Ranking Arguments With Compensation-Based Semantics

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Abstract

In almost all existing semantics in argumentation, a strong attack has a lethal effect on its target that a set of several weak attacks may not have. This paper investigates the case where several weak attacks may *compensate* one strong attack. It defines a broad class of ranking semantics, called $\alpha-\texttt{BBS}$, which satisfy compensation. $\alpha-\texttt{BBS}$ assign a burden number to each argument and order the arguments with respect to those numbers. We study formal properties of $\alpha-\texttt{BBS}$, implement an algorithm that calculates the ranking, and perform experiments that show that the approach computes the ranking very quickly. Moreover, an approximation of the ranking can be provided at any time.

Introduction

An argumentation framework consists of an *argumentation graph*, that is *arguments* and *attacks* between them, and a *semantics* for evaluating the arguments, and thus for specifying which arguments are *acceptable*.

The most dominant semantics in the literature are those that compute extensions of arguments, initially proposed by Dung (1995). Such semantics are based on the assumption that a successful attack completely destroys its target. Consequently, several successful attacks cannot destroy the target at a greater extent. There are applications where this assumption makes perfect sense (Dung 1995). In other applications, like decision making or dialogues, an attack only *weakens* its target. Think about a committee which recruits young researchers. Once an argument against a candidate is given, even if this argument is attacked, the initial argument is still considered by the members of the committee (but with a lower strength). Consequently, one attack does not necessarily have the same effect as several attacks. Consider argumentation graph $\mathcal{F}_1$ from Figure 1. Arguments a and b are both attacked by strong (i.e. non attacked) arguments. However, b is weakened by more attacks, thus a can be seen as more acceptable than b. Note that the number of attackers plays a role in this example. A similar reasoning holds for $\mathcal{F}_2$. Indeed, b should be more acceptable than a since a is weakened whereas b is not. In graph $\mathcal{F}_3$, the arguments a and b have the same number of attackers. However, the two attackers of b are weaker than the attackers of a. Consequently b should be more acceptable than a. Let us now have a look at $\mathcal{F}_4$. The argument a is attacked by one strong argument, whereas b is attacked by three weak arguments.

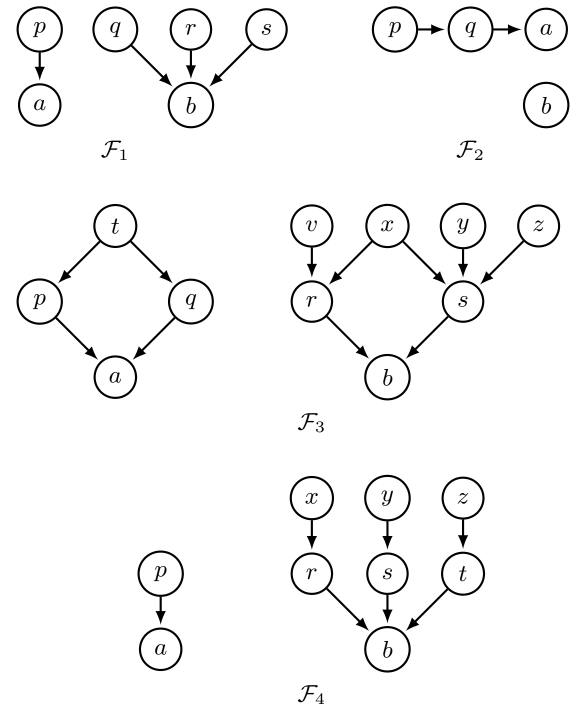

Figure 1: Four argumentation frameworks

There are three possibilities: i) give precedence to the number of attacks, in which case a would be more acceptable than b; ii) promote the quality of attacks, thus b would be more acceptable than a; iii) consider that n weak attacks are equivalent to one strong attack, thus a and b would be equally acceptable. We would like to emphasize that there is no "best" or "ideal" solution in general - the choice depends on the particular application. For example, in multi-criteria decision making context, compensation is very common (Dubois, Fargier, and Bonnefon 2008). Assume that r, s, t, p promote respectively criteria c_1, c_2, c_3 and c. Assume

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also that each c_i is less important than c, but the three criteria (c_1, c_2, c_3) together have the same importance as c. Thus, in our example, the three arguments r, s, t compensate p; so a and b would be equally acceptable.

In computational argumentation literature, there are semantics that privilege quality (case ii). Examples of such semantics are extension-based ones (Dung 1995; Baroni, Giacomin, and Guida 2005). There are also semantics that promote quantity (case i). Bbs and Dbs semantics defined by Amgoud and Ben-Naim (2013) are some examples. However, to the best of our knowledge, the compensation situation has never been considered. In this paper, we explore this intermediate position for the first time.

The contribution of the paper is threefold: First, we define the notion of compensation. We argue that it is based on two parameters: i) a parameter showing at what extent an attack is weak, and ii) a parameter indicating the number of weak attacks needed to compensate a strong one.

Second, we propose a large family of semantics, called $\alpha-$BBS (for α *burden-based semantics*), that allow compensation. α is a parameter which may take different values, each of which leads to a different semantics. This parameter allows one to choose to which extent to privilege quality of attacks over their quantity (or vice versa). Indeed, the smaller the value of α, the bigger the influence of the *number* of attacks. Conversely, the greater the value of α, the bigger the influence of the *quality* of attackers. Consequently, α is broadly related to the two parameters of compensation.
From a conceptual point of view, the new semantics do not compute extensions. They rather assign a numerical score to every argument, and then rank-order the arguments from the most acceptable to the least acceptable ones with respect to the scores. We show that the novel semantics satisfy postulates proposed in the literature.

