Deep Dive into CDCL Pseudo-Boolean Solvers

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Context

Boolean Satisfiability

The satisfiability problem (SAT) is the first problem proven to be NP-complete (Cook, 1971)

Given a CNF formula Σ, this problem is determining whether there exists an assignment of the (Boolean) variables of Σ such that this formula evaluates to true

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Modern SAT solvers (Glucose, CaDiCaL, Kissat) can now deal with problems containing millions of variables and clauses

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While modern SAT solvers perform poorly on such instances for $n > 20$, PB solvers based on cutting-planes may solve them in linear time

Pseudo-Boolean (PB) Constraints

We consider conjunctions of linear equations or inequations over Boolean variables of the form:

$$\sum_{i=1}^{n} \alpha_i \ell_i \bigtriangleup \delta$$

in which

- the coefficients α_i are integers
- ℓ_i are literals, i.e., a variable v or its negation $\bar{v} = 1 - v$
- $\bigtriangleup$ is a relational operator among $\{<, \leq, =, \geq, >\}$
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For example:

$$-3a + 4b - 7c + d \leq -5$$

Normalized PB Constraints

Without loss of generality, we consider conjunctions of normalized PB constraints of the form:

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$$-3a + 4b - 7c + d \leq -5 \equiv 3a + 4\bar{b} + 7c + \bar{d} \geq 10$$

The CDCL Architecture

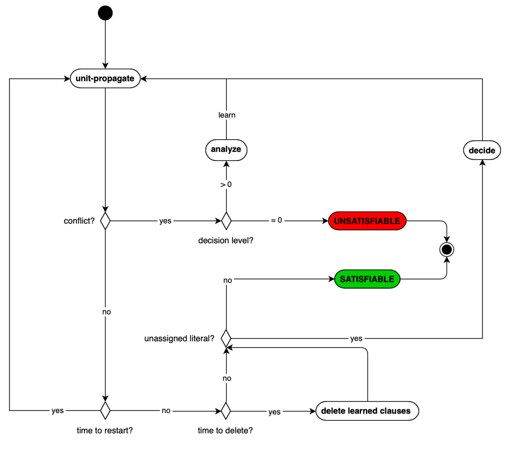

Figure 1: Overview of the CDCL algorithm

Extending the CDCL Architecture

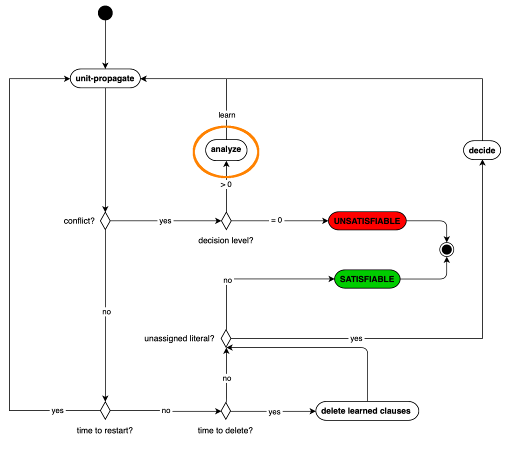

Figure 2: Use of the proof system in the CDCL algorithm

Fitting Cutting-Planes into the CDCL Architecture

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The constraint above can be rewritten as $c \wedge 3a + 4\bar{b} + \bar{d} \geq 3$
but also as $c \wedge (a \vee \bar{b})$

Cutting Planes and Generalized Resolution

Many PB solvers have been designed based on the Generalized Resolution (Hooker, 1988).

$$\frac{\alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta_1 \qquad \beta\overline{\ell} + \sum_{i=1}^{n} \beta_i \ell_i \geq \delta_2}{\sum_{i=1}^{n} (\beta\alpha_i + \alpha\beta_i)\ell_i \geq \beta\delta_1 + \alpha\delta_2 - \alpha\beta} \text{(cancellation)}$$

$$\frac{\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta}{\sum_{i=1}^{n} \min(\alpha_i, \delta)\ell_i \geq \delta} \text{(saturation)}$$

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As with the resolution rule in classical SAT solvers, these two rules can be used to learn new constraints during conflict analysis

Analyzing Conflicts

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This conflict is analyzed by applying the cancellation rule as follows:

$$\frac{6\bar{b} + 6c + 4e + f + g + h \geq 7 \qquad 5a + 4b + c + d \geq 6}{15a + 15c + 8e + 3d + 2f + 2g + 2h \geq 20}$$

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The constraint we obtain here is no longer conflicting!

