\Delta_\top(\mathcal{C})$ for improving readibility.

steps s	B_1^s	B_2^s	B_3^s
0	$p_1 \oplus p_2$	$\neg p_1 \wedge \neg p_2$	$p_1 \vee p_2$
$2k+1$	$\neg p_1 \vee \neg p_2$	$p_1 \oplus p_2$	$p_1 \oplus p_2$
$2k+2$	$p_1 \oplus p_2$	$\neg p_1 \vee \neg p_2$	$\neg p_1 \vee \neg p_2$

Table 1: The belief sequences of Alex, Beth and Chris in G_*.

- $R^1_\Delta(B^s_i, \mathcal{C}^s_i) = \Delta(\mathcal{C}^s_i)$;
- $R^2_\Delta(B^s_i, \mathcal{C}^s_i) = \Delta_{\Delta(\mathcal{C}^s_i)}(\langle B^s_i \rangle)$ $\quad [= B^s_i \circ_\Delta \Delta(\mathcal{C}^s_i)]$;
- $R^3_\Delta(B^s_i, \mathcal{C}^s_i) = \Delta(\langle B^s_i, \mathcal{C}^s_i \rangle)$;
- $R^4_\Delta(B^s_i, \mathcal{C}^s_i) = \Delta(\langle B^s_i, \Delta(\mathcal{C}^s_i) \rangle)$;
- $R^5_\Delta(B^s_i, \mathcal{C}^s_i) = \Delta_{B^s_i}(\Delta(\mathcal{C}^s_i))$ $\quad [= \Delta(\mathcal{C}^s_i) \circ_\Delta B^s_i]$;
- $R^6_\Delta(B^s_i, \mathcal{C}^s_i) = \Delta_{B^s_i}(\mathcal{C}^s_i)$.

Example 1 (continued)**.** *We consider the BRG $G_* = (V_*, A_*, \mathcal{L}_{P_*}, \mathcal{B}_*, \mathcal{R}_*)$ defined as follows. $V_* = \{1, 2, 3\}$ where 1 corresponds to Alex, 2 to Beth, and 3 to Chris. $A_* = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$ expresses that Alex knows Beth and vice-versa, and Alex knows Chris and vice-versa, but Beth and Chris are not connected. $\mathcal{L}_{\mathcal{P}_*}$ is the propositional language defined from the variables $\mathcal{P}_* = \{p_1, p_2\}$, where p_1 stands for "the first meal is healthy" and p_2 means "the second meal is healthy." The initial beliefs of the group members are $B_1 = p_1 \oplus p_2$, $B_2 = \neg p_1 \wedge \neg p_2$ and $B_3 = p_1 \vee p_2$. Assume that all agents use the same revision policy, $R_i = R^1_{\Delta^{d_H,\mathsf{GMax}}}$ for each $i \in V_*$. The belief sequences associated with the three agents are given in Table 1: the belief cycle of agent 1 (resp. 2, 3) is given by (B^0_1, B^1_1) (resp. (B^1_2, B^2_2), (B^1_3, B^2_3)). We have for each $i \in V_*$, $Cyc(i) = p_1 \oplus p_2, \neg p_1 \vee \neg p_2$ and $Acc_G(i) = \neg p_1 \vee \neg p_2$.*

[Schwind *et al.*, 2015] studied the extent to which BRGs satisfy some basic logical properties depending on the class of revision policies used by the agents, and focused on the case when all agents use the same revision policy ranging over R^k_Δ, $k \in \{1, \ldots, 6\}$. It turned out that when the revision policy is induced from the merging operator $\Delta^{d_D,\Sigma}$, i.e., the merging operator based on the drastic distance and the summation function, the underlying BRGs satisfy a number of expected properties [Schwind *et al.*, 2015]. The authors also developed a software allowing one to play BRGs, available online at *http://www.cril.univ-artois.fr/software/brg.html*.

3 On the Notion of Promotion

Belief control in a multi-agent system can take various forms, depending on the meaning given to "control." Here, we are specifically interested in control strategies that consist in promoting a certain belief φ in the beliefs of the agents. A key issue to be addressed is then to determine what "promoting" precisely means in this context. A simple view is to consider that promoting φ in the belief base B of an agent consists in replacing B by a base equivalent to φ. While such a drastic way of promoting φ makes sense, it is not the only one. Thus, for instance, revising B by φ is another approach to do the job. Considering the whole spectrum of promotion techniques is interesting because in some scenarios it could be the case that the agent under consideration can be ready to promote φ by revising her own beliefs B with it, while she would be reluctant in questioning her whole base B and replacing it by φ. In a bribery context, she could for instance ask much more money to accept to change her base B to φ than to change it to the revision of B by φ.

We now characterize the notion of "promotion" of φ through a preorder $\preceq_\varphi$ over belief bases which reflects the closeness relationship to φ; thus, $B' \preceq_\varphi B$ states that B' is at least as close to φ as B. On this ground, promoting B consists in replacing B by any B' satisfying $B' \preceq_\varphi B$:

Definition 2 (φ-promotion)**.** *For every formula $\varphi \in \mathcal{L}_\mathcal{P}$, the φ-promotion relation is the binary relation $\preceq_\varphi$ on $\mathcal{L}_\mathcal{P} \times \mathcal{L}_\mathcal{P}$ defined for all belief bases B, B' as $B' \preceq_\varphi B$ if and only if $B \wedge \varphi \models B' \models B \vee \varphi$.*

Obviously, one can check that for any $\varphi \in \mathcal{L}_\mathcal{P}$, the binary relation $\preceq_\varphi$ is reflexive and transitive. More precisely:

Proposition 1. *For every formula $\varphi \in \mathcal{L}_\mathcal{P}$, $(\mathcal{L}_\mathcal{P}, \preceq_\varphi)$ is a Boolean lattice $(\mathcal{L}_\mathcal{P}, \sqcap_\varphi, \sqcup_\varphi, \dot{\neg}, \varphi, \neg\varphi)$, where:*

- *$\sqcap_\varphi$ is the meet operation defined for each $B, B' \in \mathcal{L}_\mathcal{P}$ as $B \sqcap_\varphi B' = (B \wedge B') \vee ((B \vee B') \wedge \varphi)$,*
- *$\sqcup_\varphi$ is the join operation defined for each $B, B' \in \mathcal{L}_\mathcal{P}$ as $B \sqcup_\varphi B' = (B \wedge B') \vee ((B \vee B') \wedge \neg\varphi)$,*
- *$\dot{\neg}$ is the complement corresponding to the standard logical negation $\neg$,*
- *φ is the least element, i.e., for each $B \in \mathcal{L}_\mathcal{P}$, $\varphi \preceq_\varphi B$,*
- *$\neg\varphi$ is the top element, i.e., for each $B \in \mathcal{L}_\mathcal{P}$, $B \preceq_\varphi \neg\varphi$.*

where formulae of $\mathcal{L}_\mathcal{P}$ are considered up to equivalence.

Every base B' promoting φ in B satisfies $\varphi \preceq_\varphi B' \preceq_\varphi B$. Thus B is (up to logical equivalence) the greatest formula w.r.t. $\preceq_\varphi$ promoting φ in B, and φ is (up to logical equivalence) the least formula w.r.t. $\preceq_\varphi$ promoting φ in B. Stated otherwise, the least demanding promotion of φ w.r.t. B consists in letting B unchanged, while the promotion of φ w.r.t. B leading to a formula as close as possible to φ consists in replacing B by φ.

