On Supported Inference and Extension Selection in Abstract Argumentation Frameworks – long version–

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Abstract

We present two approaches for deriving more arguments from an abstract argumentation framework than the ones obtained using sceptical inference, that is often too cautious. The first approach consists in selecting only some of the extensions. We point out several choice criteria to achieve such a selection process. Choices are based either on the attack relation between extensions or on the support of the arguments in each extension. The second approach consists of the definition of a new inference policy, between sceptical and credulous inference, and based as well on the support of the arguments. We illustrate the two approaches on examples, study their properties, and formally compare their inferential powers.

1 Introduction

An abstract argumentation system is often represented as an *oriented graph*, where nodes correspond to *arguments* and arcs correspond to *attacks* between them [15]. Different *semantics* are used to calculate *extensions* (sets of arguments that can be accepted together). From the extensions, a *status*, accepted or rejected, is assigned to each argument, using some *acceptance policy*. They are two main acceptance policies. In the first one, the *sceptical* policy, an argument is accepted if (there are extensions and) it appears in each extension. For the second one, the *credulous* policy, an argument is accepted if it belongs to (at least) one extension.

When the number of extensions is large, using a sceptical / credulous approach can be sub-optimal. Namely, if there is a lot of extensions, only few (if any) arguments are in all of them. Thus, using sceptical inference gives almost no information. Conversely, the credulous approach may result in too many arguments.

There exist settings for abstract argumentation where preferences, weighted attacks or similar extra information are considered [19, 20, 8, 16, 11, 2]. Those additional data can be exploited to reduce the number of extensions. Contrastingly, the problem addressed in this paper is to increase the number of accepted arguments when there is no further data, i.e., other data except the arguments and the attacks between them.

We investigate this problem and present two approaches for dealing with it. The first one consists in selecting only some of the extensions (the "best" ones, for a given semantics). The idea is to discriminate the extensions by taking advantage of the attack relation. The selection achieved in this way leads to increase the number of sceptically accepted arguments. Two methods for selecting extensions are pointed out. The first one is based on a pairwise comparison of extensions. The second method is based on a global evaluation of each extension, followed by a selection of the best evaluated ones. The second approach we developed goes through the definition of a new policy for accepting arguments. We introduce a third acceptance policy, which can be viewed as a trade-off between the credulous and the sceptical policy. The very idea is to consider the number of times an argument appears in the extensions. For the sceptical policy a "good" argument is one that appears in all extensions. If no such argument exists, then it makes sense to consider that arguments that appears in every extension but one are "quite good", and better than the ones that appear in less extensions.

2 Formal Setting

This section introduces basic definitions and notations we use throughout the paper.

Definition 1 (Argumentation system)**.** *An* argumentation system *(AS) is a pair* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *where* $\mathscr{R} \subseteq \mathscr{A} \times \mathscr{A}$. $\mathscr{A}$ *is called the set of* arguments *and* $\mathscr{R}$ *is called the* attack relation*. We denote by* $\mathscr{R}_{\downarrow\mathscr{E}}$ *the restriction of attack relation* $\mathscr{R}$ *on set* $\mathscr{E}$.

In order to simplify notation, we do not explicitly mention an argumentation system when it is clear from the context which argumentation system we refer to. We restrict ourselves to the case when $\mathscr{A}$ is finite.

In order to determine mutually acceptable sets of arguments, different semantics have been introduced in argumentation. We first introduce the basic notions of conflict-freeness, defence [15] and strong defence [4].

Definition 2 (Conflict-free, strong defence)**.** *Let* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *be an AS, and let* $\mathscr{E}, \mathscr{E}', \mathscr{E}'' \subseteq \mathscr{A}$ *and* $a \in \mathscr{A}$.

- $\mathscr{E}$ *is* conflict-free *if and only if there exist no arguments* $a, b \in \mathscr{E}$ *such that* $a \mathscr{R} b$.
- $\mathscr{E}$ *defends* a *if and only if for every* $b \in \mathscr{A}$ *we have that if* $b \mathscr{R} a$ *then there exists* $c \in \mathscr{E}$ *such that* $c \mathscr{R} b$.
- *Argument* a *is strongly defended from* $\mathscr{E}'$ *by* $\mathscr{E}''$ *(written* $sd(a, \mathscr{E}', \mathscr{E}'')$*) if and only if* $(\forall b \in \mathscr{E}')$ *if* $(b \mathscr{R} a)$ *then* $(\exists c \in \mathscr{E}'' \setminus \{a\})((c \mathscr{R} b) \wedge sd(c, \mathscr{E}', \mathscr{E}'' \setminus \{a\}))$.

Let us now define usual semantics for Dung's AS, especially the complete, preferred, grounded [15], semi-stable [7] and ideal semantics [14].

Definition 3 (Acceptability semantics)**.** *Let* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *be an AS and* $\mathscr{B} \subseteq \mathscr{A}$. *We say that a set* $\mathscr{B}$ *is* admissible *iff it is conflict-free and defends all its elements.*

- $\mathscr{B}$ *is a* complete *extension iff* $\mathscr{B}$ *is admissible and contains all the arguments it defends.*
- $\mathscr{B}$ *is a* preferred *extension iff it is a maximal (w.r.t.* $\subseteq$*) admissible set.*
- $\mathscr{B}$ *is a* stable *extension iff* $\mathscr{B}$ *is conflict-free and for all* $a \in \mathscr{A} \setminus \mathscr{B}$*, there exists* $b \in \mathscr{B}$ *such that* $b \mathscr{R} a$*.*
- $\mathscr{B}$ *is a* semi-stable *extension iff* $\mathscr{B}$ *is a complete extension and the union of the set* $\mathscr{B}$ *and the set of all arguments attacked by* $\mathscr{B}$ *is maximal (for set inclusion).*
- $\mathscr{B}$ *is a* grounded extension *iff* $\mathscr{B}$ *is a minimal (for set inclusion) complete extension.*
- $\mathscr{B}$ *is an* ideal *extension iff* $\mathscr{B}$ *is a maximal (for set inclusion) admissible set contained in every preferred extension.*

We say that a semantics σ *returns conflict-free sets iff for every AS* $\mathscr{F}$*, every extension of* $\mathscr{F}$ *is conflict-free.*

A semantics σ is said to return conflict-free sets iff for every AS $\mathscr{F}$, every extension of $\mathscr{F}$ is conflict-free. For an argumentation system $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ we denote $\mathtt{Ext}_\sigma(\mathscr{F})$; or, by a slight abuse of notation, $\mathtt{Ext}_\sigma(\mathscr{A}, \mathscr{R})$ the set of its extensions with respect to semantics σ. We use abbreviations c, p, s, ss, g and i for respectively complete, preferred, stable, semi-stable, grounded and ideal semantics. For example, $\mathtt{Ext}_p(\mathscr{F})$ denotes the set of preferred extensions of $\mathscr{F}$.

Example 1. *Let* $\mathscr{F}_1 = (\mathscr{A}_1, \mathscr{R}_1)$ *be an argumentation system with* $\mathscr{A}_1 = \{a,b,c,d\}$ *and* $\mathscr{R}_1 = \{(b,c),(c,b),(b,d),(c,d)\}$*. The graphical representation of the system is shown in Figure 1. There are two preferred / stable / semi-stable extensions:* $\mathtt{Ext}_p(\mathscr{F}_1)$

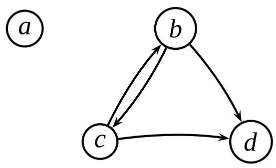

Figure 1: $\mathscr{F}_1 = (\mathscr{A}_1, \mathscr{R}_1)$: An argumentation system

$= \mathtt{Ext}_s(\mathscr{F}_1) = \mathtt{Ext}_{ss}(\mathscr{F}_1) = \{\{a,b\},\{a,c\}\}$*; three complete extensions:* $\mathtt{Ext}_c(\mathscr{F}_1) = \{\{a\},\{a,\ b\},\{a,c\}\}$*; and one ideal / grounded extension:* $\mathtt{Ext}_g(\mathscr{F}_1) = \mathtt{Ext}_i(\mathscr{F}_1) = \{\{a\}\}$. $\triangle$

Definition 4 (Acceptance policies)**.** *Let* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *be an AS,* σ *a semantics and let* $x \in \mathscr{A}$*. An* acceptance policy *is a function* $\mathtt{Inf}_\sigma : \mathtt{Ext}_\sigma(\mathscr{F}) \to 2^{\mathscr{A}}$*. The two main acceptance policies are sceptical and credulous policies. We say that x is* sceptically accepted *under semantics* σ *(or in short s-*sceptically accepted*) iff* $\texttt{Ext}_\sigma(\mathscr{F}) \neq \emptyset$ *and* $x \in \bigcap_{\mathscr{E} \in \texttt{Ext}_\sigma(\mathscr{F})} \mathscr{E}$. x *is* credulously accepted *under semantics* σ *iff* $x \in \bigcup_{\mathscr{E} \in \texttt{Ext}_\sigma(\mathscr{F})} \mathscr{E}$. *We denote the set of sceptically accepted arguments by* $\texttt{Sc}_\sigma(\mathscr{F})$ *and the set of credulously accepted arguments by* $\texttt{Cr}_\sigma(\mathscr{F})$.

3 Comparing Extensions by Pairwise Comparison

Let us see now how to select only some of the extensions provided by some semantics. This will allow us to derive more sceptically accepted arguments (and less credulously accepted ones). This section studies the way to select the "best" extensions based on the following process:

1. Compare all pairs of extensions based on a given criterion (e.g. the number of arguments in one extension not attacked by the other extension)

2. Choose the "best" extension(s) given the winners of pairwise comparisons

We first consider several criteria for pairwise comparison of extensions.

Definition 5 (Pairwise comparison criteria)**.** *Let* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *be an AS,* σ *a semantics and* $\texttt{Ext}_\sigma(\mathscr{F})$ *the set of extensions of* $\mathscr{F}$. *Let* $\mathscr{E}, \mathscr{E}' \in \texttt{Ext}_\sigma(\mathscr{F})$. *Then:*

1. $\mathscr{E} \succeq_{nonatt} \mathscr{E}'$ *if the number of arguments in* $\mathscr{E}$ *non attacked by* $\mathscr{E}'$ *is greater than or equal to the number of arguments in* $\mathscr{E}'$ *non attacked by arguments of* $\mathscr{E}$

2. $\mathscr{E} \succeq_{strdef} \mathscr{E}'$ *if the number of arguments in* $\mathscr{E}$ *strongly defended from* $\mathscr{E}'$ *by* $\mathscr{E}$ *is greater than or equal to the number of arguments in* $\mathscr{E}'$ *strongly defended from* $\mathscr{E}$ *by* $\mathscr{E}'$

3. $\mathscr{E} \succeq_{delarg} \mathscr{E}'$ *if the cardinality of any largest subset* S *of* $\mathscr{E}$ *such that* if all the attacks from S to $\mathscr{E}'$ are deleted then $\mathscr{E}$ is an extension of $(\mathscr{E} \cup \mathscr{E}', \mathscr{R}_{\downarrow \mathscr{E} \cup \mathscr{E}'})$ *is greater than or equal to the cardinality of any largest subset* S' *of* $\mathscr{E}'$ *such that* if all the attacks from S' to $\mathscr{E}$ are deleted then $\mathscr{E}$ is an extension of $(\mathscr{E} \cup \mathscr{E}', \mathscr{R}_{\downarrow \mathscr{E} \cup \mathscr{E}'})$

4. $\mathscr{E}' \succeq_{delatt} \mathscr{E}'$ *if the maximal number of attacks from* $\mathscr{E}$ *to* $\mathscr{E}'$ *that can be deleted such that* $\mathscr{E}$ *is still an extension of* $(\mathscr{E} \cup \mathscr{E}', \mathscr{R}_{\downarrow \mathscr{E} \cup \mathscr{E}'})$ *is greater than or equal to the maximal number of attacks from* $\mathscr{E}'$ *to* $\mathscr{E}$ *that can be deleted such that* $\mathscr{E}'$ *is still an extension of* $(\mathscr{E} \cup \mathscr{E}', \mathscr{R}_{\downarrow \mathscr{E} \cup \mathscr{E}'})$

The two first criteria are based on the number of non attacked or (strongly) defended arguments. The last two ones are based on a notion of robustness from attacks stemming from the other extension. One could also consider other criteria, for example by comparing the total number of attacks from $\mathscr{E}$ to $\mathscr{E}'$ and the total number of attacks from $\mathscr{E}'$ to $\mathscr{E}$. For a criterion γ, we write $\mathscr{E} \succ_\gamma \mathscr{E}'$ iff $\mathscr{E} \succeq_\gamma \mathscr{E}'$ and it is not the case that $\mathscr{E}' \succeq_\gamma \mathscr{E}$. We also write $\mathscr{E} \sim_\gamma \mathscr{E}'$ iff $\mathscr{E} \succeq_\gamma \mathscr{E}'$ and $\mathscr{E}' \succeq_\gamma \mathscr{E}$.