Weakening

To preserve the conflict, the weakening rule must be used:

$$\frac{\alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta}{\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta - \alpha} \text{ (weakening)}$$

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Weakening can be applied in many different ways (Le Berre et al., 2020b)

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Another solution is to take advantage of the following property:

As soon as the coefficient of the literal to cancel is equal to 1 in at least one of the constraints, the derived constraint is guaranteed to be conflicting (Dixon, 2004)

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This strategy is equivalent to that used by solvers such as *SATIRE* (Whittemore and Sakallah, 2001) or *Sat4j-Resolution* to lazily infer clauses to use resolution based reasoning

Weakening and Division

In *RoundingSat* (Elffers and Nordström, 2018), the coefficient is rounded to one thanks to the division rule, applied after having weakened away some unfalsified literals

$$\frac{\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta \qquad \rho \in \mathbb{N}^*}{\sum_{i=1}^{n} \lceil \frac{\alpha_i}{\rho} \rceil \ell_i \geq \lceil \frac{\delta}{\rho} \rceil} \text{(division)}$$

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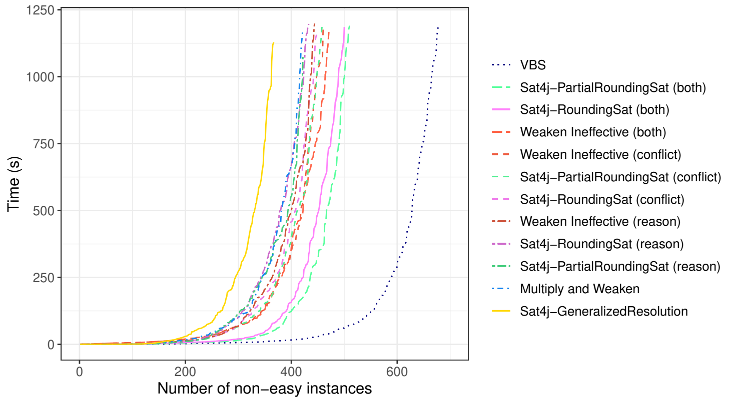
It is also possible to apply partial weakening before division to infer stronger constraints

Many Different Strategies



An Achilles Heel in the Cutting Planes Proof System

Irrelevant Literals (Le Berre et al., 2020a)

Cutting planes rules may introduce irrelevant literals

$$\frac{3d + a + b + c \geq 3 \qquad 3\bar{d} + 2a + 2b \geq 3}{3a + 3b + c \geq 3}$$

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Production of Irrelevant Literals

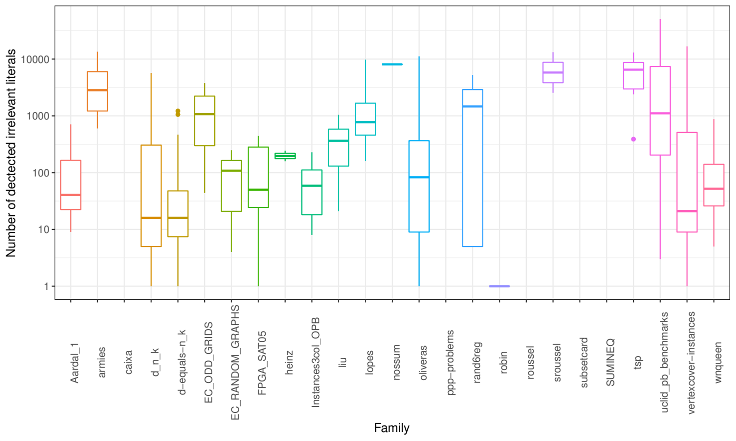

Figure 3: Statistics about the production of irrelevant literals in *Sat4j-GeneralizedResolution* for each family of benchmarks (logarithmic scale)

Artificially Relevant Literals

Irrelevant literals may become artificially relevant, in which case they may impact the strength of the derived constraints

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Detecting irrelevant literals is NP-hard, we thus introduce an incomplete algorithm for removing them

Detecting Irrelevant Literals (1)