The following model-theoretic characterization of the notion of promotion can be derived easily. $\blacktriangle$ denotes the symmetric difference between sets:

Proposition 2. *Given a formula φ, let B and B' be two belief bases such that B' promotes φ w.r.t. B. Then $\exists S \subseteq Mod(B) \blacktriangle Mod(\varphi)$ such that $Mod(B') = (Mod(B) \cap Mod(\varphi)) \cup S$.*

This proposition also illustrates the fact that B and φ play symmetric roles in the notion of promotion. Indeed, making B closer to φ is precisely the same as making φ closer to B:

Proposition 3. *$B' \preceq_\varphi B$ if and only if $B' \preceq_B \varphi$.*

When a promotion of φ in B is achieved, the set of interpretations assigning different truth values to B and to φ may only diminish. Formally:

Proposition 4. *Given a formula φ, let B and B' be two belief bases such that B' promotes φ in B. Then $Mod(B') \blacktriangle Mod(\varphi) \subseteq Mod(B) \blacktriangle Mod(\varphi)$.*

We define a *promotion operator* $\odot$ as a mapping from $\mathcal{L}_\mathcal{P} \times \mathcal{L}_\mathcal{P}$ to $\mathcal{L}_\mathcal{P}$ such that $\psi \odot \varphi \preceq_\varphi \psi$. Interestingly, some belief change operators of the literature are promotion operators. We recall below some of the postulates for rational revision operators $\circ$, and contraction operators $-$ [Katsuno and Mendelzon, 1992, Caridroit *et al.*, 2015], as well as some postulates for arbitration (alias symmetric revision) operators $\diamond$ [Liberatore and Schaerf, 1998]:

(R1) $\varphi \circ \mu \models \mu$.

(R2) If $\varphi \wedge \mu$ is consistent, then $\varphi \circ \mu \equiv \varphi \wedge \mu$.

(C1) $\varphi \models \varphi - \mu$.

(C2) If $\varphi \not\models \mu$, then $\varphi - \mu \models \varphi$.

(C4) If $\varphi \models \mu$, then $(\varphi - \mu) \wedge \mu \models \varphi$.

(LS2) $\varphi \wedge \mu \models \varphi \diamond \mu$.

(LS7) $\varphi \diamond \mu \models \varphi \vee \mu$.

Proposition 5. *Let φ be a formula and let B be a belief base. Let $\circ$ be any revision operator satisfying **(R1)** and **(R2)**, let $-$ be any KM contraction operator satisfying **(C1)**, **(C2)** and **(C4)**, and let $\diamond$ be any arbitration operator satisfying **(LS2)** and **(LS7)**. We have:*

1. *(i) $B \circ \varphi \preceq_\varphi B \wedge \varphi \preceq_\varphi B$, (ii) $B \vee \varphi \preceq_\varphi B - \neg\varphi \preceq_\varphi B$, and (iii) $B \diamond \varphi \preceq_\varphi B$;*
2. *$B \circ \varphi$, $B \wedge \varphi$, $B \vee \varphi$, $B - \neg\varphi$ and $B \diamond \varphi$ are incomparable w.r.t. $\preceq_\varphi$ in the general case.*[3]

We can now lift the relation of formula promotion to BRGs defined on the same set of variables V, acquaintance relation A, propositional language $\mathcal{L}_\mathcal{P}$ and revision policies $\mathcal{R}$. Given two BRGs $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ and $G' = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}', \mathcal{R})$ and a formula φ, we note $G' \preceq_\varphi G$ if and only if for each agent $i \in V$, $B'_i \preceq_\varphi B_i$. Finally, we note $G \odot \varphi$ any BRG G' such that $G' \preceq_\varphi G$. Observe that such a promotion operation of φ in G can be non-uniform, i.e., it is not necessarily the case that the promotion of φ in distinct bases of G' is achieved thanks to the same promotion operator. For instance, it can be the case that $B'_i = B_i$ for agent $i \in V$, while $B'_j = B_j \circ \varphi$ for agent $j \in V$, and $B'_k = \varphi$ for agent $k \in V$.

4 On Monotonicity in BRGs

In this section, we focus on the issue of monotonicity for BRGs instantiated with revision policies from the six classes defined in Section 2. Given a merging operator Δ and $E \subseteq \{1, \ldots, 6\}$ R^E_Δ denotes the set $\{R^k_\Delta \mid k \in E\}$. We use the simpler notation R^k_Δ instead of R^E_Δ when $E = \{k\}$. Given a class $\mathcal{G}$ of BRGs and $E \subseteq \{1, \ldots, 6\}$, $\mathcal{G}(R^E_\Delta)$ is the subclass of all BRGs $(V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ from $\mathcal{G}$ where for each $R_i \in \mathcal{R}$, $R_i \in R^E_\Delta$. Additionally, a set of revision policies R^E_Δ is said to satisfy a given property P on a given class $\mathcal{G}$ of BRGs if all BRGs from $\mathcal{G}(R^E_\Delta)$ satisfy P.

Given a BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$, a subset C of V of so-called "controllable agents" whose initial beliefs can be modified, and a formula φ, one is interested in determining how to modify the belief bases of agents in C in order to make φ unanimously accepted in the resulting game (when possible). The objective is thus to determine a successful "control strategy" to be implemented in order to reach the goal when it can be reached. A control strategy for G given C takes here the form of a mapping σ from C to $\mathcal{L}_\mathcal{P}$, stating for each $i \in C$ that B_i must be replaced by $\sigma(i)$; it is successful for φ if φ is unanimously accepted in the BRG obtained by applying σ to G. Typically, one wants to minimize the number of agents in C to be controlled, but the optimization problem under consideration can also be much more complex (for instance, it may take into account the cost of controlling each agent in C which may not be uniform).

[3] This is even the case for $\circ$ and $-$ when they correspond one another thanks to Levi/Harper identities.

steps s	B^s_1	B^s_2	B^s_3
0	$p_1 \oplus p_2$	$\neg p_1 \wedge \neg p_2$	$p_1 \wedge p_2$
$s \geq 1$	$p_1 \oplus p_2$	$p_1 \oplus p_2$	$p_1 \oplus p_2$

Table 2: An example of control strategy for G_* where $\varphi = p_1 \vee p_2$ is unanimously accepted.

Among the potential control strategies is the *basic strategy* σ_φ for G given C defined for any $i \in C$ by $\sigma(i) \equiv \varphi$: it simply amounts to promoting φ as much as possible in the belief base of each agent from C by just replacing it by φ. We have the following surprising result:

Proposition 6. *Given a BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$, a set $C \subseteq V$ of controllable agents, and a formula φ, it can be the case that the basic strategy σ_φ for G given C is not successful for φ, while a control strategy for G given C which is successful for φ exists.*

Example 1 (continued)**.** *Assume that the goal of the restaurant manager is to convince all protagonists that at least one of the meals is healthy, i.e., $\varphi = p_1 \vee p_2$. Note that at the beginning, Chris' beliefs coincide with this goal (since $B_3 = \varphi$) and that φ is not unanimously accepted. So if Chris is the only controllable agent, the basic strategy σ_φ is not successful. However, when we replace Chris' beliefs by $p_1 \wedge p_2$ instead, we get that $Acc_{G_*}(i) = p_1 \oplus p_2$ for each $i \in V$, so $p_1 \vee p_2$ is unanimously accepted (see Table 2). This shows that control is possible here given $C = \{3\}$ as the only controllable agent, but not with the basic strategy.*

This example illustrates the complexity of the controllability issue for BRGs in the general case. This explains why it is important to identify some conditions on BRGs for which focusing on the basic strategy would be enough to decide whether a positive answer can be given or not to the controllability question. In the following, we show that some monotonicity properties can be used as such conditions. A first property of monotonicity has been introduced in [Schwind *et al.*, 2015]. Given a BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$, a formula α and an agent $i \in V$, let us denote $G_{i \to \alpha}$ the BRG $(V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}', \mathcal{R})$ where $\mathcal{B}'$ maps V to $\{B'_1, \ldots, B'_n\}$, with $B'_i = B_i \wedge \alpha$ and for every $j \in V$, $j \neq i$, $B'_j = B_j$.