Example 2. *Consider the AS* $\mathscr{F}_2 = (\mathscr{A}_2, \mathscr{R}_2)$ *with* $\mathscr{A}_2 = \{a, b, c, d\}$ *and* $\mathscr{R}_2 = \{(a,c), (a,d), (b,c), (c,a), (d,b)\}$. *depicted in Figure 2.* $\texttt{Ext}_p(\mathscr{F}_2) = \{\mathscr{E}, \mathscr{E}'\}$ *with* $\mathscr{E} = \{a,b\}$, $\mathscr{E}' = \{c,d\}$. *All the arguments are attacked, so* $\mathscr{E} \sim_{nonatt} \mathscr{E}'$. *No argument is strongly* *defended, so $\mathscr{E} \sim_{strdef} \mathscr{E}'$. We also have $\mathscr{E} \succ_{delarg} \mathscr{E}'$ since for $S = \{b\}$ $\mathscr{E}$ is still an extension even if all the attacks from S are deleted; whereas there are no $S' \subseteq \mathscr{E}'$ with $S' \neq \emptyset$ such that $\mathscr{E}'$ is still a preferred extension even after deleting all the attacks from S'. Finally, $\mathscr{E} \succ_{delatt} \mathscr{E}'$ since even if the attack from a to c is deleted, $\mathscr{E}$ is still a preferred extension, whereas as soon as one attack from $\mathscr{E}'$ is deleted, $\mathscr{E}'$ is no longer a preferred extension.* △

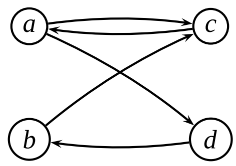

Figure 2: $\mathscr{F}_2 = (\mathscr{A}_2, \mathscr{R}_2)$: Pairwise comparison of extensions

Definition 6 (Copeland-based extensions)**.** *Let $\gamma \in \{nonatt, strdef, delarg, delatt\}$ be one of the criteria from Definition 5. Let $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ be an argumentation system, σ a semantics and $\mathtt{Ext}_\sigma(\mathscr{F})$ the set of extensions of $\mathscr{F}$ with respect to σ. We define the set of Copeland-based extensions (*CBE*) as follows*

$$\mathtt{CBE}_{\sigma,\gamma}(\mathscr{F}) = \underset{\mathscr{E} \in \mathtt{Ext}_\sigma(\mathscr{F})}{\arg\max} \; |\{\mathscr{E}' \in \mathtt{Ext}_\sigma(\mathscr{F}) \mid \mathscr{E} \succeq_\gamma \mathscr{E}'\}| - |\{\mathscr{E}'' \in \mathtt{Ext}_\sigma(\mathscr{F}) \mid \mathscr{E}'' \succeq_\gamma \mathscr{E}\}|$$

We call this selection "Copeland-based" since it is inspired by the Copeland's method from voting theory [21]. Of course, one can envisage other ways to select the extensions given criterion γ, for instance all voting methods based on the majority graph (such as Miller, Fishburn, Schwartz, Banks or Slater's methods [6]). Clearly, selecting some extensions is a way to increase the number of sceptically accepted arguments (and to decrease the number of credulously accepted arguments):

Fact 1. *For every $\gamma \in \{nonatt, strdef, delarg, delatt\}$, for every semantics σ, for every AS $\mathscr{F} = (\mathscr{A}, \mathscr{R})$, for every $x \in \mathscr{A}$:*

- $\mathtt{CBE}_{\sigma,\gamma}(\mathscr{F}) \subseteq \mathtt{Ext}_\sigma(\mathscr{F})$
- *if x is σ-sceptically accepted then x is $\mathtt{CBE}_{\sigma,\gamma}$-sceptically accepted*
- *if x is $\mathtt{CBE}_{\sigma,\gamma}$-credulously accepted then it is σ-credulously accepted.*

Example 3. *Consider the argumentation system from Example 2. For example, we have that $\mathtt{CBE}_{\sigma,delarg}(\mathscr{F}_2) = \mathtt{CBE}_{\sigma,delatt}(\mathscr{F}_2) = \{\mathscr{E}\}$.* △

Baroni and Giacomin [4, Section 3] pointed out a set of extension evaluation criteria that can be seen as properties for characterizing good semantics. We now show that the semantics defined in this section satisfy the same properties as the underlying semantics they are built from, with the exception of directionality.

Proposition 1. *Let x be any property among I-maximality, Admissibility, Strong Admissibility, Reinstatement, Weak Reinstatement, CF-Reinstatement [4].*
If the semantics σ *satisfies property x, then the semantics* $\mathtt{CBE}_{\sigma,\gamma}$ *satisfies property x.*

Proof. Follows directly from the definitions of properties. □

The next example shows that directionality is not always satisfied by CBE approach.

Example 4. *Consider the AS* $\mathscr{F}_4 = (\mathscr{A}_4, \mathscr{R}_4)$ *with* $\mathscr{A}_4 = \{a, b, c, d\}$ *and* $\mathscr{R}_4 = \{(a,b), (a,c), (b,a), (b,d), (c,d)\}$*, depicted in Figure 3. Directionality is satisfied by preferred semantics [4]. Let us show that it is not satisfied by* CBE *approach using preferred semantics. Let* $U = \{a,b\}$*. We have* $\mathtt{Ext}_p(\mathscr{F}) = \{\{a,d\},\{b,c\}\}$ *and* $\mathtt{Ext}_p(\mathscr{F}\downarrow_U) = \{\{a\},\{b\}\}$*. Let us use the criterion* $delatt$*. We obtain that* $\mathtt{CBE}_{p,delatt}(\mathscr{F}) = \{\{b,c\}\}$ *and* $\mathtt{CBE}_{p,delatt}(\mathscr{F}\downarrow_U) = \{\{a\},\{b\}\}$*. Note that we have* $\{\mathscr{E} \cap U \mid \mathscr{E} \in \mathtt{CBE}_{p,delatt}\} = \{\{b\}\}$*. Thus, directionality does not hold, since we have* $\{\mathscr{E} \cap U \mid \mathscr{E} \in \mathtt{CBE}_{p,delatt}\} \neq \mathtt{CBE}_{p,delatt}(\mathscr{F}\downarrow_U)$*.*

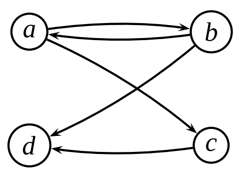

Figure 3: $\mathscr{F}_4 = (\mathscr{A}_4, \mathscr{R}_4)$: Directionality is not satisfied by CBE approach

△

Note that the relations among different semantics do not carry over in case of CBE approach. For instance, it is not guaranteed that each CBE-stable extension is also a CBE-preferred extension. Consider the following example.

Remark: for the complex examples, instead of providing graphical representation, we provide machine readable code in the appendix. This allows the reader to check them using one of the existing software tools, e.g. Aspartix[1].

Example 5. *Let* $\mathscr{F}_5 = (\mathscr{A}_5, \mathscr{R}_5)$ *with* $\mathscr{A}_5 = \{a,b,c,d,e,x_1,x_2\}$ *and* $\mathscr{R}_5 = \{(x_1,x_1), (x_2,x_2), (a,c), (a,x_1), (x_1,b), (a,d), (a,e), (a,x_2), (b,x_2), (b,c), (c,a), (c,b), (c,d), (c,e), (c,x_2), (c,x_1), (d,a), (d,c), (e,a), (e,c)\}$*. Denote* $\mathscr{E}_1 = \{a,b\}$*,* $\mathscr{E}_2 = \{c\}$*,* $\mathscr{E}_3 = \{d,e\}$*. There are exactly two stable extensions,* $\mathscr{E}_1$ *and* $\mathscr{E}_2$*. There are exactly three preferred extensions:* $\mathscr{E}_1, \mathscr{E}_2$ *and* $\mathscr{E}_3$*. Since all arguments of* $\mathscr{E}_1$ *are attacked by* $\mathscr{E}_2$ *and vice versa,* $\mathtt{CBE}_{s,nonatt}(\mathscr{F}) = \{\mathscr{E}_1, \mathscr{E}_2\}$*. However,* $\mathscr{E}_1 \succ_{nonatt} \mathscr{E}_3$ *whereas* $\mathscr{E}_2 \succeq_{s,nonatt} \mathscr{E}_3$ *and* $\mathscr{E}_3 \succeq_{s,nonatt} \mathscr{E}_2$*. Thus,* $\mathtt{CBE}_{p,nonatt}(\mathscr{F}) = \{\mathscr{E}_1\}$*.* △

In the next example there is a CBE-preferred extension that is not a CBE-complete extension.

[1] `rull.dbai.tuwien.ac.at:8080/ASPARTIX/`

Example 6. *Let* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *with* $\mathscr{A} = \{a, b, c, d, e, x_1, x_2\}$ *and* $\mathscr{R} = \{(x_1, x_1), (x_2, x_2), (a, b), (a, c), (a, d), (a, e), (b, a), (b, x_1), (c, a), (d, a), (d, x_2), (e, a), (x_1, b), (x_2, d), (x_1, c), (x_2, e)\}$. *Denote* $\mathscr{E}_1 = \{a\}$, $\mathscr{E}_2 = \{b, c\}$, $\mathscr{E}_3 = \{d, e\}$. *There are exactly two preferred extensions:* $\mathscr{E}_1$ *and* $\mathscr{E}_2 \cup \mathscr{E}_3$. *The same argumentation system has five complete extensions:* $\emptyset$, $\mathscr{E}_1$, $\mathscr{E}_2$, $\mathscr{E}_3$, $\mathscr{E}_2 \cup \mathscr{E}_3$. *Since* $\mathscr{E}_1$ *attacks all arguments of* $\mathscr{E}_2 \cup \mathscr{E}_3$ *and vice versa,* $\mathtt{CBE}_{p,nonatt}(\mathscr{F}) = \{\mathscr{E}_1, \mathscr{E}_2 \cup \mathscr{E}_3\}$. *However, it can be checked that, out of all complete extensions,* $\mathscr{E}_2 \cup \mathscr{E}_3$ *has the biggest Copeland score, since it is strictly stronger than both* $\mathscr{E}_2$ *and* $\mathscr{E}_3$ *with respect to* $\succeq_{nonatt}$. *We obtain* $\mathtt{CBE}_{c,nonatt}(\mathscr{F}) = \{\mathscr{E}_2 \cup \mathscr{E}_3\}$. △

4 Comparing Extensions by Global Evaluation

In Section 3 we considered different criteria for *pairwise comparison* of extensions. In this section we define the score of an argument as the number of extensions it appears in. One may justify this choice of score as some kind of generalization of the principles behind sceptical acceptance. For sceptical acceptance a "good" argument is an argument that appears in all extensions. But, if no such argument exists, it could make sense to consider that arguments that appears in every extension but one are "good", and typically better than the ones that appears in less extensions. Note that one can use other scores in the construction and obtain similar results.

Definition 7 (Scores and support vectors). *Let* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *be an argumentation system,* σ *a semantics,* x *be an argument, and* $\mathtt{Ext}_\sigma(\mathscr{F})$ *the set of extensions of* $\mathscr{F}$ *with respect to* σ. *We define* $\mathtt{ne}$ *as the number of extensions* x *appears in. Formally,* $\mathtt{ne}_\sigma(x, \mathscr{F}) = |\{\mathscr{E} \in \mathtt{Ext}_\sigma(\mathscr{F}) \mid x \in \mathscr{E}\}|$. *For an extension* $\mathscr{E} \in \mathtt{Ext}_\sigma(\mathscr{F})$, *with* $\mathscr{E} = \{a_1, \ldots, a_n\}$ *we define its support as* $\mathtt{vsupp}_\sigma(\mathscr{E}, \mathscr{F}) = (\mathtt{ne}_\sigma(a_1, \mathscr{F}), \ldots, \mathtt{ne}_\sigma(a_n, \mathscr{F}))$.

When $\mathscr{F}$ *and* σ *are clear from the context, we write* $\mathtt{ne}(x)$ *and* $\mathtt{vsupp}(\mathscr{E})$ *instead of* $\mathtt{ne}_\sigma(x, \mathscr{F})$ *and* $\mathtt{vsupp}_\sigma(\mathscr{E}, \mathscr{F})$.