Irrelevant literals can be detected thanks to this reduction to subset-sum

$$\ell \text{ is irrelevant in } \alpha\ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta$$

$$\Leftrightarrow \sum_{i=1}^{n} \alpha_i \ell_i = \delta - k \text{ has no solution for } k \in \{1, \ldots, \alpha\}$$

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For instance, c is irrelevant in $3a + 3b + 2c \geq 3$ because there is no solution for neither of the equalities

$$3a + 3b = 1 \text{ and } 3a + 3b = 2$$

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$$3a + 3b = 1 \text{ and } 3a + 3b = 2$$

A dynamic programming algorithm can decide whether any of the equalities has a solution in pseudo-polynomial time with a single run

Detecting Irrelevant Literals (2)

As coefficients and degrees may be very big in the derived PB constraints, solving subset-sum on the corresponding instances would be very inefficient

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Our algorithm solves subset-sum modulo a fixed number, or even several numbers

Removing Irrelevant Literals

We can remove any irrelevant literal while preserving equivalence, by taking advantage that their truth value does not affect the constraint

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Or, we can locally assign it to 1, and simplify the constraint accordingly:

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In practice, we use a heuristic to decide which strategy to apply, as none of them is better than the other

Impact of the Removal of Irrelevant Literals on the Proof

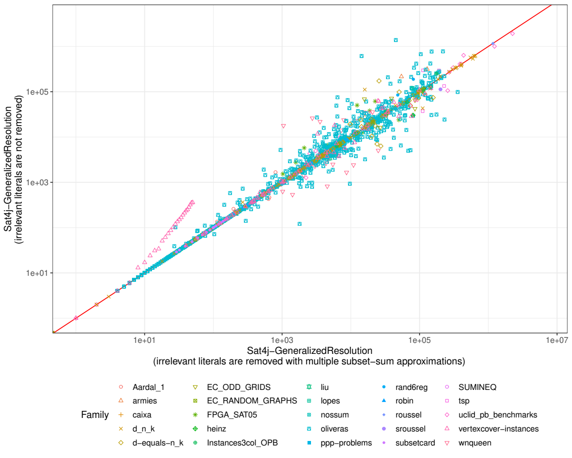

Figure 4: Comparison of the size of the proofs (number of cancellations) built by *Sat4j-GeneralizedResolution* with and without the removal of irrelevant literals on all benchmarks (logarithmic scale)

Focus on the Vertex-Cover Family: Experimental Results

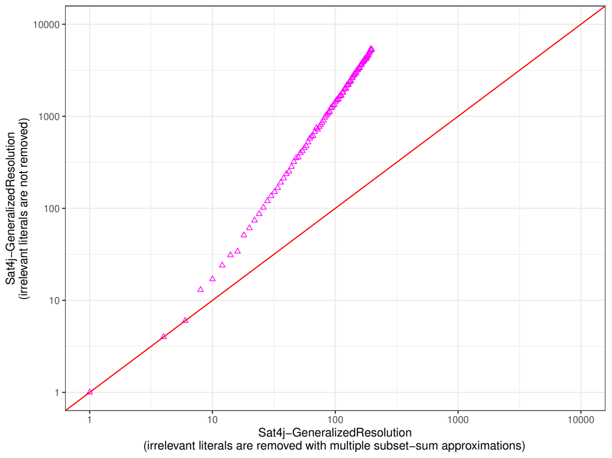

Figure 5: Comparison of the size of the proofs (number of cancellations) built by *Sat4j-GeneralizedResolution* with and without the removal of irrelevant literals on vertex-cover instances (logarithmic scale)

Focus on the Vertex-Cover Family: *Sat4j*'s Behavior

When given an instance of this family, the first constraint learned by *Sat4j* has the form

$$nx + x_1 + \ldots + x_{n-1} \geq n$$

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Even few irrelevant literals can lead to the production of an exponentially larger proof

Impact of the Removal of Irrelevant Literals on the Runtime

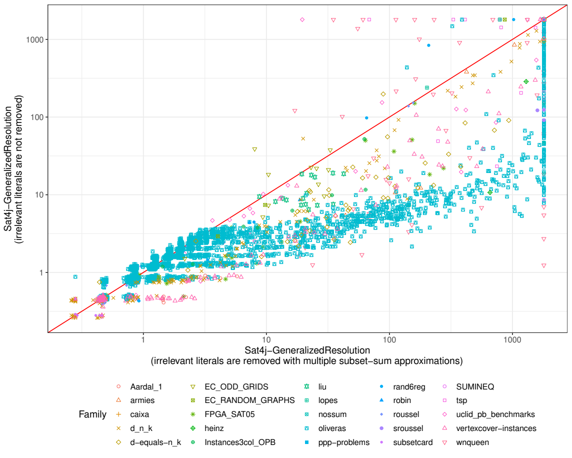

Figure 6: Comparison of the runtime of *Sat4j-GeneralizedResolution* with and without the removal of irrelevant literals on all benchmarks (logarithmic scale)