Definition 3 (Monotonicity (Mon))**.** *A BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ satisfies **(Mon)** if whenever φ is unanimously accepted in G, φ is unanimously accepted in $G_{i \to \varphi}$ for every $i \in V$.*

	$B_i^s \wedge \varphi_{\max}(B^s) \models \bot$ (case (i))	$\varphi_{\max}(B^s) \models B_i^s$ (case (ii))	Otherwise (case (iii))
$R^1_{\Delta^{d_D},\Sigma}$	$\varphi_{\max}(B^s)$	$\varphi_{\max}(B^s) \vee (\varphi_{\max-1}(B^s) \wedge \neg B_i^s)$	$\varphi_{\max}(B^s) \wedge \neg B_i^s$
$R^2_{\Delta^{d_D},\Sigma}$	$\varphi_{\max}(B^s)$	$\varphi_{\max}(B^s)$	$\varphi_{\max}(B^s) \wedge \neg B_i^s$
$R^3_{\Delta^{d_D},\Sigma}$	$\varphi_{\max}(B^s)$	$\varphi_{\max}(B^s)$	$\varphi_{\max}(B^s)$
$R^4_{\Delta^{d_D},\Sigma}$	$\varphi_{\max}(B^s) \vee B_i^s$	$\varphi_{\max}(B^s)$	$\varphi_{\max}(B^s) \vee B_i^s$
$R^5_{\Delta^{d_D},\Sigma}$	B_i^s	$\varphi_{\max}(B^s)$	B_i^s
$R^6_{\Delta^{d_D},\Sigma}$	B_i^s	$\varphi_{\max}(B^s)$	$\varphi_{\max}(B^s) \wedge B_i^s$

Table 3: B_i^{s+1} depending on the revision policy applied by agent i in any BRG from $\mathcal{G}_{com}(R^{\{1,\ldots,6\}}_{\Delta^{d_D},\Sigma})$.

(Mon) is similar to the *monotonicity criterion* in Social Choice Theory. In the BRG context, a formula φ that is unanimously accepted should still be so if some agent's initial beliefs were "strengthened" by φ. Note that the strengthening of a belief base by a formula φ involved here consists in just expanding the base by φ. However, one could consider weaker versions than expansion, i.e., the promotion operators $\odot$ we formalized in the previous section. So we introduce now a stronger version of the monotonicity property (cf. Definition 3) on BRGs based on the promotion relation:

Definition 4 (Strong Monotonicity **(SMon)**). *A BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ satisfies **(SMon)** if for each $i \in V$, if φ is unanimously accepted in G, then φ is unanimously accepted in any BRG $G \odot \varphi$.*

It is easy to see that **(SMon)** implies **(Mon)** and that the converse does not hold in general. BRGs satisfying **(SMon)** are interesting in terms of strategy-proofness. Indeed, Proposition 1 tells us that φ is the least element of $(\mathcal{L}_\mathcal{P}, \preceq_\varphi)$. As a consequence, for such BRGs, determining whether it is possible to convince all the agents involved in the BRG to accept some piece of belief φ simply amounts to determining whether φ is unanimously accepted in the BRG obtained by the promotion of φ in every B_i associated with a controllable agent so that $B_i \odot \varphi = \varphi$. Stated otherwise:

Proposition 7. *Let $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ satisfying **(SMon)**, a set $C \subseteq V$ of controllable agents, and a formula φ. If the basic strategy σ_φ for G given C is not successful for φ then no control strategy for G given C is successful for φ.*

An interesting issue now is to know whether there are BRGs satisfying **(SMon)**. We provide a positive answer to this question in the next section.

5 The Case of Complete Graphs

We now study the extent to which **(SMon)** is satisfied by BRGs whose acquaintance graph is a *complete* graph. This simple topology is adequate to the cases when all the agents of V know each other (for instance, this is the case in meetings where all agents are sitting around a table). We simply call the corresponding class of BRGs the complete BRGs:

Definition 5 (Complete BRG). *A BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ is* complete *if (V, A) is a complete graph, i.e., for all $i, j \in V$, $i \neq j$, $(i, j) \in A$. Given a class $\mathcal{G}$ of BRGs, $\mathcal{G}_{com}$ denotes the subclass of complete BRGs from $\mathcal{G}$.*

In the general case, **(SMon)** is not satisfied by BRGs from $\mathcal{G}_{com}(R^k_\Delta)$ with $k \in \{1, \ldots, 6\}$, even when Δ is "fully" rational in the sense that it satisfies all IC postulates. Indeed:

Proposition 8. *For $\Delta \in \{\Delta^{d_H,\Sigma}, \Delta^{d_H,\mathsf{GMax}}\}$, for any $k \in \{1, \ldots, 6\}$, R^k_Δ does not satisfy **(SMon)** on $\mathcal{G}_{com}$.*

In the following, we investigate the monotonicity issue for complete BRGs when the merging operator used for defining the revision policies is the distance-based merging operator based on the *drastic distance* (and the summation function) $\Delta^{d_D,\Sigma}$. Computing $\Delta^{d_D,\Sigma}_\mu(\mathcal{C})$ consists in selecting in the models of the integrity constraint μ those satisfying as many bases of the profile $\mathcal{C}$ as possible. Several works have proved this specific operator to satisfy a number of expected properties, e.g., some (language) independence conditions [Konieczny *et al.*, 2011, Marquis and Schwind, 2014]. In particular, $\Delta^{d_D,\Sigma}$ is robust from the point of view of strategy-proofness [Everaere *et al.*, 2007], this is why this operator appears as a good candidate for the monotonicity issue.

The following notations are used in the rest of this section. Let $\mathcal{C}$ be a profile and ω be an interpretation. $\bigwedge \mathcal{C} = \bigwedge\{B_i \mid B_i \in \mathcal{C}\}$. $\#E$ denotes the number of elements from a finite set E. $\#_{\max}(\mathcal{C})$ is the maximal number of bases in $\mathcal{C}$ that are consistent when interpreted conjunctively, i.e., $\#_{\max}(\mathcal{C}) = \max(\#\{\mathcal{C}' \mid \mathcal{C}' \subseteq \mathcal{C}, \bigwedge \mathcal{C}' \not\models \bot\})$. $\varphi_{\max}(\mathcal{C}) = \Delta^{d_D,\Sigma}(\mathcal{C})$. And $\varphi_{\max-1}(\mathcal{C})$ is any formula φ such that $Mod(\varphi) = \{\omega \mid \#\{B_i \in \mathcal{C} \mid \omega \models B_i\} = \#_{\max}(\mathcal{C}) - 1\}$.