Definition 8 (Aggregation functions). *Let* $v = (v_1, \ldots, v_n)$ *be a vector of natural numbers. We denote by* $sum(v)$ *the sum of all elements of* v, *by* $max(v)$ *the maximal element of* v, *by* $min(v)$ *the minimal element of* v, *by* $leximax(v)$ *the re-arranged version of* v *where* $v_1, \ldots, v_n$ *are put in decreasing order, by* $leximin(v)$ *the re-arranged version of* v *where* $v_1, \ldots, v_n$ *are put in increasing order.*

For example, if $v = (2, 1, 4, 2, 5)$, then we have $sum(v) = 14$ and $leximin(v) = (1, 2, 2, 4, 5)$. Note that there exist other ways to aggregate vectors [13].

For the next definition we need the notion of lexicographic order $<_{lex}$ (for *leximin* and *leximax*). Let $v = (v_1, \ldots, v_n)$ and $v' = (v'_1, \ldots, v'_n)$ be two vectors of natural numbers. We have $v <_{lex} v'$ iff $\exists j \in 1, \ldots, n (\forall i \in 1, \ldots, j-1, v_i = v'_i)$ and $v_j < v'_j$. We also have $v <_{leximin} v'$ iff $leximin(v) <_{lex} leximin(v')$ and $v <_{leximax} v'$ iff $leximax(v) <_{lex} leximax(v')$.

Definition 9 (Order-based extensions). *Let* $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ *be an argumentation system,* σ *a semantics,* $\mathtt{Ext}_\sigma(\mathscr{F})$ *be the set of extensions of* $\mathscr{F}$ *with respect to* σ, *and* γ *be an aggregation function. We have* $\mathtt{OBE}_{\sigma,\gamma}(\mathscr{F}) = \arg\max_{\mathscr{E} \in \mathtt{Ext}_\sigma(\mathscr{F})} \gamma(\mathtt{vsupp}_\sigma(\mathscr{E}, \mathscr{F}))$.

The idea of the previous definition is to calculate the popularity of an extension by taking into account the popularity of the arguments it contains.

	vsupp	*max*	*min*	*sum*	*leximax*	*leximin*
$\{a,e,g,c\}$	3324	$\underline{4}$	$\underline{2}$	$\underline{12}$	$\underline{4332}$	$\underline{2334}$
$\{a,e,g,d\}$	3321	3	1	9	3321	1233
$\{a,f,h,c\}$	3234	$\underline{4}$	$\underline{2}$	$\underline{12}$	$\underline{4332}$	$\underline{2334}$
$\{b,h,c,e\}$	2343	$\underline{4}$	$\underline{2}$	$\underline{12}$	$\underline{4332}$	$\underline{2334}$
$\{b,h,c,f\}$	2342	$\underline{4}$	$\underline{2}$	11	4322	2234

Table 1: Computations of $\mathtt{OBE}_{\sigma,\oplus}$

Example 7. *Let $\mathscr{F}_7 = (\mathscr{A}_7, \mathscr{R}_7)$ be AS $\mathscr{F}_7 = (\mathscr{A}_7, \mathscr{R}_7)$ with $\mathscr{A}_7 = \{a,b,c,d,e,f,g,h\}$ and $\mathscr{R}_7 = \{(a,b), (b,a), (e,f), (f,e), (b,g), (f,g), (g,h), (h,d), (d,c), (c,d)\}$. There*

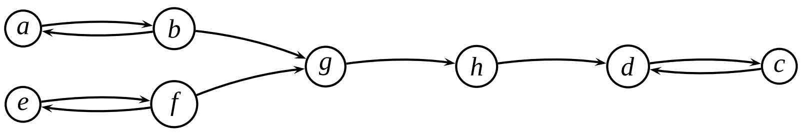

Figure 4: $\mathscr{F}_7 = (\mathscr{A}_7, \mathscr{R}_7)$

are five preferred extensions: $\{a,e,g,c\}$, $\{a,e,g,d\}$, $\{a,f,h,c\}$, $\{b,h,c,e\}$, $\{b,h,c,f\}$. So $\mathtt{ne}_p(a,\mathscr{F}_7) = 3$, $\mathtt{ne}_p(b,\mathscr{F}_7) = 2$, $\mathtt{ne}_p(c,\mathscr{F}_7) = 4$, $\mathtt{ne}_p(d,\mathscr{F}_7) = 1$, $\mathtt{ne}_p(e,\mathscr{F}_7) = 3$, $\mathtt{ne}_p(f,\mathscr{F}_7) = 2$, $\mathtt{ne}_p(g,\mathscr{F}_7) = 2$, $\mathtt{ne}_p(h,\mathscr{F}_7) = 3$. The computation of the support vectors of each extension and the selection (underlined) made by $sum, max, min, leximin$, $leximax$ are indicated in Table 1.

We obtain $\mathtt{OBE}_{\sigma,max}(\mathscr{F}_7) = \mathtt{OBE}_{\sigma,min}(\mathscr{F}_7) = \{\{a,e,g,c\},\{a,f,h,c\},\{b,h,c,e\},\{b,h,c,f\}\}$. So, whereas $\mathtt{Sc}_p(\mathscr{F}_7) = \emptyset$, we have $\mathtt{Sc}_{\mathtt{OBE}_{p,min}}(\mathscr{F}_7) = \{c\}$. Similarly, we have $\mathtt{OBE}_{\sigma,sum}(\mathscr{F}_7) = \mathtt{OBE}_{\sigma,leximin}(\mathscr{F}_7) = \mathtt{OBE}_{\sigma,leximax}(\mathscr{F}_7) = \{\{a,e,g,c\},\{a,f,h,c\},\{b,h,c,e\}\}$.

$\triangle$

Fact 2. *For every $\gamma \in \{sum, max, min, leximin, leximax\}$, for every semantics σ, for every AS $\mathscr{F} = (\mathscr{A}, \mathscr{R})$, for every $x \in \mathscr{A}$:*

- $\mathtt{OBE}_{\sigma,\gamma}(\mathscr{F}) \subseteq \mathtt{Ext}_\sigma(\mathscr{F})$
- *if x is σ-sceptically accepted then x is $\mathtt{OBE}_{\sigma,\gamma}$-sceptically accepted*
- *if x is $\mathtt{OBE}_{\sigma,\gamma}$-credulously accepted then it is σ-credulously accepted.*

Here also we can show that these semantics keep the same properties as the underlying semantic they are built from.

Proposition 2. *Let x be any property among I-maximality, Admissibility, Strong Admissibility, Reinstatement, Weak Reinstatement, CF-Reinstatement [4].*
If the semantics σ satisfies property x, then the semantics $\mathtt{OBE}_{\sigma,\gamma}$ satisfies property x.

Proof. Follows directly from the corresponding definitions. □

Like in Section 3, directionality is not always satisfied by the OBE approach.

Example 8. *Consider the AS $\mathscr{F}_8 = (\mathscr{A}_8, \mathscr{R}_8)$ defined as $\mathscr{A}_8 = \{a,b,c,d,e\}$ and $\mathscr{R}_8 = \{(a,b), (a,e), (b,a), (b,c), (b,d), (c,d), (c,e), (d,c), (d,e), (e,c), (e,d)\}$. depicted in Figure 5. Directionality is satisfied by preferred semantics [4]. Let us show that it is not satisfied by OBE approach using preferred semantics. Let $U = \{a,b\}$. We have $\mathtt{Ext}_p(\mathscr{F}) = \{\{a,c\},\{a,d\},\{b,e\}\}$ and $\mathtt{Ext}_p(\mathscr{F}\downarrow_U) = \{\{a\},\{b\}\}$. Let us use the criterion leximax. We obtain $\mathtt{OBE}_{p,leximax}(\mathscr{F}) = \{\{a,c\},\{a,d\}\}$ and $\mathtt{OBE}_{p,leximax}(\mathscr{F}\downarrow_U) = \{\{a\},\{b\}\}$. Note that $\{\mathscr{E}\cap U \mid \mathscr{E} \in \mathtt{OBE}_{p,leximax}\} = \{\{a\}\}$. Thus, directionality does not hold, since $\{\mathscr{E}\cap U \mid \mathscr{E} \in \mathtt{OBE}_{p,leximax}\} \neq \mathtt{OBE}_{p,leximax}(\mathscr{F}\downarrow_U)$.*

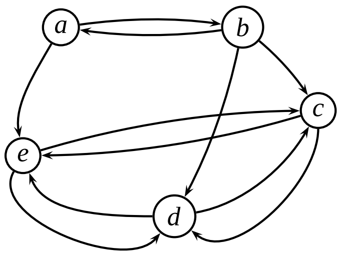

Figure 5: $\mathscr{F}_8 = (\mathscr{A}_8, \mathscr{R}_8)$: Directionality is not satisfied by CBE approach

△

A natural issue is to determine how the proposed criteria are connected. Do some of the rules coincide? Are some of them refinements of others? In the rest of this section we provide the answer to this question. Essentially, all the criteria give different results; the exceptions come from the obvious fact that *leximin* (resp. *leximax*) refines min (resp. max). We used the preferred semantics to construct the counter-examples; a similar study can be conducted for the other semantics. Let us first formalise what we mean by inclusion between the criteria.

Definition 10. *Let Γ and Γ' be two functions. We write $\Gamma \sqsubseteq \Gamma'$ iff for every $\mathscr{F}, \Gamma(\mathscr{F}) \subseteq \Gamma'(\mathscr{F})$. The relation $\sqsubseteq$ is a pre-order. Let us denote its strict part by $\sqsubset$, its symmetric part by $\doteq$ and its negation by $\not\sqsubseteq$. We write Γ $\mathtt{ind}$ Γ' iff $\Gamma \not\sqsubseteq \Gamma'$ and $\Gamma \not\sqsubseteq \Gamma'$.*

Proposition 3. *For every acceptability semantics σ,*

$$\mathtt{OBE}_{\sigma,leximin} \sqsubseteq \mathtt{OBE}_{\sigma,\min} \text{ and } \mathtt{OBE}_{\sigma,leximax} \sqsubseteq \mathtt{OBE}_{\sigma,\max}$$

Proof. Let us show that for every semantics σ, for every argumentation system $\mathscr{F}$, for every extension $\mathscr{E}$ of $\mathscr{F}$ under σ, if $\mathscr{E} \in \mathtt{OBE}_{\sigma,leximin}(\mathscr{F})$ then $\mathscr{E} \in \mathtt{OBE}_{\sigma,\min}(\mathscr{F})$. Suppose that $\mathscr{E} \in \mathtt{OBE}_{\sigma,leximin}(\mathscr{F})$. If $\mathscr{E}$ is the only extension of $\mathscr{F}$, the proof is over. Else, let $\mathscr{E}'$ be another extension of $\mathscr{F}$. Since $\mathscr{E} \in \mathtt{OBE}_{\sigma,leximin}(\mathscr{F})$, then $min(\mathtt{vsupp}_\sigma(\mathscr{E},\mathscr{F})) \geq min(\mathtt{vsupp}_\sigma(\mathscr{E},\mathscr{F}'))$. This is true for every extension $\mathscr{E}' \in \mathtt{Ext}_\sigma(\mathscr{F})$, thus $\mathscr{E} \in \mathtt{OBE}_{\sigma,\min}(\mathscr{F})$. Hence, $\mathtt{OBE}_{\sigma,leximin} \sqsubseteq \mathtt{OBE}_{\sigma,\min}$.

Let us show that for every semantics σ, for every argumentation system $\mathscr{F}$, for every extension $\mathscr{E}$ of $\mathscr{F}$ under σ, if $\mathscr{E} \in \mathtt{OBE}_{\sigma,leximax}(\mathscr{F})$ then $\mathscr{E} \in \mathtt{OBE}_{\sigma,\max}(\mathscr{F})$. Suppose $\mathscr{E} \in \mathtt{OBE}_{\sigma,leximax}(\mathscr{F})$. If $\mathscr{E}$ is the only extension of $\mathscr{F}$, the proof is over. Else, let $\mathscr{E}'$ be another extension of $\mathscr{F}$. Since $\mathscr{E} \in \mathtt{OBE}_{\sigma,leximax}(\mathscr{F})$, then $max(\mathtt{vsupp}_\sigma(\mathscr{E},\mathscr{F})) \geq max(\mathtt{vsupp}_\sigma(\mathscr{E},\mathscr{F}'))$. This is true for every $\mathscr{E}' \in \mathtt{Ext}_\sigma(\mathscr{F})$, thus $\mathscr{E} \in \mathtt{OBE}_{\sigma,\max}(\mathscr{F})$. Hence, $\mathtt{OBE}_{\sigma,leximax} \subseteq \mathtt{OBE}_{\sigma,\max}$. □

We now provide a complete comparison between pairs of criteria under preferred semantics.