Weakening Ineffective Literals

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Ineffective literals can be seen as locally irrelevant, as opposed to the globally irrelevant literals presented before

In the context of the current partial assignment, it is easy to detect ineffective literals, but they can only be weakened away (as ineffective literals may be relevant)

Adapting further PB Solvers to CDCL

CDCL Architecture Recap

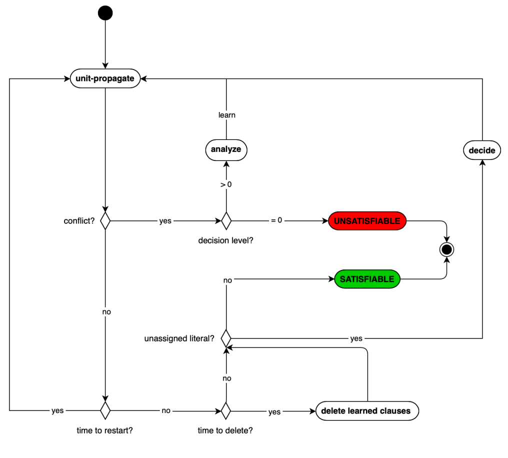

Figure 7: Overview of the CDCL Algorithm

CDCL Architecture Recap

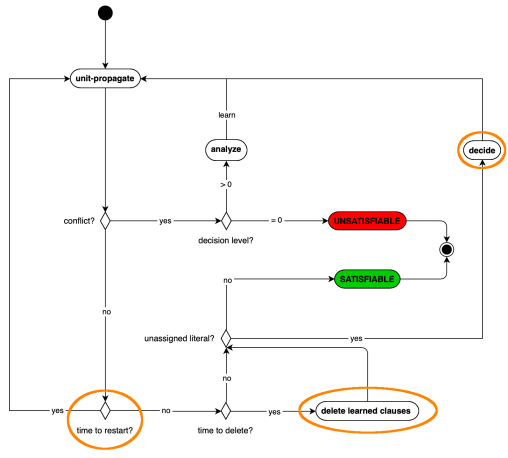

Figure 8: Use of other strategies in the CDCL Algorithm

Motivation

It is well known that, in addition to conflict analysis, several features of SAT solvers are crucial for solving problems efficiently, such as:

- branching heuristic
- learned constraint deletion strategy
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These features are mostly reused as is by current PB solvers, without taking into account the particular properties of PB constraints

Our main finding (Le Berre and Wallon, 2021) is that considering the size of the coefficients and the current partial assignment allows to significantly improve the solver

Comparison of different variants (decision)

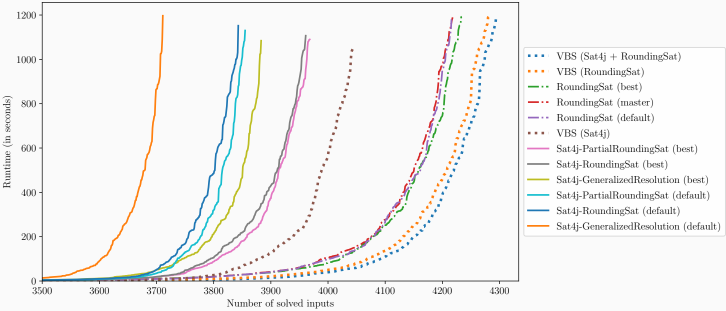

Figure 9: Cactus plot of the best configurations of different solvers

Comparison of different variants (optimization)