Let us first show that a stronger version of the agreement preservation property **(AP)** (see [Schwind *et al.*, 2015]) is satisfied by complete BRGs:

Proposition 9. *Let $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^{\{1,\ldots,6\}}_{\Delta^{d_D},\Sigma})$. If there is a step s where $\bigwedge\{B_i^s \mid i \in V\}$ is consistent, then for each $i \in V$, $Acc_G(B_i) = \bigwedge\{B_i \mid i \in V\}$. Moreover, G converges at step $s+2$ at most. It converges at step $s+1$ at most if $G \in \mathcal{G}_{com}(R^{\{2,\ldots,6\}}_{\Delta^{d_D},\Sigma})$.*

We now intend to characterize the outcome of a BRG from its initial state. Beforehand, let us introduce an intermediary result characterizing the belief base of any agent in a BRG from $\mathcal{G}_{com}(R^k_{\Delta^{d_D},\Sigma})$ where $k \in \{1, \ldots, 6\}$, at some step $s+1$ given the belief bases of all agents at step s:

Lemma 1. *Let $E \subseteq \{1, \ldots, 6\}$ and G be a BRG from $\mathcal{G}_{com}(R^E_{\Delta^{d_D},\Sigma})$. Then for each agent $i \in V$ and each step s, her base B_i^{s+1} can be characterized as shown in Table 3.*

Obviously enough, a consequence of Table 3 is that all six revision policies from $R^k_{\Delta^{d_D},\Sigma}$, $k \in \{1, \ldots, 6\}$, remain distinct. Moreover, in the case when all agents have the same revision policy among $R^k_{\Delta^{d_D},\Sigma}$, $k \in \{1, \ldots, 4\}$, the outcome of complete BRGs can be fully characterized:

Proposition 10. *Let $k \in \{1, \ldots, 4\}$ and $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^k_{\Delta^{d_D},\Sigma})$. Then for each $i \in V$, $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle)$.*

The cases when all agents have the same revision policy among $R^k_{\Delta^{d_D},\Sigma}$, $k \in \{5, 6\}$ is simpler to characterize:

Proposition 11. *Let $k \in \{5, 6\}$ and $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^k_{\Delta^{d_D},\Sigma})$.*
- *If $k = 5$, then for all $i \in V$, $Acc_G(i) = B_i$ if $\Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle) \not\models B_i$, otherwise $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle)$.*
- *If $k = 6$, then for all $i \in V$, $Acc_G(i) = B_i$ if $\Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle) \wedge B_i \models \bot$, otherwise $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle) \wedge B_i$.*

These results allow us now to derive monotonicity results for the class of complete BRGs we are dealing with:

Proposition 12. *Let $k \in \{1, \ldots, 6\}$. Then all BRGs from the class $\mathcal{G}_{com}(R^k_{\Delta^{d_D},\Sigma})$ satisfy **(SMon)**.*

A consequence of Propositions 7 and 12 is that considering the basic strategy is enough for BRGs from $\mathcal{G}_{com}(R^k_{\Delta^{d_D},\Sigma})$, i.e., to decide whether a control strategy exists for making a formula φ accepted by all agents in such a BRG, it is enough to focus on the basic strategy.

Let us illustrate how these monotonicity results apply to the following slightly modified version of our running example:

Example 1 (continued). *Consider again the BRG $G_* = (V_*, A_*, \mathcal{L}_{P_*}, \mathcal{B}_*, \mathcal{R}_*)$ from our running example, and let G'_* be the BRG $(V_*, A'_*, \mathcal{L}_{P_*}, \mathcal{B}'_*, \mathcal{R}_*)$ where:*
- *A' is the total relation over V_*, i.e., Beth and Chris know each other as well;*
- *$\mathcal{B}'_*$ maps V_* to $\{B'_1, B'_2, B'_3\}$, with $B'_1 = B_1$, $B'_2 = B_2$ and $B'_3 = \neg p_1 \wedge \neg p_2$, i.e., the beliefs of Alex and Beth in G'_* remain the same as in G_*, but Chris now initially believes in G'_* that none of the two meals is healthy.*

From Proposition 10, we get that $Acc_{G'_*}(i) = \Delta^{d_D,\Sigma}(\langle B'_1, B'_2, B'_3\rangle) \equiv \neg p_1 \wedge \neg p_2$ for each $i \in V_*$. Note that $p_1 \vee p_2$ is not accepted by any agent. By applying the basic strategy σ_φ where $\varphi = p_1 \vee p_2$ and when Chris is the only controllable agent, one gets that the outcome of each agent in the resulting BRG becomes $p_1 \oplus p_2$. We have now φ being unanimously accepted, which means that σ_φ is successful.

6 Related Work

As far as we know, the control problem for multi-agent systems has not been investigated to date in settings similar to BRGs. Though there are many works on opinion dynamics where some control or influence issues have been studied, the opinion of an agent takes typically the form of a real number, as in Hegselmann-Krause's model [Hegselmann and Krause, 2005]. Let us mention among others [Tsang and Larson, 2014, Bloembergen *et al.*, 2014, Ranjbar-Sahraei *et al.*, 2014, Blondel *et al.*, 2009]. The very nature of the agents opinions (real number vs. propositional formula) makes these studies and our own approach quite unrelated. Indeed, unlike real numbers that can be simply averaged, merging propositional formulae is a much more complex process (this is why many works on the topic in the knowledge representation area have been done for two decades). The quite paradoxical result presented in the paper (roughly, promoting some belief is not guaranteed to be the safest way to make it unanimously accepted) echoes paradoxes pointed out in other settings for multi-agent systems, in particular social network games [Apt *et al.*, 2013]. However, the resemblance is only at a shallow level since the BRG model departs a lot from social network games. Especially, social network games are "true" games, unlike BRGs (in the BRG model, agents do not have strategies and there is no payoff based on the adequacy of an agent's beliefs with the beliefs of her acquaintances).

On the other hand, some control issues (under various forms: manipulation, bribery, etc.) have received much attention recently in voting theory, especially from the viewpoint of computational complexity. See among many others [Amanatidis *et al.*, 2015, Bredereck *et al.*, 2015, Faliszewski *et al.*, 2015, Loreggia *et al.*, 2015, Faliszewski *et al.*, 2014, Bouveret and Lang, 2014, Christian *et al.*, 2007]. However, the results obtained in voting theory cannot be exploited directly for BRGs since in BRGs the agents beliefs may evolve after each communication step (when voting is considered, the users preferences are typically considered as fixed and agents do not influence each other). An exception is [Hassanzadeh *et al.*, 2013] which shows how to reach a consensus on rankings (encoding the agents preferences) by iterative voting. However, the corresponding model is quite different from the BRG one since it deals with preferences, and not with beliefs, and the system dynamics is ruled by distinct mechanisms (voting vs. revision). Furthermore, belief control has not been investigated in this model (the focus is on convergence issues).

Finally, other approaches have been considered for modeling the belief dynamics of a group of agents (see dynamic epistemic logic [Harel *et al.*, 2000, Hendricks and Symons, 2006], or in propositional logic [Gauwin *et al.*, 2007, Delgrande *et al.*, 2007, Grandi *et al.*, 2015]), but the belief control issue for such settings has not been considered as well.

7 Conclusion

We pointed out a quite paradoxical result in belief diffusion: in a network of agents, replacing some agents' belief bases by a piece of belief φ may not be the "optimal" way to make φ unanimously accepted. However, we have identified a class of BRGs satisfying a property of strong monotonicity, and thus, avoiding this paradox, making for these BRGs the belief control issue easier to manage.

As perspectives for further research, we plan to identify additional classes of BRGs for which the strong monotonicity property holds, e.g., by considering other topologies as the line, single loop and more generally regular graph topologies. For all these BRGs, the next step will be to search for control strategies by focusing now on the "who" issue, i.e., which agents from a predefined set C should be considered. Another interesting perspective is to study further the notion of "promotion," as an attempt to provide a uniform axiomatization of a large class of standard belief change operators.