Proposition 4. *The inclusion links between rules for preferred semantics are as depicted in Figure 6. Namely,* $\mathtt{OBE}_{p,leximin} \sqsubset \mathtt{OBE}_{p,\min}$ *and* $\mathtt{OBE}_{p,leximax} \sqsubset \mathtt{OBE}_{p,\max}$ *whereas the other pairs of rules* (x,y) *with* $x,y \in \{\mathtt{OBE}_{p,sum}, \mathtt{OBE}_{p,\min}, \mathtt{OBE}_{p,\max}, \mathtt{OBE}_{p,leximin}, \mathtt{OBE}_{p,leximax}\}$, $x \neq y$ *are incomparable, i.e.,* x `ind` y.

	$\mathtt{OBE}_{p,sum}$	$\mathtt{OBE}_{p,\min}$	$\mathtt{OBE}_{p,\max}$	$\mathtt{OBE}_{p,leximin}$	$\mathtt{OBE}_{p,leximax}$
$\mathtt{OBE}_{p,min}$	ind				
$\mathtt{OBE}_{p,max}$	ind	ind			
$\mathtt{OBE}_{p,leximin}$	ind	$\sqsubset$	ind		
$\mathtt{OBE}_{p,leximax}$	ind	ind	$\sqsubset$	ind	

Figure 6: Inclusion relationships between rules for preferred semantics. If $\mathscr{R}$ is the row rule and $\mathscr{R}'$ is the column rule, symbol $\sqsubset$ means that $\mathscr{R} \sqsubset \mathscr{R}'$; symbol `ind` means that $\mathscr{R}$ `ind` $\mathscr{R}'$.

Proof. Let us first construct three counter examples that will be used in this proof.

Example 9. *Let* $\mathscr{F}_9 = (\mathscr{A}_9,\mathscr{R}_9)$ *with* $\mathscr{A}_9 = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, a, b\}$ *and* $\mathscr{R}_9 = \{(x_1,x_2), (x_2,x_1), (x_3,x_4), (x_4,x_3), (x_1,x_5), (x_2,x_5), (x_3,x_5), (x_4,x_5), (x_5,x_5), (x_4,x_6), (x_6,x_6), (x_2,x_7), (x_7,x_7), (x_7,b), (x_5,a), (x_6,b)\}$.

We have $\mathtt{Ext}_p(\mathscr{F}_9) = \{\mathscr{E}_1,\mathscr{E}_2,\mathscr{E}_3,\mathscr{E}_4\}$ *with* $\mathscr{E}_1 = \{x_1,x_4,a\}$, $\mathscr{E}_2 = \{x_2,x_3,a\}$, $\mathscr{E}_3 = \{x_1,x_3,a\}$, $\mathscr{E}_4 = \{x_2,x_4,a,b\}$. *Their supports are:* $\mathtt{vsupp}_p(\mathscr{E}_1,\mathscr{F}_9) = \mathtt{vsupp}_p(\mathscr{E}_2,\mathscr{F}_9) = \mathtt{vsupp}_p(\mathscr{E}_3,\mathscr{F}_9) = (4,2,2)$, $\mathtt{vsupp}_p(\mathscr{E}_4,\mathscr{F}_9) = (4,2,2,1)$.

Hence, we obtain that $\mathtt{OBE}_{p,sum}(\mathscr{F}_9) = \{\mathscr{E}_4\}$, $\mathtt{OBE}_{p,\min}(\mathscr{F}_9) = \{\mathscr{E}_1,\mathscr{E}_2,\mathscr{E}_3\}$, $\mathtt{OBE}_{p,\max}(\mathscr{F}_9) = \{\mathscr{E}_1,\mathscr{E}_2,\mathscr{E}_3,\mathscr{E}_4\}$, $\mathtt{OBE}_{p,leximin}(\mathscr{F}_9) = \{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3\}$, $\mathtt{OBE}_{p,leximax}(\mathscr{F}_9) = \{\mathscr{E}_4\}$. △

Example 10. *Let* $\mathscr{F}_{10} = (\mathscr{A}_{10},\mathscr{R}_{10})$ *with* $\mathscr{A}_{10} = \{a, x_1, x_2, x_3, x_4, x_5, b_1, b_2, b_3, b_4, y_1, y_2, y_3\}$ *and* $\mathscr{R}_{10} = \{(x_1,x_2), (x_2,x_1), (x_3,x_4), (x_4,x_3), (x_1,x_5), (x_2,x_5), (x_3,x_5), (x_4,x_5), (x_1,x_3), (x_3,x_1), (x_1,x_4), (x_4,x_1), (x_2,x_3), (x_3,x_2), (x_2,x_4), (x_4,x_2), (x_5,x_5), (x_5,a), (y_1,y_2), (y_2,y_1), (y_1,y_3), (y_2,y_3), (y_3,y_3), (y_3,b_1), (y_3,b_2), (y_3,b_3), (y_3,b_4), (x_1,y_1), (x_1,y_2), (x_2,y_1), (x_2,y_2), (x_3,y_1), (x_3,y_2), (x_4,y_1), (x_4,y_2), (y_1,x_1), (y_2,x_1), (y_1,x_2), (y_2,x_2), (y_1,x_3), (y_2,x_3), (y_1,x_4), (y_2,x_4)\}$.

There are exactly six preferred extensions: $\mathscr{E}_1 = \{a, x_4\}$, $\mathscr{E}_2 = \{a, x_3\}$, $\mathscr{E}_3 = \{a, x_2\}$, $\mathscr{E}_4 = \{a, x_1\}$, $\mathscr{E}_5 = \{b_1, b_2, b_3, b_4, y_2\}$, $\mathscr{E}_6 = \{b_1, b_2, b_3, b_4, y_1\}$. *Their supports are:* $\mathtt{vsupp}_p(\mathscr{E}_1, \mathscr{F}_{10}) = \mathtt{vsupp}_p(\mathscr{E}_2, \mathscr{F}_{10}) = \mathtt{vsupp}_p(\mathscr{E}_3, \mathscr{F}_{10}) = \mathtt{vsupp}_p(\mathscr{E}_4, \mathscr{F}_{10}) = (4, 1)$, $\mathtt{vsupp}_p(\mathscr{E}_5, \mathscr{F}_{10}) = \mathtt{vsupp}_p(\mathscr{E}_6, \mathscr{F}_{10}) = (2, 2, 2, 2, 1)$.

We obtain $\mathtt{OBE}_{p,sum}(\mathscr{F}_{10}) = \{\mathscr{E}_5, \mathscr{E}_6\}$, $\mathtt{OBE}_{p,\min}(\mathscr{F}_{10}) = \{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \mathscr{E}_4, \mathscr{E}_5, \mathscr{E}_6\}$, $\mathtt{OBE}_{p,\max}(\mathscr{F}_{10}) = \mathtt{OBE}_{p,leximin}(\mathscr{F}_{10}) = \mathtt{OBE}_{p,leximax}(\mathscr{F}_{10}) = \{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \mathscr{E}_4\}$.

$\triangle$

Example 11. *Let* $\mathscr{F}_{11} = (\mathscr{A}_{11}, \mathscr{R}_{11})$ *with* $\mathscr{A}_{11} = \{x_1, x_2, x_3, x_4, x_5, a, y_1, y_2, y_3, y_4\}$ *and* $\mathscr{R}_{11} = \{(x_1,x_2), (x_2,x_1), (x_1,x_3), (x_3,x_1), (x_1,x_4), (x_4,x_1), (x_2,x_3), (x_3,x_2), (x_2,x_4), (x_4,x_2), (x_3,x_4), (x_4,x_3), (x_1,x_5), (x_2,x_5), (x_3,x_5), (x_4,x_5), (x_5,x_5), (x_5,a), (y_1,y_2), (y_2,y_1), (y_3,y_4), (y_4,y_3), (x_1,y_1), (x_1,y_2), (x_1,y_3), (x_1,y_4), (x_2,y_1), (x_2,y_2), (x_2,y_3), (x_2,y_4), (x_3,y_1), (x_3,y_2), (x_3,y_3), (x_3,y_4), (x_4,y_1), (x_4,y_2), (x_4,y_3), (x_4,y_4), (y_1,x_1), (y_1,x_2), (y_1,x_3), (y_1,x_4), (y_2,x_1), (y_2,x_2), (y_2,x_3), (y_2,x_4), (y_3,x_1), (y_3,x_2), (y_3,x_3), (y_3,x_4), (y_4,x_1), (y_4,x_2), (y_4,x_3), (y_4,x_4)\}$.

There are exactly eight preferred extensions: $\mathscr{E}_1 = \{x_4, a\}$, $\mathscr{E}_2 = \{x_3, a\}$, $\mathscr{E}_3 = \{x_2, a\}$, $\mathscr{E}_4 = \{x_1, a\}$, $\mathscr{E}_5 = \{y_2, y_4\}$, $\mathscr{E}_6 = \{y_2, y_3\}$, $\mathscr{E}_7 = \{y_1, y_3\}$, $\mathscr{E}_8 = \{y_1, y_4\}$.

Their supports are: $\mathtt{vsupp}_p(\mathscr{E}_1, \mathscr{F}_{11}) = \mathtt{vsupp}_p(\mathscr{E}_2, \mathscr{F}_{11}) = \mathtt{vsupp}_p(\mathscr{E}_3, \mathscr{F}_{11}) = \mathtt{vsupp}_p(\mathscr{E}_4, \mathscr{F}_{11}) = (4, 1)$; $\mathtt{vsupp}_p(\mathscr{E}_5, \mathscr{F}_{11}) = \mathtt{vsupp}_p(\mathscr{E}_6, \mathscr{F}_{11}) = \mathtt{vsupp}_p(\mathscr{E}_7, \mathscr{F}_{11}) = \mathtt{vsupp}_p(\mathscr{E}_8, \mathscr{F}_{11}) = (2, 2)$.

Thus, we obtain $\mathtt{OBE}_{p,sum}(\mathscr{F}_{11}) = \mathtt{OBE}_{p,\max}(\mathscr{F}_{11}) = \mathtt{OBE}_{p,leximax}(\mathscr{F}_{11}) = \{\mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3, \mathscr{E}_4\}$; $\mathtt{OBE}_{p,\min}(\mathscr{F}_{11}) = \mathtt{OBE}_{p,leximin}(\mathscr{F}_{11}) = \{\mathscr{E}_5, \mathscr{E}_6, \mathscr{E}_7, \mathscr{E}_8\}$.

$\triangle$

Let us now show the proposition by using the previous three examples.

- $\mathtt{OBE}_{p,\min}$ ind $\mathtt{OBE}_{p,sum}$, follows from Example 9.
- $\mathtt{OBE}_{p,\max}$ ind $\mathtt{OBE}_{p,sum}$, follows from Example 10.
- $\mathtt{OBE}_{p,\max}$ ind $\mathtt{OBE}_{p,\min}$, follows from Example 11.
- $\mathtt{OBE}_{p,leximin}$ ind $\mathtt{OBE}_{p,sum}$, follows from Example 9.
- $\mathtt{OBE}_{p,leximin} \sqsubset \mathtt{OBE}_{p,\min}$. From Proposition 3 $\mathtt{OBE}_{p,leximin} \sqsubseteq \mathtt{OBE}_{p,\min}$. Example 10 shows that $\mathtt{OBE}_{p,leximin} \not\equiv \mathtt{OBE}_{p,\min}$. Consequently, $\mathtt{OBE}_{p,leximin} \sqsubset \mathtt{OBE}_{\min}$.
- $\mathtt{OBE}_{p,leximin}$ ind $\mathtt{OBE}_{p,\max}$, follows from Example 11.
- $\mathtt{OBE}_{p,leximax}$ ind $\mathtt{OBE}_{p,sum}$, follows from Example 10.
- $\mathtt{OBE}_{p,leximax}$ ind $\mathtt{OBE}_{p,\min}$, follows from Example 11.
- $\mathtt{OBE}_{p,leximax} \sqsubset \mathtt{OBE}_{p,\max}$. From Proposition 3, $\mathtt{OBE}_{p,leximax} \sqsubseteq \mathtt{OBE}_{p,\max}$. Example 9 shows that $\mathtt{OBE}_{p,leximax} \not\equiv \mathtt{OBE}_{p,\max}$. Consequently, $\mathtt{OBE}_{p,leximax} \sqsubset \mathtt{OBE}_{p,\max}$.
- $\mathtt{OBE}_{p,leximax}$ ind $\mathtt{OBE}_{p,leximin}$, follows from Example 11.