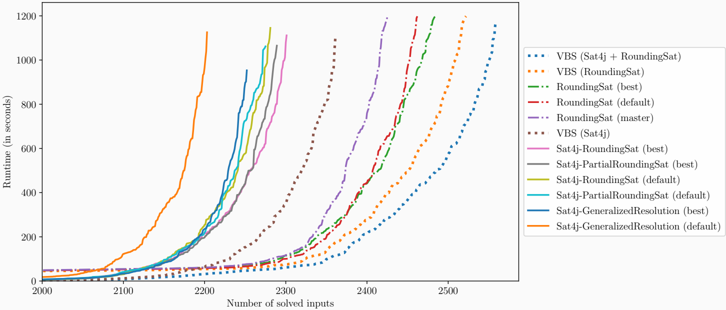

Figure 10: Cactus plot of the best configurations of different solvers

Conclusion and Perspectives

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- Implementations of the cutting planes proof system in PB solvers are not fully satisfactory, as its strength is not fully exploited
- Irrelevant literals may be produced during conflict analysis, and lead to the inference of weaker constraints
- Applying the weakening rule on ineffective literals is a possible (aggressive) counter-measure
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Conclusion

- Implementations of the cutting planes proof system in PB solvers are not fully satisfactory, as its strength is not fully exploited
- Irrelevant literals may be produced during conflict analysis, and lead to the inference of weaker constraints
- Applying the weakening rule on ineffective literals is a possible (aggressive) counter-measure
- Applying partial weakening and division gives better performance

- Complementary heuristics implemented in CDCL PB solvers can be adapted to take into account properties of PB constraints and to improve the performance of *Sat4j*

Perspectives

- Find other strategies for applying cutting planes rules so as to exploit more power of this proof system
- Design such strategies so as to prevent the production of irrelevant literals instead of removing them

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- Improve the detection of the optimal backjump level during conflict analysis

- Improve the detection of conflicts to deal with the conflictual reasons encountered during conflict analysis

Deep Dive into CDCL Pseudo-Boolean Solvers

Romain Wallon

May 20th, 2021

Laboratoire d'Informatique de l'X (LIX), École Polytechnique

Adapting further PB Solvers

Leveraging Properties of PB Constraints for Fine Tuning Sat4j

Let us consider again a confict analysis

$$3\bar{a}(?@?) + 3\bar{f}(?@?) + d(?@?) + e(?@?) \geq 5$$

$$6a(?@?) + 3b(?@?) + 3c(?@?) + 3d(?@?) + 3f(?@?) \geq 9$$

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$$3\bar{a}(?@?) + 3\bar{f}(?@?) + \color{red}{d(0@3)} + e(?@?) \geq 5$$

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We now apply the cancellation rule between these two constraints:

$$\frac{3\bar{a} + 3\bar{f} + d + e \geq 5 \qquad 6a + 3b + 3c + 3d + 3f \geq 9}{3a(0@3) + 3b(1@1) + 3c(0@2) + 2\bar{d}(1@3) + e(?@?) \geq 7}$$

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The PB constraints involved in this conflict analysis have very different properties compared to clauses!

(E)VSIDS for Making Decisions: Classical Implementation

All variables encountered during conflict analysis are bumped

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This is the case for all the variables appearing in the previous reason:

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(E)VSIDS for Making Decisions: Classical Implementation

All variables encountered during conflict analysis are bumped

This is the case for all the variables appearing in the previous reason:

$$3\bar{a} + 3\bar{f} + d + e \geq 5$$

This means that the scores of the variables a, f, d and e are incremented

(E)VSIDS for Making Decisions: Coefficients

A first approach for adapting VSIDS to PB constraints has been proposed in (Dixon, 2004), but it only takes into account the original cardinality constraints, and thus not the reason we have here:

$$3\bar{a} + 3\bar{f} + d + e \geq 5$$

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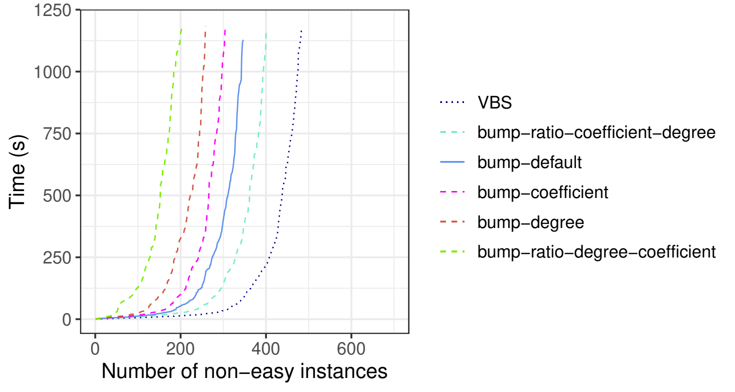
(E)VSIDS for Making Decisions: Experiments