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Appendix: Proofs of Propositions

Proposition 1
For every formula $\varphi \in \mathcal{L}_\mathcal{P}$, $(\mathcal{L}_\mathcal{P}, \preceq_\varphi)$ is a Boolean lattice $(\mathcal{L}_\mathcal{P}, \sqcap_\varphi, \sqcup_\varphi, \dot{\neg}, \varphi, \neg\varphi)$, where:

- $\sqcap_\varphi$ is the meet operation defined for each $B, B' \in \mathcal{L}_\mathcal{P}$ as $B \sqcap_\varphi B' = (B \wedge B') \vee ((B \vee B') \wedge \varphi)$,
- $\sqcup_\varphi$ is the join operation defined for each $B, B' \in \mathcal{L}_\mathcal{P}$ as $B \sqcup_\varphi B' = (B \wedge B') \vee ((B \vee B') \wedge \neg\varphi)$,
- $\dot{\neg}$ is the complement corresponding to the standard logical negation $\neg$,
- φ is the least element, i.e., for each $B \in \mathcal{L}_\mathcal{P}$, $\varphi \preceq_\varphi B$,
- $\neg\varphi$ is the top element, i.e., for each $B \in \mathcal{L}_\mathcal{P}$, $B \preceq_\varphi \neg\varphi$.

where formulae of $\mathcal{L}_\mathcal{P}$ are considered up to equivalence.

Proof. Let $\varphi \in \mathcal{L}_\mathcal{P}$. Since the standard inference relation $\models$ on $\mathcal{L}_\mathcal{P}$ induces a Boolean lattice where formulae from $\mathcal{L}_\mathcal{P}$ are considered up to equivalence, it is enough to build an isomorphism f_φ of Boolean lattices from $(\mathcal{L}_\mathcal{P}, \preceq_\varphi)$ to $(\mathcal{L}_\mathcal{P}, \models)$. Let us define f_φ as follows, for each $B \in \mathcal{L}_\mathcal{P}$, $f_\varphi(B) = (B \vee \varphi) \wedge (\neg B \vee \neg\varphi)$. Let $B, B' \in \mathcal{L}_\mathcal{P}$. We have to show that $B' \preceq_\varphi B$ if and only if $f_\varphi(B') \models f_\varphi(B)$. Yet we have $B' \preceq_\varphi B$ if and only if (i) $B' \models B \vee \varphi$ and (ii) $B \wedge \varphi \models B'$. On the one hand, (i) is equivalent to $B' \wedge \neg\varphi \models B \wedge \neg\varphi$. On the other hand, (ii) is equivalent to $\neg B' \models \neg B \vee \neg\varphi$, which is equivalent $\neg B' \wedge \varphi \models \neg B \wedge \varphi$. Thus $B' \preceq_\varphi B$ if and only if $B' \wedge \neg\varphi \models B \wedge \neg\varphi$ and $\neg B' \wedge \varphi \models \neg B \wedge \varphi$ if and only if $(B' \wedge \neg\varphi) \vee (\neg B' \wedge \varphi) \models (B \wedge \neg\varphi) \vee (\neg B \wedge \varphi)$ if and only if $f_\varphi(B') \models f_\varphi(B)$. Then it is easy to verify that:

- $f_\varphi(B \sqcap_\varphi B') = f_\varphi(B) \wedge f_\varphi(B')$,
- $f_\varphi(B \sqcup_\varphi B') = f_\varphi(B) \vee f_\varphi(B')$,
- $f_\varphi(\neg B) = \dot{\neg} f_\varphi(B)$,
- $f_\varphi(\varphi) = \bot$,
- $f_\varphi(\neg\varphi) = \top$.

□

Proposition 2
Given a formula φ, let B and B' be two belief bases such that B' promotes φ w.r.t. B. Then $\exists S \subseteq Mod(B) \blacktriangle Mod(\varphi)$ such that $Mod(B') = (Mod(B) \cap Mod(\varphi)) \cup S$.

Proof. Let φ be a formula, B and B' be two belief bases and assume that B' promotes φ w.r.t. B. Then we have (a) $B \wedge \varphi \models B'$ and (b) $B' \models B \vee \varphi$. Let α_S be the formula defined as $\alpha_S = B' \wedge (\neg B \vee \neg\varphi)$. From (b) we get that $\alpha_S \models (B \vee \varphi) \wedge (\neg B \vee \neg\varphi)$, so $Mod(\alpha_S) \subseteq Mod(B) \Delta Mod(\varphi)$. Yet B' can be written as the formula $(B' \wedge (B \wedge \varphi)) \vee \alpha_S$. so from (a) B' is equivalent to the formula $(B \wedge \varphi) \vee \alpha_S$. Hence, $Mod(B') = (Mod(B) \cap Mod(\varphi)) \cup Mod(\phi_S)$. □

Proposition 3
$B' \preceq_\varphi B$ if and only if $B' \preceq_B \varphi$.

Proof. Direct by definition. □

Proposition 4
Given a formula φ, let B and B' be two belief bases such that B' promotes φ in B. Then $Mod(B') \blacktriangle Mod(\varphi) \subseteq Mod(B) \blacktriangle Mod(\varphi)$.

Proof. If B' promotes φ in B, then we have $B \wedge \varphi \models B' \models B \vee \varphi$. By contraposition, we have $\neg B \wedge \neg\varphi \models \neg B' \models \neg B \vee \neg\varphi$, i.e., $\neg B' \preceq_{\neg\varphi} \neg B$. Let ω be a model of $\neg B' \wedge \varphi$. Then since $\neg B' \models \neg B \vee \neg\varphi$ holds, we also have that ω is a model of $(\neg B \vee \neg\varphi) \wedge \varphi$, i.e., a model of $\neg B \wedge \varphi$. Let ω be a model of $B' \wedge \neg\varphi$. Then since $B' \models B \vee \varphi$ holds, we also have that ω is a model of $(B \vee \varphi) \wedge \neg\varphi$, i.e., a model of $B \wedge \neg\varphi$. This concludes the proof. □

Proposition 5
Let φ be a formula and let B be a belief base. Let $\circ$ be any revision operator satisfying **(R1)** and **(R2)**, let $-$ be any KM contraction operator satisfying **(C1)**, **(C2)** and **(C4)**, and let $\diamond$ be any arbitration operator satisfying **(LS2)** and **(LS7)**. We have:

1. (i) $B \circ \varphi \preceq_\varphi B \wedge \varphi \preceq_\varphi B$, (ii) $B \vee \varphi \preceq_\varphi B - \neg\varphi \preceq_\varphi B$, and (iii) $B \diamond \varphi \preceq_\varphi B$;
2. $B \circ \varphi$, $B \wedge \varphi$, $B \vee \varphi$, $B - \neg\varphi$ and $B \diamond \varphi$ are incomparable w.r.t. $\preceq_\varphi$ in the general case.

Proof.