$\square$

5 Support-Based Acceptance Policy

This section presents a completely different approach for selecting arguments. We focus on arguments that have the greatest supports among extensions to construct what we call "candidate sets". Then, an argument is called *supportedly accepted* if it is in all the candidate sets.

Definition 11 (Candidate sets)**.** *Let $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ be an AS and let σ be a semantics. Let $\succeq$ be any pre-order defined on $\mathscr{A}$. Let $|\mathscr{A}| = m$. For a permutation θ of $\{1, \ldots, m\}$, let $>_\theta$ be the linear order on $\mathscr{A}$ defined by $a_{\theta(1)} >_\theta \ldots >_\theta a_{\theta(m)}$. $>_\theta$ is said to be compatible with $\succeq$ iff $a_{\theta(1)} \succeq \ldots \succeq a_{\theta(m)}$. A set $\mathscr{E} \subseteq \mathscr{A}$ is a candidate set of $\mathscr{F}$ under semantics σ w.r.t. $\succeq$ iff there exists a permutation θ of $\{1, \ldots, m\}$ such that $>_\theta$ is compatible with $\succeq$ and $\mathscr{E}$ is obtained by the following greedy procedure:*

$S := \emptyset$;
for $j = 1, \ldots, m$ *do*
  if $(\mathtt{ne}_\sigma(a_{\theta(j)}, \mathscr{F}) \geq 1)$ *and* $(S \cup \{a_{\theta(j)}\}$ *is conflict-free*$)$
  then $S := S \cup \{a_{\theta(j)}\}$
end for;
$\mathscr{E} := S$.

Note that $\succeq$ from previous definition depends on $\mathscr{F}$ and σ but we do not make it explicit (e.g. by writing $\succeq_{\mathscr{F},\sigma}$) in order to simplify the notation.

Roughly speaking, the previous definition says that to construct a candidate set, we first take the arguments with the maximal score s; then, we add as many arguments with score $s-1$ as possible (by taking into account that the resulting sets stays conflict-free); then arguments with score $s-2$ are added etc. There might be several possibilities (e.g. incompatible arguments having the same score) thus there might be several candidate sets.

In the following, we consider the pre-order $\succeq$ on $\mathscr{A}$ defined by for all $x, y \in \mathscr{A}$, $x \succeq y$ iff $\mathtt{ne}_\sigma(x, \mathscr{F}) \geq \mathtt{ne}_\sigma(y, \mathscr{F})$. We denote the set of candidate sets of $\mathscr{F}$ under σ w.r.t. this pre-order by $\mathtt{CS}_\sigma(\mathscr{F})$.

Note that, in general, neither each candidate set is an extension nor each extension is a candidate set. Observe also that the construction of candidate sets is reminiscent to the one of preferred subbases from a stratified belief base with respect to the inclusion-based ordering [5]; here the belief base consists of all the arguments and the stratification is based on the $\mathtt{ne}_\sigma(., \mathscr{F})$ score.

The notion of candidate set allows us to define a new inference mechanism that we call *supported inference* since for an argument, being "supported" by more extensions means more chance to be accepted.

Definition 12 (Supported acceptance)**.** *Let $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ be an AS, σ be a semantics and let $x \in \mathscr{A}$. We say that x is supportedly accepted under semantics σ iff $x \in \bigcap_{\mathscr{E} \in \mathtt{CS}_\sigma(\mathscr{F})}$. We denote the set of supportedly accepted arguments $\mathtt{Sp}_\sigma(\mathscr{F})$.*

We can show that supported inference is "between" sceptical and credulous inference. Namely, every sceptically accepted argument is supportedly accepted, and every supportedly accepted argument is credulously accepted.

Proposition 5. *For every AS $\mathscr{F} = (\mathscr{A}, \mathscr{R})$, for every semantics σ returning conflict-free extensions:*

$$\mathtt{Sc}_\sigma(\mathscr{F}) \subseteq \mathtt{Sp}_\sigma(\mathscr{F}) \subseteq \mathtt{Cr}_\sigma(\mathscr{F}).$$

Proof. Suppose that there are exactly m extensions, i.e. that $|\mathtt{Ext}_\sigma(\mathscr{F})| = m$. The case $m = 0$ is trivial; in the rest of the proof we suppose that $m \geq 1$. Observe that we have $\mathtt{Sc}_\sigma(\mathscr{F}) = \{x \in \mathscr{A} \mid \mathtt{ne}_\sigma(x, \mathscr{F}) = m\}$. Since σ returns conflict-free sets, all elements of $\mathtt{Ext}_\sigma(\mathscr{F})$ are conflict-free. Hence, their intersection is conflict-free. Thus, every candidate extension contains $\mathtt{Sc}_\sigma(\mathscr{F})$, formally for every $\mathscr{E}_i \in \mathtt{CS}_\sigma(\mathscr{F})$, we have $\mathtt{Sc}_\sigma(\mathscr{F}) \subseteq \mathscr{E}_i$. Therefore, $\mathtt{Sc}_\sigma(\mathscr{F}) \subseteq \mathtt{Sp}_\sigma(\mathscr{F})$.

Let $x \in \mathtt{Sp}_\sigma(\mathscr{F})$; then x is in all candidate sets. From Definition 11 we conclude that $\mathtt{ne}_\sigma(x, \mathscr{F}) \geq 1$ (since only arguments appearing in at least one extension can be added to candidate sets). This means that $x \in \mathtt{Cr}_\sigma(\mathscr{F})$. □

Note that the condition telling that σ returns conflict-free extensions is necessary to ensure the link between sceptical and supported acceptance. To show why, consider the following example.

Example 12. *Let $\mathscr{F} = (\mathscr{A}, \mathscr{R})$ with $\mathscr{A} = \{a, b\}$ and $\mathscr{R} = \{(a, b), (b, a)\}$. Suppose that $\mathtt{Ext}_\sigma(\mathscr{F}) = \{\{a, b\}\}$. Thus, $\mathtt{Sc}_\sigma(\mathscr{F}) = \{a, b\}$. Note that $\mathtt{CS}_\sigma(\mathscr{F}) = \{\{a\}, \{b\}\}$; hence, $\mathtt{Sp}_\sigma(\mathscr{F}) = \emptyset$.* △

However, this is not an issue, since all the well-known semantics return conflict-free sets. Let us now illustrate the results one can obtain with these candidate sets.

Example 13. *Let $\mathscr{F}_{13} = (\mathscr{A}_{13}, \mathscr{R}_{13})$ be an AS with $\mathscr{A}_{13} = \{a, b, c, d, e, f, g, h\}$ and $\mathscr{R}_{13} = \{(a, b), (b, a), (b, g), (c, d), (d, c), (d, g), (e, f), (f, e), (f, g), (g, h)\}$, shown in Figure 7. There are eight preferred extensions: $\{a, c, e, g\}$, $\{a, d, e, h\}$, $\{a, c, f, h\}$, $\{a, d, f, h\}$,*

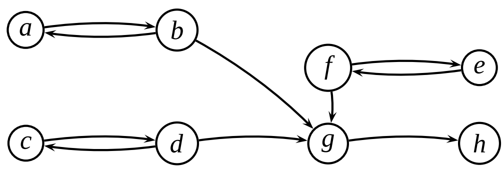

Figure 7: $\mathscr{F}_{13} = (\mathscr{A}_{13}, \mathscr{R}_{13})$: Argument h is almost sceptically accepted

$\{b, c, e, h\}$, $\{b, d, e, h\}$, $\{b, c, f, h\}$, $\{b, d, f, h\}$. There are no sceptically accepted arguments, i.e. $\mathtt{Sc}_p(\mathscr{F}_{13}) = \emptyset$. But h is accepted by seven out of the eight extensions, and it is supportedly accepted, i.e., $\mathtt{Sp}_p(\mathscr{F}_{13}) = \{h\}$. △

Let us illustrate the behaviour of supported inference on a more complex example:

Example 14. *Let $\mathscr{F}_{14} = (\mathscr{A}_{14}, \mathscr{R}_{14})$ be an argumentation system shown in Figure 8. There are four preferred extensions: $\{b, d, e, j, k\}$, $\{a, i, g, f, k\}$, $\{c, b, j, e, h\}$ and*

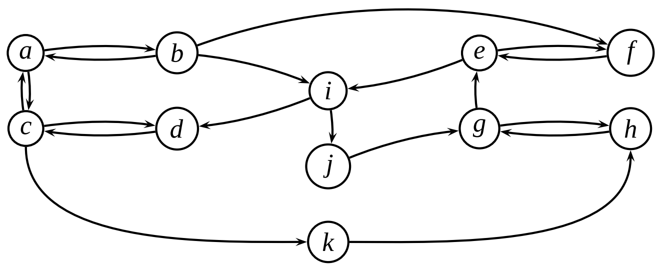

Figure 8: $\mathscr{F}_{14} = (\mathscr{A}_{14}, \mathscr{R}_{14})$

$\{a,d,e,j,k\}$. *There is no sceptically accepted argument, i.e.* $\mathtt{Sc}_p(\mathscr{F}_{14}) = \emptyset$. *We can build two candidate sets:* $\{b,d,e,j,k\}$ *and* $\{a,d,e,j,k\}$. *So we can supportedly accept four arguments:* $\mathtt{Sp}_p(\mathscr{F}_{14}) = \{d,e,j,k\}$.

△

Note that the set of candidate set is not always a subset of the set of extensions. Consider for instance the AS from Example 7, where there is only one candidate set $\{c,a,e,h\}$, that is *not* an extension. It is interesting to note that in that example there are four supportedly inferred arguments, whereas with the OBE methods only *c* is inferred.

A major drawback of credulous inference is that the set of inferred arguments is not always conflict-free. This is problematic since all these arguments cannot be accepted together in such a case. Sceptical inference does not suffer from this problem since the set of inferred arguments is ensured to be conflict-free. Interestingly, supported inference offers the same important property:

Fact 3. *For any $\mathscr{F}$, the set of supportedly accepted arguments is conflict-free.*

Note that this set is not necessarily admissible. This should not be shocking since the same observation can be made for the set of sceptically accepted arguments. Consider the following example:

Example 15. *Let* $\mathscr{F}_{15} = (\mathscr{A}_{15}, \mathscr{R}_{15})$ *with* $\mathscr{A}_{15} = \{a,b,c,d\}$ *and* $\mathscr{R}_{15} = \{(a,b), (a,c), (b,a), (b,c), (c,d)\}$*, shown in Figure 9.*

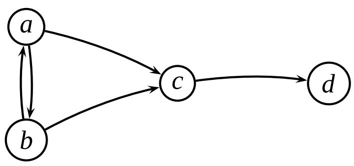

Figure 9: $\mathscr{F}_{15} = (\mathscr{A}_{15}, \mathscr{R}_{15})$: The set of sceptical arguments is not admissible

There are two preferred extensions: $\{a,d\}$ *and* $\{b,d\}$, $\mathtt{Sc}_p(\mathscr{F}_{15}) = \{d\}$, *but* $\{d\}$ *is not an admissible set. Observe that in this particular case we have* $\mathtt{Sp}_p(\mathscr{F}_{15}) = \mathtt{Sc}_p(\mathscr{F}_{15}) = \{d\}$. △

Finally, an interesting issue is to determine whether some connections exist between supported inference and the approach presented in the previous section. We provide a systematic study of the links between the two approaches under preferred semantics.

Proposition 6. *For every* $\gamma \in \{sum, min, max, leximin, leximax\}$, $\mathtt{OBE}_\gamma$ *and* $\mathtt{CS}$ *are incomparable under preferred semantics, i.e.,* $\mathtt{OBE}_{p,\gamma}$ $\mathtt{ind}$ $\mathtt{CS}_p$.

	$\mathtt{OBE}_{p,sum}$	$\mathtt{OBE}_{p,\min}$	$\mathtt{OBE}_{p,\max}$	$\mathtt{OBE}_{p,leximin}$	$\mathtt{OBE}_{p,leximax}$
$\mathtt{CS}_p$	ind	ind	ind	ind	ind

Figure 10: Inclusion relationships between rules for preferred semantics: $\mathtt{OBE}_{p,\gamma}$ $\mathtt{ind}$ $\mathtt{CS}_p$ for all γ from Definition 9.

Proof. The proof of this proposition follows from Example 16. Namely, that example shows that for every $\gamma \in \{sum, \text{min}, \text{max}, leximin, leximax\}$, $\mathtt{OBE}_{p,\gamma}$ $\mathtt{ind}$ $\mathtt{CS}_p$.