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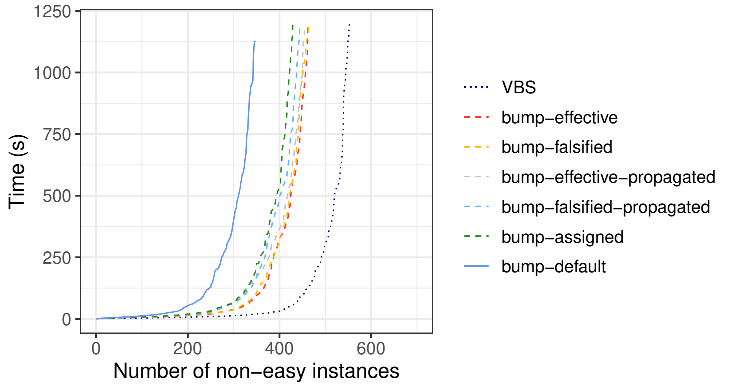

Quality of Learned Constraints: Classical Implementations

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The quality measures used by SAT solvers do not take into account the properties of PB constraints

Quality of Learned Constraints: Size and Coefficients

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$$3a + 3b + 3c + 2\bar{d} + e \geq 7$$

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There are satisfied and unassigned literals in this constraint!

We thus introduce 5 new definitions of LBD:

- `lbd-a`: the LBD is computed over assigned literals only
- `lbd-s`: the LBD is computed over assigned literals, and unassigned literals are considered assigned at the same (dummy) decision level
- `lbd-d`: the LBD is computed over assigned literals, and unassigned literals are considered assigned at different (dummy) decision levels
- `lbd-f`: the LBD is computed over falsified literals only
- `lbd-e`: the LBD is computed over effective literals only

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This is also true, but to a lesser extent, for PB solvers

The constraints to delete are those having a bad score w.r.t. the quality measure used in the solver

We thus introduce the following deletion strategies, based on the different quality measures we presented:

- `delete-degree`
- `delete-degree-size`
- `delete-lbd-a`
- `delete-lbd-s`
- `delete-lbd-d`
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- `restart-degree-size`
- `restart-lbd-a`
- `restart-lbd-s`
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Comparison of different variants (decision)

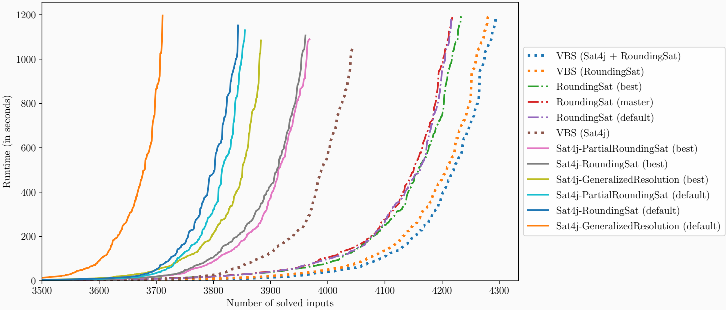

Figure 11: Cactus plot of the best configurations of different solvers

Comparison of different variants (optimization)

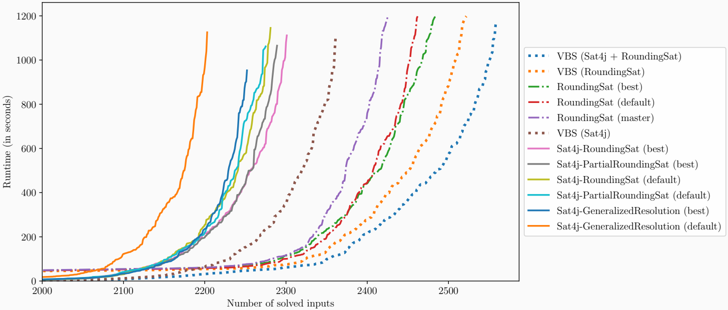

Figure 12: Cactus plot of the best configurations of different solvers

Deep Dive into CDCL Pseudo-Boolean Solvers

Romain Wallon

May 20th, 2021

Laboratoire d'Informatique de l'X (LIX), École Polytechnique

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