- $B \wedge \varphi \preceq_\varphi B$ holds since $B \wedge \varphi \models B \wedge \varphi \models B \vee \varphi$.
- $B \circ \varphi \preceq_\varphi B \wedge \varphi \preceq_\varphi B$ since $B \wedge \varphi \wedge \varphi \models B \circ \varphi$ is equivalent to state that $B \wedge \varphi \models B \circ \varphi$ (which holds as soon as $\circ$ satisfies **(R2)**) and $B \circ \varphi \models (B \wedge \varphi) \vee \varphi$ holds as soon as $\circ$ satisfies **(R1)**).
- $B \vee \varphi \preceq_\varphi B - \neg\varphi$ holds since $(B - \neg\varphi) \wedge \varphi \models B \vee \varphi$ holds, and $B \vee \varphi \models (B - \neg\varphi) \vee \varphi$ holds since $B \models B - \neg\varphi$ holds, as soon as $-$ satisfies **(C1)**.
- $B - \neg\varphi \preceq_\varphi B$ holds since $B \wedge \varphi \models B - \neg\varphi$ because $B \wedge \varphi \models B$ holds and $B \models B - \neg\varphi$ holds, as soon as $-$ satisfies **(C1)**, and $B - \neg\varphi \models B \vee \varphi$ holds because $B - \neg\varphi \models B$ when $B \not\models \neg\varphi$ and **(C2)** is satisfied by $-$, and $(B - \neg\varphi) \wedge \neg\varphi \models B$ when $B \models \neg\varphi$ and **(C4)** is satisfied by $-$.
- $B \diamond \varphi \preceq_\varphi B$ trivially holds when $\diamond$ satisfies **(LS2)** and **(LS7)**.

Incomparabilities are easy to show as well using counter-examples. □

Proposition 6
Given a BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$, a set $C \subseteq V$ of controllable agents, and a formula φ, it can be the case that the basic strategy σ_φ for G given C is not successful for φ, while

a control strategy for G given C which is successful for φ exists.

Proof. The example directly following the proposition serves as a proof. □

Proposition 7
Let $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ satisfying **(SMon)**, a set $C \subseteq V$ of controllable agents, and a formula φ. If the basic strategy σ_φ for G given C is not successful for φ then no control strategy for G given C is successful for φ.

Proof. First, let us point out that for every formula φ, since φ is the least element of $(\mathcal{L}_\mathcal{P}, \preceq_\varphi)$ for any formula φ, then by definition of a promotion operator we get that $\varphi \preceq_\varphi B \odot \varphi$ for every base B and every promotion operator $\odot$. This means that for every BRG $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$, for any set $C \subseteq V$ and any formula φ, we have that

$$G_{\sigma_\varphi} \preceq_\varphi G_\sigma \odot \varphi, \quad (1)$$

where σ_φ is the basic strategy for G given C, σ is any strategy for G given C, and G_{σ_φ} (resp. G_σ) is the BRG obtained by applying σ_φ (resp. σ) to G.

Now, let $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG satisfying **(SMon)**, $C \subseteq V$ be a set of controllable agents, and φ be a formula. Let us show the contraposite of the statement to be proved. Assume that there is a control strategy σ for G given C which is successful for φ, that is, there exists a mapping σ from C to $\mathcal{L}_\mathcal{P}$ stating for each $i \in C$ that B_i is replaced by $\sigma(i)$, such that φ is unanimously accepted in the BRG G_σ obtained by applying σ to G. But since G satisfies **(SMon)**, φ must be also be unanimously accepted in any BRG $G_sigma \odot \varphi$. In particular, from Equation 1 above φ is unanimously accepted in the BRG B_{σ_φ} obtained by applying the basic strategy σ_φ to G. So the basic strategy σ_φ for G given C is successful for φ. This concludes the proof. □

Proposition 8
For $\Delta \in \{\Delta^{d_H,\Sigma}, \Delta^{d_H,\mathsf{GMax}}\}$, for any $k \in \{1, \ldots, 6\}$, R^k_Δ does not satisfy **(SMon)** on $\mathcal{G}_{com}$.

Proof. Let $\Delta \in \{\Delta^{d_H,\Sigma}, \Delta^{d_H,\mathsf{GMax}}, \Delta^{d_H,\mathsf{GMin}}\}$ and $k \in \{1, \ldots, 6\}$. Let $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be the BRG from $\mathcal{G}_{com}(R^k_\Delta)$ defined as $V = \{1, 2, 3, 4\}$, $\mathcal{L}_\mathcal{P}$ is the propositional language defined from $\mathcal{P} = \{p, q, r\}$, $B_1 = p \wedge q \wedge r$, $B_2 = p \wedge \neg q \wedge r$, $B_3 = \neg p \wedge \neg q \wedge r$ and $B_4 = \top$. Let $\varphi \equiv (p \oplus q) \wedge r$. Then one can verify that B_4 is stable and consists of the single belief base $p \wedge \neg q \wedge r$, i.e., φ is accepted by agent 4 in G. On the other hand, consider the BRG $G_{1 \to q \wedge r} = (V, A, \mathcal{L}_\mathcal{P}, B', \mathcal{R})$ and note that we have $q \wedge r \preceq_\varphi p \wedge q \wedge r$, i.e., B'_1 promotes φ w.r.t. B_1, however B'_4 is stable and consists of the single belief base $\neg q \wedge r$, i.e., φ is no longer accepted by agent 4 in $G_{1 \to q \wedge r}$. □

Proposition 9
Let $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^{\{1,\ldots,6\}}_{\Delta^{d_D},\Sigma})$. If there is a step s where $\bigwedge\{B^s_i \mid i \in V\}$ is consistent, then for each $i \in V$, $Acc_G(B_i) = \bigwedge\{B_i \mid i \in V\}$. Moreover, G converges at step $s + 2$ at most. It converges at step $s + 1$ at most if $G \in \mathcal{G}_{com}(R^{\{2,\ldots,6\}}_{\Delta^{d_D},\Sigma})$.

Proof. Let $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^{\{2,\ldots,6\}}_{\Delta^{d_D},\Sigma})$, $s \geq 0$ and assume $\varphi = \bigwedge\{B^s_i \mid i \in V\}$ is consistent. Let $i \in V$. It is easy to see that B^s_i can be expressed as a formula $\varphi \vee \alpha^s_i$ and $\Delta^{d_D,\Sigma}(\langle \mathcal{C}^s_i \rangle)$ as a formula $\varphi \vee \beta^s_i$, where both formulae $\varphi \wedge \alpha^s_i$, $\varphi \wedge \beta^s_i$ and $\alpha^s_i \wedge \beta^s_i$ are inconsistent. We use these notations in the rest of the proof. We fall into the following cases for each $i \in V$:

- $R_i = R^2_{\Delta^{d_D},\Sigma}$. Then $B^{s+1}_i = B_i \circ^{d_D} \Delta^{d_D,\Sigma}(\langle \mathcal{C}^s_i \rangle) = (\varphi \vee \alpha^s_i) \circ^{d_D} (\varphi \vee \beta^s_i) = \varphi$.
- $R_i = R^3_{\Delta^{d_D},\Sigma}$. Then $B^{s+1}_i = \Delta^{d_D,\Sigma}(\langle B^s_i, \mathcal{C}^s_i \rangle) = \varphi$.
- $R_i = R^4_{\Delta^{d_D},\Sigma}$. Then $B^{s+1}_i = \Delta^{d_D,\Sigma}(\langle B_i, \Delta^{d_D,\Sigma}(\langle \mathcal{C}^s_i \rangle) \rangle) = \Delta^{d_D,\Sigma}(\langle \varphi \vee \alpha^s_i, \varphi \vee \beta^s_i \rangle) = \varphi$.
- $R_i \in \{R^5_{\Delta^{d_D},\Sigma}, R^6_{\Delta^{d_D},\Sigma}\}$. Then $B^{s+1}_i = \Delta^{d_D,\Sigma}(\langle \mathcal{C}^s_i \rangle) \circ^{d_D} B_i = (\varphi \vee \beta^s_i) \circ^{d_D} (\varphi \vee \alpha^s_i) = \varphi$.

We just showed that if $G \in \mathcal{G}_{com}(R^{\{2,\ldots,6\}}_{\Delta^{d_D},\Sigma})$, then for each $i \in V$, $B^{s+1}_i = \varphi$ and G converges at step $s + 1$ at most.