Example 16. *Let* $\mathscr{F}_{16} = (\mathscr{A}_{16}, \mathscr{R}_{16})$ *with* $\mathscr{A}_{16} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, a, b, c\}$ *and* $\mathscr{R}_{16} = \{(x_1,x_2), (x_1,x_3), (x_1,x_4), (x_1,x_5), (x_1,x_6), (x_2,x_1), (x_2,x_3), (x_2,x_4), (x_2,x_5), (x_2,x_6), (x_3,x_1), (x_3,x_2), (x_3,x_4), (x_3,x_5), (x_3,x_7), (x_4,x_1), (x_4,x_2), (x_4,x_3), (x_4,x_5), (x_4,x_7), (x_5,x_5), (x_5,a), (x_6,x_6), (x_6,b), (x_7,x_7), (x_7,c)\}$.

There are exactly four preferred extensions: $\mathscr{E}_1 = \{x_1,a,b\}$, $\mathscr{E}_2 = \{x_2,a,b\}$, $\mathscr{E}_3 = \{x_3,a,c\}$, $\mathscr{E}_4 = \{x_4,a,c\}$. *Their supports are:* $\mathtt{vsupp}_p(\mathscr{E}_1,\mathscr{F}_{16}) = \mathtt{vsupp}_p(\mathscr{E}_2,\mathscr{F}_{16}) = \mathtt{vsupp}_p(\mathscr{E}_3,\mathscr{F}_{16}) = \mathtt{vsupp}_p(\mathscr{E}_4,\mathscr{F}_{16}) = (4,2,1)$. *Thus,* $\mathtt{OBE}_{p,sum}(\mathscr{F}_{16}) = \mathtt{OBE}_{p,\min}(\mathscr{F}_{16}) = \mathtt{OBE}_{p,\max}(\mathscr{F}_{16}) = \mathtt{OBE}_{p,leximin}(\mathscr{F}_{16}) = \mathtt{OBE}_{p,leximax}(\mathscr{F}_{16}) = \{\mathscr{E}_1,\mathscr{E}_2,\mathscr{E}_3,\mathscr{E}_4\}$.

Note that $\mathtt{CS}_p(\mathscr{F}_{16}) = \{\mathscr{E}'_1,\mathscr{E}'_2,\mathscr{E}'_3,\mathscr{E}'_4\}$ *with* $\mathscr{E}'_1 = \{a,b,c,x_1\}$, $\mathscr{E}'_2 = \{a, b, c, x_2\}$, $\mathscr{E}'_3 = \{a,b,c,x_3\}$, $\mathscr{E}'_4 = \{a,b,c,x_4\}$. △

□

The previous proposition shows that, in general case, the set of extensions in OBE approach is not comparable with the set of candidate sets. However, in Example 16, the set of sceptically accepted arguments with respect to OBE (independently of γ used) is $\{a\}$; the set of supportedly accepted arguments in this example is $\{a,b,c\}$. One could ask whether the set of γ-sceptically accepted arguments is always a subset of the set of supportedly accepted arguments for some γ from OBE approach? The next result provides the answer to this question. Namely, the set of *max*-sceptically accepted arguments is always a subset of the set of supportedly accepted arguments under hypothesis that the argumentation semantics returns conflict-free sets. For other criteria used in OBE approach, the sets of γ-sceptically accepted arguments and supportedly accepted arguments are independent. Let us first show that every $\mathtt{OBE}_{\sigma,\max}$-sceptically accepted argument is also supportedly accepted, for every semantics that returns conflict-free extensions.

Proposition 7. *Let σ be a semantics returning conflict-free extensions. We have*

$$\mathtt{Sc}_{\mathtt{OBE}_{\sigma,\max}} \sqsubseteq \mathtt{Sp}_{\sigma}.$$

Proof. Let us show that for every argumentation system $\mathscr{F} = (\mathscr{A}, \mathscr{R})$, for every semantics σ that returns conflict-free extensions, $\mathtt{Sc}_{\mathtt{OBE}_{\sigma,\max}}(\mathscr{F}) \subseteq \mathtt{Sp}_{\sigma}(\mathscr{F})$. Let $a \in \mathtt{Sc}_{\mathtt{OBE}_{\sigma,\max}}(\mathscr{F})$; this means that for every $\mathscr{E} \in \mathtt{OBE}_{\sigma,\max}(\mathscr{F})$, $a \in \mathscr{E}$.

Denote $s = max_{x\in\mathscr{A}}\mathtt{ne}_{\sigma}(x,\mathscr{F})$, and let us show that no argument has better score than a, i.e. $\mathtt{ne}_{\sigma}(a,\mathscr{F}) = s$. By means of contradiction, suppose the contrary, i.e. let $b \in \mathscr{A}$ be such that $\mathtt{ne}_{\sigma}(b,\mathscr{F}) = s > \mathtt{ne}_{\sigma}(a,\mathscr{F})$. This means that there exists $\mathscr{E}' \in \mathtt{Ext}_{\sigma}(\mathscr{F})$ such that $b \in \mathscr{E}$ and $a \notin \mathscr{E}$. Since $b \in \mathscr{E}$, $\mathscr{E} \in \mathtt{OBE}_{\sigma,\max}(\mathscr{F})$. Contradiction with hypothesis $a \in \bigcap_{\mathscr{E}\in\mathtt{OBE}_{\sigma,\max}(\mathscr{F})} \mathscr{E}$. Hence, by reductio ad absurdum, we conclude $\mathtt{ne}_{\sigma}(a,\mathscr{F}) = s$.

Denote $strat_1 = \{x \in \mathscr{A} \mid \mathtt{ne}_{\sigma}(x,\mathscr{F}) = s\}$ and let us show that for every $x \in strat_1$, set $\{a,x\}$ is conflict-free. By means of contradiction, suppose the contrary. Let $b \in strat_1$ be an argument such that $\{a,b\}$ is not conflict-free. Since $a \in \mathtt{Sc}_{\mathtt{OBE}_{\sigma,\max}}(\mathscr{F})$, $\mathscr{F}$ has at least one extension; consequently $s \geq 1$. This means also that b is in at least one extension, say $\mathscr{E}'$. Because of the conflict between a and b, and since σ returns conflict-free extensions, $a \notin \mathscr{E}'$. But since $b \in strat_1$, $\mathscr{E}' \in \mathtt{OBE}_{\sigma,\max}(\mathscr{F})$. Contradiction with hypothesis that $a \in \bigcap_{\mathscr{E}\in\mathtt{OBE}_{\sigma,\max}(\mathscr{F})} \mathscr{E}$. By reductio ad absurdum, we conclude that no argument from $strat_1$ is in conflict with a. Thus, a is in all candidate sets, $a \in \bigcap_{\mathscr{E}\in\mathtt{CS}_{\sigma}(\mathscr{F})} \mathscr{E}$. In other words, $a \in \mathtt{Sp}_{\sigma}(\mathscr{F})$. We conclude that $\mathtt{Sc}_{\mathtt{OBE}_{\sigma,\max}} \sqsubseteq \mathtt{Sp}_{\sigma}$, for all semantics σ returning conflict-free sets. ☐

Let us now illustrate the indifference between γ-sceptical acceptance and supported acceptance for $\gamma \neq max$, again on the case of preferred semantics.

Proposition 8. *The links between* $\mathtt{Sc}_{\gamma}$ *and* $\mathtt{Sp}$ *under preferred semantics are as follows:*

1. $\mathtt{Sc}_{\mathtt{OBE}_{p,\max}} \sqsubset \mathtt{Sp}_{p}$.
2. *for every* $\gamma \in \{sum, min, leximin, leximax\}$, $\mathtt{Sc}_{\mathtt{OBE}_{p,\gamma}}$ $\mathtt{ind}$ $\mathtt{Sp}_{p}$.

	$\mathtt{Sc}_{\mathtt{OBE}_{p,sum}}$	$\mathtt{Sc}_{\mathtt{OBE}_{p,min}}$	$\mathtt{Sc}_{\mathtt{OBE}_{p,max}}$	$\mathtt{Sc}_{\mathtt{OBE}_{p,leximin}}$	$\mathtt{Sc}_{\mathtt{OBE}_{p,leximax}}$
$\mathtt{Sp}_{p}$	ind	ind	$\sqsupset$	ind	ind

Figure 11: Inclusion relationships (for different γ) between sceptical inference using $\mathtt{OBE}_{p,\gamma}$ and supported inference ($\mathtt{Sp}_{p}$). If $\mathscr{R}$ is the row rule and $\mathscr{R}'$ is the column rule, symbol $\sqsupset$ means that $\mathscr{R} \sqsupset \mathscr{R}'$, i.e. $\mathscr{R}' \sqsubset \mathscr{R}$; symbol ind means that $\mathscr{R}$ ind $\mathscr{R}'$.

Proof. To prove this proposition, we need several counter examples.

Example 17. *Let* $\mathscr{F}_{17} = (\mathscr{A}_{17}, \mathscr{R}_{17})$ *with* $\mathscr{A}_{17} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, a, b, c\}$ *and* $\mathscr{R}_{17} = \{(x_1,x_2), (x_1,x_3), (x_1,x_4), (x_1,x_5), (x_1,x_6), (x_1,x_7), (x_1,x_8), (x_1,x_9), (x_2,x_1), (x_2,x_3), (x_2,x_4), (x_2,x_5), (x_2,x_6), (x_2,x_7), (x_2,x_8), (x_2,x_9), (x_3,x_1), (x_3,x_2), (x_3,x_4), (x_3,x_5), (x_3,x_6), (x_3,x_7), (x_3,x_8), (x_3,x_9), (x_4,x_1), (x_4,x_2), (x_4,x_3),$

(x_4,x_5), (x_4,x_6), (x_4,x_7), (x_4,x_8), (x_4,x_9), (x_5,x_1), (x_5,x_2), (x_5,x_3), (x_5,x_4), (x_5,x_6), (x_5,x_7), (x_5,x_8), (x_5,x_{10}), (x_6,x_1), (x_6,x_2), (x_6,x_3), (x_6,x_4), (x_6,x_5), (x_6,x_7), (x_6,x_8), (x_6,x_{10}), (x_7,x_1), (x_7,x_2), (x_7,x_3), (x_7,x_4), (x_7,x_5), (x_7,x_6), (x_7,x_8), (x_7,x_{10}), (x_7,x_{11}), (x_8,x_1), (x_8,x_2), (x_8,x_3), (x_8,x_4), (x_8,x_5), (x_8,x_6), (x_8,x_7), (x_8,x_{10}), (x_8,x_{11}), (x_9,x_9), (x_9,a), (x_{10},x_{10}), (x_{10},b), (x_{11},x_{11}), (x_{11},c), (a,b), $(b,a)\}$.

There are exactly eight preferred extensions: $\mathscr{E}_1 = \{x_1,a\}$, $\mathscr{E}_2 = \{x_2,a\}$, $\mathscr{E}_3 = \{x_3,a\}$, $\mathscr{E}_4 = \{x_4,a\}$, $\mathscr{E}_5 = \{x_5,b\}$, $\mathscr{E}_6 = \{x_6,b\}$, $\mathscr{E}_7 = \{x_7,b,c\}$, $\mathscr{E}_8 = \{x_8,b,c\}$.

Their supports are: $\mathtt{vsupp}_p(\mathscr{E}_1,\mathscr{F}_{17}) = \mathtt{vsupp}_p(\mathscr{E}_2,\mathscr{F}_{17}) = \mathtt{vsupp}_p(\mathscr{E}_3,\mathscr{F}_{17}) = \mathtt{vsupp}_p(\mathscr{E}_4,\mathscr{F}_{17}) = \mathtt{vsupp}_p(\mathscr{E}_5,\mathscr{F}_{17}) = \mathtt{vsupp}_p(\mathscr{E}_6,\mathscr{F}_{17}) = (4,1)$*;* $\mathtt{vsupp}_p(\mathscr{E}_7,\mathscr{F}_{17}) = \mathtt{vsupp}_p(\mathscr{E}_8,\mathscr{F}_{17}) = (4,2,1)$.

Thus, we obtain $\mathtt{OBE}_{p,sum}(\mathscr{F}_{17}) = \{\mathscr{E}_7,\mathscr{E}_8\}$*. Therefore* $\mathtt{Sc}_{p,sum}(\mathscr{F}_{17}) = \{b,c\}$.