Now, if $G \in \mathcal{G}_{com}(R^{\{1,\ldots,6\}}_{\Delta^{d_D},\Sigma})$, then we may fall into the additional case where for an agent $i \in V$, $R_i = R^1_{\Delta^{d_D},\Sigma}$; then $B^{s+1}_i = \Delta^{d_D,\Sigma}(\langle \mathcal{C}^s_i \rangle) = \varphi \vee \beta^s_i \models \varphi$. Yet one can remark that $\bigwedge\{\beta^s_i \mid i \in V\}$ is inconsistent. Indeed, towards a contradiction if there were an interpretation $\omega \models \bigwedge\{\beta^s_i \mid i \in V\}$, this would mean that $\omega \models \varphi_{\max}(\mathcal{C}^{s+1}_i)$ for every $i \in V$, i.e., $\omega \models \varphi_{\max}(B)$, that is, $\omega \models \Delta^{d_D,\Sigma}(\langle B^s_i, \mathcal{C}^s_i \rangle) \models \varphi$, which contradicts the fact that $\varphi \wedge \beta^s_i$ is inconsistent for each $i \in V$. This shows that at step $s + 2$, $B^{s+2}_i = \varphi$. We just showed that if $G \in \mathcal{G}_{com}(R^{\{1,\ldots,6\}}_{\Delta^{d_D},\Sigma})$, then for each $i \in V$, $B^{s+2}_i = \varphi$ and G converges at step $s + 2$ at most. □

Proposition 10
Let $k \in \{1, \ldots, 4\}$ and $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^k_{\Delta^{d_D},\Sigma})$. Then for each $i \in V$, $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n \rangle)$.

Proof.

- Let $k = 3$, then the result is direct. Then for each $i \in V$, we have $B^1_i = \varphi_{\max}(B^0) = \Delta^{d_D,\Sigma}(\langle B \rangle)$. So for each $i \in V$, $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B \rangle)$.
- Let $k = 2$. First, Table 3 shows that for each $i \in V$, $B^1_i \models \varphi_{\max}(B^0)$, so by **(UP)** we get that for each step $s \geq 1$ and for each $i \in V$,

$$B^s_i \models \varphi_{\max}(B^0). \quad (2)$$

Then, let us show that for each step $s \geq 0$, $\varphi_{\max}(B^s) \models \varphi_{\max}(B^{s+1})$ by induction on s. In the case where there is no $\omega' \models \varphi_{\max}(B^s)$, $\omega' \neq \omega$, then according to Table 3, $\omega \models \varphi_{\max}(B^{s+1})$ since no base B_i^s is in case (iii). So assume there is $\omega' \models \varphi_{\max}(B^s)$, $\omega' \neq \omega$. From Table 3, one can see that ω and ω' are models of the same number of bases B^{s+1} for bases B_i^s that are in the cases (i) or (ii). So one only needs to show that ω and ω' are models of the same number of bases B^{s+1} for bases B_i^s that are in the case (iii). Yet from Table 3 we get that for each interpretation $\omega'' \models \varphi_{\max}(B^s)$, we have that $\omega'' \models B_i^s$ if and only if $\omega'' \not\models B_i^{s+1}$ when B_i^s is in the case (iii). Since both ω, ω' are in the same number of bases B^{s+1} for bases B_i^s that are in the cases (i) or (ii), they are as well in the same number of bases B^{s+1} for bases B_i^s that are in the case (iii). Hence, both ω, ω' are in the same number of bases B^{s+1} for bases B_i^s that are in the case (iii). This shows that $\#\{B^{s+1} \mid \omega \models B^{s+1}\} = \#\{B^{s+1} \mid \omega' \models B^{s+1}\}$. Yet by Equation 2, $\#\neg\varphi_{\max}(B^{s+1}) = 0$. Thus $\omega \models \varphi_{\max}(B^{s+1})$. We just proved that for each step $s \geq 0$,

$$\varphi_{\max}(B^s) \models \varphi_{\max}(B^{s+1}). \tag{3}$$

What is left to be shown is that each interpretation $\omega \models \varphi_{\max}(B^0)$ is a model of one of the belief bases in the belief cycle of each agent i. It is enough to show that for each $s \geq 0$, $\omega \models \varphi_{\max}(B^s)$ and each $i \in V$, if $\omega \not\models B_i^s$ then $\omega \models B_i^{s+1}$. By Table 3 B_i^s cannot be in the case (ii), so B_i^s is in the case (i) or (iii), and one directly get that $\omega \models B_i^{s+1}$ from the table. This last result together with Equation 2 show that for each $i \in V$, $Acc_G(i) = \varphi_{\max}(B^0)$, which means that $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B\rangle)$ and concludes the proof.

• Let $k = 1$. If $\varphi_{\max-1}(B^0) \models \bot$ then the BRG at step 0 is equivalent to the case where $k = 2$; Equation 2 then holds and the BRG is played similarly to the case where $k = 2$. Then assume $\varphi_{\max-1}(B^0) \not\models \bot$ and let us prove that (a) $\#\neg\varphi_{\max}(B^1) \leq \#_{\max}(B^1) - 1$ and (b) $\#\neg\varphi_{\max}(B^2) \leq \#_{\max}(B^2) - 2$. For (a) we show that if $\omega \models \neg\varphi_{\max}(B^0)$, then $\omega \not\models \varphi_{\max}(B^1)$. Toward a contradicton, assume $\omega \models \varphi_{\max}(B^1)$. This means that all bases B_i^0 are in the case (ii), which implies that for each $i \in V$, $\varphi_{\max}(B^0) \models B_i^0$, and this contradicts Proposition 9. So $\omega \not\models \varphi_{\max}(B^1)$ and (a) holds. Let us prove (b). If $\varphi_{\max-1}(B^1) \models \bot$ then Table 3 shows that for each $i \in V$, $B_i^2 \models \varphi_{\max}(B^1)$, so by **(UP)** we get that for each step $s \geq 2$ and for each $i \in V$, $B_i^s \models \varphi_{\max}(B^1) = \varphi_{\max}(B^1)$. Then one can use the same proof as for $k = 2$, where Equation 3 is proved by using (a) instead of Equation 2 at step $s = 1$. Assume then $\varphi_{\max-1}(B^1) \not\models \bot$ and let $\omega \models \varphi_{\max-1}(B^1)$. From Table 3 the only case we may have is that the BRG has 3 agents $V = \{1, 2, 3\}$, with $B_1^0 = B_2^0$. and B_3^0 satisfies $B_3^0 \wedge B_1^0 \models \bot$. B_1^0 and B_2^0 are in the case (ii), B_3^0 is in the case (i) and then $\omega \models B_3^0$. In this case one can check that $\#\neg\varphi_{\max}(B^2) \leq \#_{\max}(B^2) - 2$, so (b) holds. Thus $\varphi_{\max-1}(B^2) \models \bot$, so Table 3 shows that for each $i \in V$, $B_i^3 \models \varphi_{\max}(B^2)$, and by **(UP)** we get that for each step $s \geq 3$ and for each $i \in V$, $B_i^s \models \varphi_{\max}(B^2) = \varphi_{\max}(B^0)$. Then one can use the same proof as for $k = 2$, where Equation 3 is proved by using (b) instead of Equation 2 at step $s = 2$.

• Let $k = 4$. By Table 3, for each $i \in V$ we have that $\varphi_{\max}(B^0) \models B_i^1$, i.e., $\varphi_{\max}(B^0) \models \bigwedge\{B_i^1 \mid i \in V\}$. Let us show that $\bigwedge\{B_i^1 \mid i \in V\} \models \varphi_{\max}(B^0)$. Towards a contradiction, let $\omega \not\models \varphi_{\max}(B^0)$ and assume that $\omega \models \bigwedge\{B_i^1 \mid i \in V\}$. Then $\omega \models B_i^1$ for each $i \in V$. Yet according to Table 3 all bases B_i^0 are necessarily in cases (i) or (iii), and in both cases this means that $\omega \models B_i^0$ for each $i \in V$. This contradicts $\omega \not\models \varphi_{\max}(B^0)$. Hence, $\bigwedge\{B_i^1 \mid i \in V\} \models \varphi_{\max}(B^0)$ and so $\bigwedge\{B_i^1 \mid i \in V\} = \varphi_{\max}(B^0)$. On the other hand, by Proposition 9 we get that for each $i \in V$, $Acc_G(i) = \bigwedge\{B_i^1 \mid i \in V\}$, which means that $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B\rangle)$. □

Proposition 11
Let $k \in \{5, 6\}$ and $G = (V, A, \mathcal{L}_\mathcal{P}, \mathcal{B}, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^k_{\Delta^{d_D,\Sigma}})$.

• If $k = 5$, then for all $i \in V$, $Acc_G(i) = B_i$ if $\Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle) \not\models B_i$, otherwise $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle)$.

• If $k = 6$, then for all $i \in V$, $Acc_G(i) = B_i$ if $\Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle) \wedge B_i \models \bot$, otherwise $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B_1, \ldots, B_n\rangle) \wedge B_i$.

Proof. The set of belief bases B^1 can be directly derived from Table 3. Then one can easily verify that $\varphi_{\max}(B^0) = \varphi_{\max}(B^1)$, which means that for each $i \in V$, $B_i^2 = B_i^1$. □

Proposition 12
Let $k \in \{1, \ldots, 6\}$. Then all BRGs from the class $\mathcal{G}_{com}(R^k_{\Delta^{d_D,\Sigma}})$ satisfy **(SMon)**.

Proof. Let $k \in \{1, \ldots, 6\}$, $G = (V, A, \mathcal{L}_\mathcal{P}, B, \mathcal{R})$ be a BRG from $\mathcal{G}_{com}(R^k_{\Delta^{d_D,\Sigma}})$, φ be a formula and $G' = (V, A, \mathcal{L}_\mathcal{P}, B', \mathcal{R})$ be a BRG such that $G' \preceq_\varphi G$. Since for each $i \in V$, $B'_i \preceq_\varphi B_i$, we get that for every interpretation ω, if $\omega \models \varphi$ then for each $i \in V$, if $\omega \models B_i$ then $\omega \models B'_i$, so

$$\omega \models \varphi \implies \#\{B \mid \omega \models B\} \leq \#\{B' \mid \omega \models B'\} \tag{4}$$

and if $\omega \not\models \varphi$ and $\omega \models B'_i$, then $\omega \models B_i$, so

$$\omega \not\models \varphi \implies \#\{B' \mid \omega \models B'\} \leq \#\{B \mid \omega \models B\} \tag{5}$$

Let us now prove that if $\varphi_{\max}(B) \wedge \varphi$ is consistent, then $\varphi_{\max}(B') \models \varphi_{\max}(B) \vee \varphi$. Let $\omega \models \varphi_{\max}(B')$, $\omega \models \neg\varphi$ and let us show that $\omega \models \varphi_{\max}(B)$. Toward a contradiction, assume $\omega \not\models \varphi_{\max}(B)$. This means that there is an interpretation $\omega' \models \varphi_{\max}(B)$, $\omega' \models \varphi$ (since $\varphi_{\max}(B) \wedge \varphi$ is consistent) such that $\#\{B \mid \omega \models B\} < \#\{B \mid \omega' \models B\}$. Yet on the one hand, $\omega' \models \varphi$, so by Equation 4 we get that $\#\{B \mid \omega' \models B\} \leq \#\{B' \mid \omega' \models B'\}$. On the other hand, $\omega \not\models \varphi$, so by Equation 5 we get that $\#\{B' \mid \omega \models B'\} \leq \#\{B \mid \omega \models B\}$. Thus overall we have that $\#\{B' \mid \omega \models B'\} < \#\{B' \mid \omega' \models B'\}$, which means that $\omega \models \varphi_{\max}(B')$ and leads to a contradiction. Therefore, we proved that

$$\varphi_{\max}(B) \wedge \varphi \not\models \bot \implies \varphi_{\max}(B') \models \varphi_{\max}(B) \vee \varphi \tag{6}$$

We are ready to prove the strong monotonicity result. We need to show that for each agent $i \in V$, if φ is accepted by i in G then φ is accepted by i in G'. Let $i \in V$ and assume that φ is accepted by i in G, i.e., $Acc_G(i) \models \varphi$.

- If $k \in \{1, \dots 4\}$, from Proposition 10 we get that $Acc_G(i) = \Delta^{d_D,\Sigma}(B)$, i.e., $Acc_G(i) = \varphi_{\max}(B)$. Since $Acc_G(i) \models \varphi$, we have that $\varphi_{\max}(B) \models \varphi$. Thus from Equation 6 we get that $\varphi_{\max}(B') \models \varphi$. By using Proposition 10 again, since $\varphi_{\max}(B') = \Delta^{d_D,\Sigma}(B')$ we get that $Acc_{G'}(i) \models \varphi$, that is, φ is accepted by i in G'.
- If $k = 5$ and $\Delta^{d_D,\Sigma}(\langle B \rangle) \models B_i$, then $Acc_G(i) = \Delta^{d_D,\Sigma}(\langle B \rangle)$ from Proposition 11, i.e., $Acc_G(i) = \varphi_{\max}(B)$. Since $Acc_G(i) \models \varphi$, we have that $\varphi_{\max}(B) \models \varphi$. Thus from Equation 6 we get that $\varphi_{\max}(B') \models \varphi$. By using Proposition 10 again, in any case we get that $Acc_{G'}(i) \models \varphi$, so φ is accepted by i in G'.
- If $k = 6$ and $\Delta^{d_D,\Sigma}(\langle B \rangle) \wedge B_i \not\models \bot$, then from Proposition 11 we have that $Acc_G(i) = \Delta^{d_D,\Sigma}(B) \wedge B_i$. On the one hand, since $\varphi_{\max}(B) \wedge B_i \not\models \bot$ from Equation 6 we get that $\varphi_{\max}(B') \models \varphi_{\max}(B) \vee \varphi$, so $\varphi_{\max}(B') \models \varphi$, i.e., $\Delta^{d_D,\Sigma}(B') \models \varphi$. On the other hand, $B'_i \preceq_\varphi B_i$ so $B'_i \models B_i \vee \varphi$, yet $B_i \models \varphi$ so $B'_i \models \varphi$. We have that $\Delta^{d_D,\Sigma}(B') \wedge B'_i \models \varphi$, and from Proposition 11 we get that $Acc_{G'}(i) \models \varphi$, so φ is accepted by i in G'.
- The remaining cases are when ($k = 5$ and $\Delta^{d_D,\Sigma}(\langle B \rangle) \not\models B_i$) or ($k = 6$ and $\Delta^{d_D,\Sigma}(\langle B \rangle) \wedge B_i \models \bot$. In both cases, from Proposition 11 we get that $Acc_G(i) = B_i$. Yet $B'_i \preceq_\varphi B_i$ so $B'_i \models B_i \vee \varphi$, yet $B_i \models \varphi$ so $B'_i \models \varphi$, which means that φ is accepted by i in G'. □