We have $\mathtt{CS}_p = \{\mathscr{E}'_1, \mathscr{E}'_2, \mathscr{E}'_3, \mathscr{E}'_4, \mathscr{E}'_5, \mathscr{E}'_6, \mathscr{E}'_7, \mathscr{E}'_8, \mathscr{E}'_9, \mathscr{E}'_{10}, \mathscr{E}'_{11}, \mathscr{E}'_{12}, \mathscr{E}'_{13}, \mathscr{E}'_{14}, \mathscr{E}'_{15}, \mathscr{E}'_{16}\}$ *with* $\mathscr{E}'_1 = \{a,c,x_1\}$, $\mathscr{E}'_2 = \{a,c,x_2\}$, $\mathscr{E}'_3 = \{a,c,x_3\}$, $\mathscr{E}'_4 = \{a,c,x_4\}$, $\mathscr{E}'_5 = \{a,c,x_5\}$, $\mathscr{E}'_6 = \{a,c,x_6\}$, $\mathscr{E}'_7 = \{a,c,x_7\}$, $\mathscr{E}'_8 = \{a,c,x_8\}$, $\mathscr{E}'_9 = \{b,c,x_1\}$, $\mathscr{E}'_{10} = \{b,c,x_2\}$, $\mathscr{E}'_{11} = \{b,c,x_3\}$, $\mathscr{E}'_{12} = \{b,c,x_4\}$, $\mathscr{E}'_{13} = \{b,c,x_5\}$, $\mathscr{E}'_{14} = \{b,c,x_6\}$, $\mathscr{E}'_{15} = \{b,c,x_7\}$, $\mathscr{E}'_{16} = \{b,c,x_8\}$*. Thus* $\mathtt{Sp}_p(\mathscr{F}_{17}) = \{c\}$*. This example shows that* $\mathtt{Sc}_{\mathtt{OBE}_{p,sum}} \not\sqsubseteq \mathtt{Sp}_p$. △

Example 18. *Let* $\mathscr{F}_{18} = (\mathscr{A}_{ex:17},\mathscr{R}_{ex:17})$ *with* $\mathscr{A}_{18} = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5, a, b\}$*and* $\mathscr{R}_{18} = \{(x_1,x_2)$, (x_1,x_5), (x_1,y_1), (x_1,y_2), (x_1,y_3), (x_1,y_4), (x_2,x_1), (x_2,x_5), (x_2,y_1), (x_2,y_2), (x_2,y_3), (x_2,y_4), (x_3,x_4), (x_3,x_5), (x_3,y_1), (x_3,y_2), (x_3,y_3), (x_4,x_3), (x_4,x_5), (x_4,y_1), (x_4,y_2), (x_4,y_3), (x_4,y_4), (x_5,x_5), (x_5,a), (y_1,x_1), (y_1,x_2), (y_1,x_3), (y_1,x_4), (y_1,y_2), (y_1,y_3), (y_1,y_4), (y_1,y_5), (y_2,x_1), (y_2,x_2), (y_2,x_3), (y_2,x_4), (y_2,y_1), (y_2,y_3), (y_2,y_4), (y_2,y_5), (y_3,x_1), (y_3,x_2), (y_3,x_3), (y_3,x_4), (y_3,y_1), (y_3,y_2), (y_3,y_4), (y_3,y_5), (y_4,x_1), (y_4,x_2), (y_4,x_3), (y_4,x_4), (y_4,y_1), (y_4,y_2), (y_4,y_3), (y_4,y_5), (y_5,y_5), (y_5,b), (a,b), $(b,a)\}$*. We obtain* $\mathtt{Ext}_p(\mathscr{F}_{18}) = \{\{x_1,x_3,a\}$, $\{x_1,x_4,a\}$, $\{x_2,x_3,a\}$, $\{x_2,x_4,a\}$, $\{y_1,b\}$, $\{y_2,b\}$, $\{y_3,b\}$, $\{y_4,b\}\}$*. Thus, we have* $\mathtt{Sc}_{\mathtt{OBE}_{p,\min}}(\mathscr{F}_{18}) = \mathtt{Sc}_{\mathtt{OBE}_{p,leximin}}(\mathscr{F}_{18}) = \{a\}$*. We obtain* $\mathtt{CS}_p(\mathscr{F}_{18}) = \{\{a,x_1,x_3\}$, $\{a,x_1,x_4\}$, $\{a,x_2,x_3\}$, $\{a,x_2,x_4\}$, $\{b,x_1,x_3\}$, $\{b,x_1,x_4\}$, $\{b,x_2,x_3\}$, $\{b,x_2,x_4\}\}$*. We have* $\mathtt{Sp}_p(\mathscr{F}_{18}) = \emptyset$*. This shows that* $\mathtt{Sc}_{\mathtt{OBE}_{p,\min}} \not\sqsubseteq \mathtt{Sp}_p$ *and that* $\mathtt{Sc}_{\mathtt{OBE}_{p,leximin}} \not\sqsubseteq \mathtt{Sp}_p$. △

Example 19. *Let* $\mathscr{F}_{19} = (\mathscr{A}_{19},\mathscr{R}_{19})$*, with* $\mathscr{A}_{19} = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, a, b, c\}$ *and* $\mathscr{R}_{19} = \{(x_1,x_2)$, (x_1,x_3), (x_1,x_4), (x_1,y_1), (x_1,y_2), (x_1,y_3), (x_2,x_1), (x_2,x_3), (x_2,x_4), (x_2,x_5), (x_2,y_1), (x_2,y_2), (x_2,y_3), (x_3,x_1), (x_3,x_2), (x_3,x_4), (x_3,x_5), (x_3,y_1), (x_3,y_2), (x_3,y_3), (x_4,x_4), (x_4,a), (x_5,x_5), (x_5,c), (y_1,x_1), (y_1,x_2), (y_1,x_3), (y_1,y_2), (y_1,y_3), (y_1,y_4), (y_2,x_1), (y_2,x_2), (y_2,x_3), (y_2,y_1), (y_2,y_3), (y_2,y_4), (y_3,x_1), (y_3,x_2), (y_3,x_3), (y_3,y_1), (y_3,y_2), (y_3,y_4), (y_4,y_4), (y_4,b), (a,b), (b,a), (b,c), $(c,b)\}$.

There are exactly six preferred extensions: $\mathscr{E}_1 = \{x_1,a\}$, $\mathscr{E}_2 = \{x_2,a,c\}$, $\mathscr{E}_3 = \{x_3,a,c\}$, $\mathscr{E}_4 = \{y_1,b\}$, $\mathscr{E}_5 = \{y_2,b\}$, $\mathscr{E}_6 = \{y_3,b\}$*. We have* $\mathtt{OBE}_{p,leximax}(\mathscr{F}_{19}) = \{\mathscr{E}_2,\mathscr{E}_3\}$*. Hence,* $\mathtt{Sc}_{\mathtt{OBE}_{p,leximax}} = \{a,c\}$*. However,* $\{a,c,x_1\} \in \mathtt{CS}_p(\mathscr{F}_{19})$ *and* $\{b,y_1\} \in \mathtt{CS}_p(\mathscr{F}_{19})$*, thus* $\mathtt{Sp}_p(\mathscr{F}_{19}) = \emptyset$*. This example shows that* $\mathtt{Sc}_{\mathtt{OBE}_{p,leximax}} \not\sqsubseteq \mathtt{Sp}_p$. △

Let us now present the proof:

- $\mathtt{Sc}_{\mathtt{OBE}_{p,sum}}$ ind $\mathtt{Sp}_p$. From Example 17, $\mathtt{Sc}_{\mathtt{OBE}_{p,sum}} \not\sqsubseteq \mathtt{Sp}_p$; from Example 16 we have $\mathtt{Sp}_p \not\sqsubseteq \mathtt{Sc}_{\mathtt{OBE}_{p,sum}}$.
- $\mathtt{Sc}_{\mathtt{OBE}_{p,\min}}$ ind $\mathtt{Sp}_p$. From Example 18, $\mathtt{Sc}_{\mathtt{OBE}_{p,\min}} \not\sqsubseteq \mathtt{Sp}_p$; from Example 16 we have $\mathtt{Sp}_p \not\sqsubseteq \mathtt{Sc}_{\mathtt{OBE}_{p,\min}}$.

- $\mathtt{Sc}_{\mathtt{OBE}_{p,\max}} \sqsubset \mathtt{Sp}_p$. From Proposition 7, we have that $\mathtt{Sc}_{\mathtt{OBE}_{p,\max}} \sqsubseteq \mathtt{Sp}_p$. To see that $\mathtt{Sp}_p \not\sqsubseteq \mathtt{Sc}_{\mathtt{OBE}_{p,\max}}$, consider Example 16.
- $\mathtt{Sc}_{\mathtt{OBE}_{p,leximin}}$ $\mathtt{ind}$ $\mathtt{Sp}_p$. From Example 18, $\mathtt{Sc}_{\mathtt{OBE}_{p,leximin}} \not\sqsubseteq \mathtt{Sp}_p$; Example 16 shows that $\mathtt{Sp}_p \not\sqsubseteq \mathtt{Sc}_{\mathtt{OBE}_{p,leximin}}$.
- $\mathtt{Sc}_{\mathtt{OBE}_{p,leximax}}$ $\mathtt{ind}$ $\mathtt{Sp}_p$. From Example 19, $\mathtt{Sc}_{\mathtt{OBE}_{p,leximax}} \not\sqsubseteq \mathtt{Sp}_p$. From Example 16 we have $\mathtt{Sp}_p \not\sqsubseteq \mathtt{Sc}_{\mathtt{OBE}_{p,leximax}}$.

□

The two previous propositions show that OBE and supported inference, although both using the scores of arguments defined as the number of extensions they belong to, induce intrinsically different reasoning mechanisms.

6 Conclusion and Related Work

This paper aimed at defining approaches for a better inference from abstract argumentation framework. Indeed, a large number of extensions results in a low number of sceptically accepted arguments. Several approaches have been described for dealing with this problem. First, different criteria for pairwise comparison of extensions and a method for selecting only the best extensions given the winners of pairwise duels have been pointed out. Second, several criteria for ordering the extensions have been presented. Both approaches result in a decrease of the number of extensions; consequently, the number of sceptical arguments increases (and the number of credulous arguments diminishes). The third approach we have put forward does not choose between existing extensions. Instead, it uses extensions to assign a score to every argument (the score of an argument is the number of extensions it belongs to). Then, starting from the arguments having the maximal score, candidate sets can be generated and on this ground supportedly accepted arguments have been defined.

Several papers in the literature are relevant to our work in the sense that their objectives are somehow similar. Thus, some previous work aimed at defining different levels of acceptability for arguments [9, 22, 18, 3]. Such levels can be obtained by attaching numerical scores between 0 and 1 to each argument, or by ranking arguments over an ordinal scale. Contrastingly, the goal of the present paper is not to tackle the problem of gradual acceptance. In this work our objective is not to question the classical binary framework for inference, where an argument is inferred or not, but to define inference relations allowing to infer more arguments than sceptical inference; to make a parallel with logical inference, a similar distinction exists between paraconsistent logics and some weighted logics (such as possibilistic or fuzzy logics).

Settings where argumentation systems are based on preferences or attack weights can also be exploited for reducing the number of extensions. However, those approaches suppose the availability of some extra information such as weights or preferences, whereas our approach is based solely on the argumentation system $\mathscr{F} = (\mathscr{A}, \mathscr{R})$.

Other approaches calculate arguments' scores / statuses without relying on the notion of extension [1, 12]. Unlike our approach, semantics (e.g., stable, preferred) are not used at all. Here, we suppose the use of an (arbitrary) semantics to calculate extensions and then point out a way to augment the number of arguments which are accepted. Our criteria are orthogonal to the notion of semantics, so that each criterion can be combined with each semantics.

Another related work is [10] which addresses the problem of defining more prudent inference relations for Dung's argumentation frameworks (i.e., the objective is to derive less arguments). Contrariwise to the present paper, instead of selecting some extensions or defining a new inference policy, the approach consists in strengthening the usual (direct) conflict-freeness property to indirect conflict-freenesss. Thus a prudent extension cannot contain two arguments when there exists an indirect attack among the first one and the second one. When the credulous policy and the preferred semantics (or the stable semantics) are considered, the set of derivable arguments from prudent extensions is included in the set of arguments derivable from the standard extensions.

Baroni et al. [3] show how to define some fine-grained argument justification statuses for abstract argumentation frameworks. For extension-based semantics, the justification status of an argument basically depends on the existence of extensions containing it and the existence of extensions attacking it. Clearly enough, the problem of selecting extensions is orthogonal to the problem of defining argument justification statuses; thus, Baroni's et al. results can be exploited as soon as some extensions exist, even if they come from a selection process. Our notion of supported inference is closer to their proposal since it induces an intermediate argument status, supported acceptance, between sceptical acceptance and credulous acceptance. However, the mechanisms at work for defining this intermediate status and its rationale are quite different from those considered in Baroni's at al. paper: in our work, the support of an argument is based on the number of extensions containing it.

Our approach also departs from the work by Dunne et el. [17] which focusses on ideal semantics. Indeed, ideal acceptance is more demanding than sceptical acceptance. As such, it proves useful when sceptical acceptance is not prudent enough, i.e. when unexpected arguments are sceptically accepted. Contrastingly, our work is motivated by the remaining cases, when sceptical inference is too cautious and discards some expected arguments.

Caminada and Wu [22] defined different labelling-based justification statuses of arguments. Indeed, they propose to attach to each argument the set of its possible labels (i.e. the collection of all labels it obtains in all complete labellings). Whereas Dung-based approach allows to split the arguments into three classes (sceptically accepted, credulously accepted, rejected), their contribution provides a way for fine-graded classification, by defining six different justification statuses: $\{\mathtt{in}\}$, $\{\mathtt{in}, \mathtt{undec}\}$, $\{\mathtt{undec}\}$, $\{\mathtt{in}, \mathtt{out}, \mathtt{undec}\}$, $\{\mathtt{out}, \mathtt{undec}\}$ and $\{\mathtt{out}\}$. The work of Caminada and Wu is related to our work since it could also be used to reason in cases when there are no (or when there are not enough) accepted arguments. However, the actual way to do it is drastically different from our approach.

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Appendix

Example 5:

```
arg(a).
arg(b).
arg(c).
arg(d).
arg(e).
arg(x2).
arg(x1).
att(x1,x1).
att(x2,x2).
att(a,c).
att(a,x1).
att(x1,b).
att(a,d).
att(a,e).
att(a,x2).
att(b,x2).
att(b,c).
att(c,a).
att(c,b).
att(c,d).
att(c,e).
att(c,x2).
att(c,x1).
att(d,a).
att(d,c).
att(e,a).
att(e,c).
```

Example 6:

```
arg(a).
arg(b).
arg(c).
arg(d).
arg(e).
arg(x1).
arg(x2).
att(x1,x1).
att(x2,x2).
att(a,b).
att(a,c).
att(a,d).
att(a,e).
```

```
att(b,a).
att(b,x1).
att(c,a).
att(d,a).
att(d,x2).
att(e,a).
att(x1,b).
att(x2,d).
att(x1,c).
att(x2,e).
```

Example 7:

```
arg(a).
arg(b).
arg(c).
arg(d).
arg(e).
arg(f).
arg(g).
arg(h).
att(a,b).
att(b,a).
att(e,f).
att(f,e).
att(b,g).
att(f,g).
att(g,h).
att(h,d).
att(d,c).
att(c,d).
```

Example 8:

```
arg(a).
arg(b).
arg(c).
arg(d).
arg(e).
att(a,b).
att(a,e).
att(b,a).
att(b,c).
att(b,d).
att(c,d).
att(c,e).
att(d,c).
```

```
att(d,e).
att(e,c).
att(e,d).
```

Example 9:

```
arg(x1).
arg(x2).
arg(x3).
arg(x4).
arg(x5).
arg(x6).
arg(x7).
arg(a).
arg(b).
att(x1,x2).
att(x2,x1).
att(x3,x4).
att(x4,x3).
att(x1,x5).
att(x2,x5).
att(x3,x5).
att(x4,x5).
att(x5,x5).
att(x4,x6).
att(x6,x6).
att(x2,x7).
att(x7,x7).
att(x7,b).
att(x5,a).
att(x6,b).
```

Example 10:

```
arg(a).
arg(x1).
arg(x2).
arg(x3).
arg(x4).
arg(x5).
arg(b1).
arg(b2).
arg(b3).
arg(b4).
arg(y1).
arg(y2).
arg(y3).
```

```
att(x1,x2).
att(x2,x1).
att(x3,x4).
att(x4,x3).
att(x1,x5).
att(x2,x5).
att(x3,x5).
att(x4,x5).
att(x1,x3).
att(x3,x1).
att(x1,x4).
att(x4,x1).
att(x2,x3).
att(x3,x2).
att(x2,x4).
att(x4,x2).
att(x5,x5).
att(x5,a).
att(y1,y2).
att(y2,y1).
att(y1,y3).
att(y2,y3).
att(y3,y3).
att(y3,b1).
att(y3,b2).
att(y3,b3).
att(y3,b4).
att(x1,y1).
att(x1,y2).
att(x2,y1).
att(x2,y2).
att(x3,y1).
att(x3,y2).
att(x4,y1).
att(x4,y2).
att(y1,x1).
att(y2,x1).
att(y1,x2).
att(y2,x2).
att(y1,x3).
att(y2,x3).
att(y1,x4).
att(y2,x4).
```

Example 11:

```
arg(x1).
```

```
arg(x2).
arg(x3).
arg(x4).
arg(x5).
arg(a).
arg(y1).
arg(y2).
arg(y3).
arg(y4).
att(x1,x2).
att(x2,x1).
att(x1,x3).
att(x3,x1).
att(x1,x4).
att(x4,x1).
att(x2,x3).
att(x3,x2).
att(x2,x4).
att(x4,x2).
att(x3,x4).
att(x4,x3).
att(x1,x5).
att(x2,x5).
att(x3,x5).
att(x4,x5).
att(x5,x5).
att(x5,a).
att(y1,y2).
att(y2,y1).
att(y3,y4).
att(y4,y3).
att(x1,y1).
att(x1,y2).
att(x1,y3).
att(x1,y4).
att(x2,y1).
att(x2,y2).
att(x2,y3).
att(x2,y4).
att(x3,y1).
att(x3,y2).
att(x3,y3).
att(x3,y4).
att(x4,y1).
att(x4,y2).
att(x4,y3).
```

```
att(x4,y4).
att(y1,x1).
att(y1,x2).
att(y1,x3).
att(y1,x4).
att(y2,x1).
att(y2,x2).
att(y2,x3).
att(y2,x4).
att(y3,x1).
att(y3,x2).
att(y3,x3).
att(y3,x4).
att(y4,x1).
att(y4,x2).
att(y4,x3).
att(y4,x4).
```

Example 13:

```
arg(a).
arg(b).
arg(c).
arg(d).
arg(e).
arg(f).
arg(g).
arg(h).
att(a,b).
att(b,a).
att(b,g).
att(c,d).
att(d,c).
att(d,g).
att(e,f).
att(f,e).
att(f,g).
att(g,h).
```

Example 14:

```
arg(a).
arg(b).
arg(c).
arg(d).
arg(e).
arg(f).
```

```
arg(g).
arg(h).
arg(i).
arg(j).
arg(k).
att(a,b).
att(b,a).
att(c,d).
att(d,c).
att(e,f).
att(f,e).
att(g,h).
att(h,g).
att(g,e).
att(a,c).
att(c,a).
att(b,f).
att(b,i).
att(e,i).
att(i,d).
att(i,j).
att(j,g).
att(c,k).
att(k,h).
```

Example 16:

```
arg(x1).
arg(x2).
arg(x3).
arg(x4).
arg(x5).
arg(x6).
arg(x7).
arg(a).
arg(b).
arg(c).
att(x1,x2).
att(x1,x3).
att(x1,x4).
att(x2,x1).
att(x2,x3).
att(x2,x4).
att(x3,x1).
att(x3,x2).
att(x3,x4).
```

```
att(x4,x1).
att(x4,x2).
att(x4,x3).
att(x5,x5).
att(x6,x6).
att(x7,x7).
att(x1,x5).
att(x2,x5).
att(x3,x5).
att(x4,x5).
att(x1,x6).
att(x2,x6).
att(x3,x7).
att(x4,x7).
att(x5,a).
att(x6,b).
att(x7,c).
```

Example 17:

```
arg(x1).
arg(x2).
arg(x3).
arg(x4).
arg(x5).
arg(x6).
arg(x7).
arg(x8).
arg(x9).
arg(x10).
arg(x11).
arg(a).
arg(b).
arg(c).
att(x1,x2).
att(x1,x3).
att(x1,x4).
att(x1,x5).
att(x1,x6).
att(x1,x7).
att(x1,x8).
att(x2,x1).
att(x2,x3).
att(x2,x4).
att(x2,x5).
att(x2,x6).
att(x2,x7).
```

```
att(x2,x8).
att(x3,x1).
att(x3,x2).
att(x3,x4).
att(x3,x5).
att(x3,x6).
att(x3,x7).
att(x3,x8).
att(x4,x1).
att(x4,x2).
att(x4,x3).
att(x4,x5).
att(x4,x6).
att(x4,x7).
att(x4,x8).
att(x5,x1).
att(x5,x2).
att(x5,x3).
att(x5,x4).
att(x5,x6).
att(x5,x7).
att(x5,x8).
att(x6,x1).
att(x6,x2).
att(x6,x3).
att(x6,x4).
att(x6,x5).
att(x6,x7).
att(x6,x8).
att(x7,x1).
att(x7,x2).
att(x7,x3).
att(x7,x4).
att(x7,x5).
att(x7,x6).
att(x7,x8).
att(x8,x1).
att(x8,x2).
att(x8,x3).
att(x8,x4).
att(x8,x5).
att(x8,x6).
att(x8,x7).
att(x1,x9).
att(x2,x9).
att(x3,x9).
```

```
att(x4,x9).
att(x5,x10).
att(x6,x10).
att(x7,x10).
att(x8,x10).
att(x7,x11).
att(x8,x11).
att(x9,x9).
att(x10,x10).
att(x11,x11).
att(x9,a).
att(x10,b).
att(x11,c).
att(a,b).
att(b,a).
```

Example 18:

```
arg(x1).
arg(x2).
arg(x3).
arg(x4).
arg(x5).
arg(y1).
arg(y2).
arg(y3).
arg(y4).
arg(y5).
arg(a).
arg(b).
att(x1,x2).
att(x2,x1).
att(x3,x4).
att(x4,x3).
att(y1,y2).
att(y1,y3).
att(y1,y4).
att(y2,y1).
att(y2,y3).
att(y2,y4).
att(y3,y1).
att(y3,y2).
att(y3,y4).
att(y4,y1).
att(y4,y2).
att(y4,y3).
att(x1,y1).
```

```
att(x1,y2).
att(x1,y3).
att(x1,y4).
att(x2,y1).
att(x2,y2).
att(x2,y3).
att(x2,y4).
att(x3,y1).
att(x3,y2).
att(x3,y3).
att(x3,y3).
att(x4,y1).
att(x4,y2).
att(x4,y3).
att(x4,y4).
att(y1,x1).
att(y1,x2).
att(y1,x3).
att(y1,x4).
att(y2,x1).
att(y2,x2).
att(y2,x3).
att(y2,x4).
att(y3,x1).
att(y3,x2).
att(y3,x3).
att(y3,x4).
att(y4,x1).
att(y4,x2).
att(y4,x3).
att(y4,x4).
att(x1,x5).
att(x2,x5).
att(x3,x5).
att(x4,x5).
att(x5,x5).
att(y1,y5).
att(y2,y5).
att(y3,y5).
att(y4,y5).
att(y5,y5).
att(x5,a).
att(y5,b).
att(a,b).
att(b,a).
```

Example 19:

```
arg(x1).
arg(x2).
arg(x3).
arg(x4).
arg(x5).
arg(y1).
arg(y2).
arg(y3).
arg(y4).
arg(a).
arg(b).
arg(c).
att(x1,x2).
att(x1,x3).
att(x2,x1).
att(x2,x3).
att(x3,x1).
att(x3,x2).
att(y1,y2).
att(y1,y3).
att(y2,y1).
att(y2,y3).
att(y3,y1).
att(y3,y2).
att(x1,y1).
att(x1,y2).
att(x1,y3).
att(x2,y1).
att(x2,y2).
att(x2,y3).
att(x3,y1).
att(x3,y2).
att(x3,y3).
att(y1,x1).
att(y1,x2).
att(y1,x3).
att(y2,x1).
att(y2,x2).
att(y2,x3).
att(y3,x1).
att(y3,x2).
att(y3,x3).
att(x1,x4).
att(x2,x4).
```

```
att(x3,x4).
att(x4,x4).
att(x2,x5).
att(x3,x5).
att(x5,x5).
att(y1,y4).
att(y2,y4).
att(y3,y4).
att(y4,y4).
att(x4,a).
att(x5,c).
att(y4,b).
att(a,b).
att(b,a).
att(b,c).
att(c,b).
```