Third, we implemented an algorithm that computes the ranking induced by $\alpha-$BBS, and ran several experiments using various values of α. We used a publicly available benchmark proposed by ICCMA (International Competition on Computational Models of Argumentation). The results show that our approach is very efficient as the ranking on the set of arguments can be calculated quickly. Moreover, an approximation of the final ranking can be provided at any time (during the execution of the algorithm).

Basic Concepts

An argumentation graph consists of a set of arguments and a set of attacks between them. Arguments represent reasons for believing statements, doing actions, etc. Attacks express conflicts between pairs of arguments.

Definition 1 (Argumentation graph) *An* argumentation graph *(AG) is an ordered pair $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, where $\mathcal{A}$ is a finite set of* arguments *and $\mathcal{R}$ is a binary attack relation on $\mathcal{A}$. The notation $a\mathcal{R}b$ or $(a, b) \in \mathcal{R}$ means that a attacks b.*

Since arguments are conflicting, one needs an acceptability semantics for identifying acceptable ones. In this paper, we focus on semantics that rank arguments with regard to acceptability. Let us first introduce the notion of ranking.

Definition 2 (Ranking) *A* ranking *on a set $\mathcal{A}$ is a binary relation $\succeq$ on $\mathcal{A}$ such that $\succeq$ is* total *(i.e., $\forall\, a, b \in \mathcal{A}$, $a \succeq b$ or $b \succeq a$) and* transitive *(i.e., $\forall\, a, b, c \in \mathcal{A}$, if $a \succeq b$ and $b \succeq c$, then $a \succeq c$). Intuitively, $a \succeq b$ means that a is* at least as acceptable *as b. We write $a \succ b$ when $b \not\succeq a$ (a is* strictly more acceptable *than b) and $a \equiv b$ when $a \succeq b$ and $b \succeq a$ (a and b are equally acceptable).*

Definition 3 (Ranking semantics) *A* ranking semantics *is a function $\mathbf{S}$ that maps any argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ into a ranking on $\mathcal{A}$.*

Notations: Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AG. For all $a \in \mathcal{A}$, we denote by $\mathtt{Att}^{\mathcal{F}}(a)$ the set of all attackers of a (i.e., $\mathtt{Att}^{\mathcal{F}}(a) = \{b \in \mathcal{A} \mid b\mathcal{R}a\}$), and by $\mathtt{Def}^{\mathcal{F}}(a)$ the set of all defenders of a (i.e., $\mathtt{Def}^{\mathcal{F}}(a) = \{b \in \mathcal{A} \mid \exists c \in \mathcal{A}$ such that $c\mathcal{R}a$ and $b\mathcal{R}c\}$). When the graph $\mathcal{F}$ is clear from the context, we write $\mathtt{Att}(a)$ and $\mathtt{Def}(a)$ for short.

Compensation

This section introduces a formalization of the notion of compensation, meaning that several weak attacks have the same impact as one strong attack. By strong attack, we mean an attack from an unattacked argument. To make precise what we mean by a weak attack, we use a parameter i. Roughly speaking, an attack is said to be weak if it comes from an argument that is attacked by i unattacked arguments.

Let us provide the formal definitions. First, $\mathcal{C}^i(\mathcal{F})$ is the set of all arguments that are attacked by exactly i unattacked arguments (and only by those arguments).

Definition 4 *For every argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{C}^i(\mathcal{F}) = \{a \in \mathcal{A}$ such that $|\mathtt{Att}(a)| = i$ and $\forall b \in \mathtt{Att}(a), \mathtt{Att}(b) = \emptyset\}$.*

We are now ready to define compensation. It says that an argument receiving one strong attack is as good as an argument receiving n weak attacks.

Definition 5 (Compensation) *Let $n, i \in \{1, 2, 3, \ldots\}$. A ranking semantics $\mathbf{S}$ satisfies* compensation *at degree (n, i) iff for every argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, for all $a, b \in \mathcal{A}$, the following holds: if*

- $|\mathtt{Att}(a)| = n$, $\mathtt{Att}(a) \subseteq \mathcal{C}^i(\mathcal{F})$, *and*
- $|\mathtt{Att}(b)| = 1$, $\mathtt{Att}(b) \subseteq \mathcal{C}^0(\mathcal{F})$,

then $a \equiv b$.

Our definition of compensation allows to discriminate between existing (and new) ranking-based semantics. We believe that it is a good starting point for studying this complex and important phenomenon.

A New Family of Semantics: α-BBS

This section proposes a large family of semantics, $\alpha-$BBS, that satisfy compensation. These semantics differ in parameter α which is linked to the two parameters of compensation.

The new semantics assign to every argument a score which represents how heavily the argument is attacked. Indeed, the score of an argument depends on the scores of its direct attackers, the scores of those direct attackers themselves depend on the scores of the attackers of the direct attackers and so on. Thus, the score function takes into account both the (direct and indirect) attackers of an argument and its (direct and indirect) defenders. Furthermore, a score increases when the number and / or the quality of the attackers increase.

Definition 6 **(s_α)** *Let $\alpha \in (0, +\infty)$ and $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation graph. We define the function s_α as follows: $s_\alpha : \mathcal{A} \to [1, +\infty)$ such that $\forall a \in \mathcal{A}$,*

$$s_\alpha(a) = 1 + \left(\sum_{b \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(b))^\alpha} \right)^{1/\alpha}$$

If $\mathtt{Att}(a) = \emptyset$ then $s_\alpha(a) = 1$. $s_\alpha(a)$ is called the burden number *of a.*

The inner exponent α is crucial for the compensation: the bigger the value of α, the bigger the influence of the quality of attackers. The outer exponent $1/\alpha$ is a correction factor (inspired by p-norms in L^p spaces), added in order to insure the uniqueness of the solution of the corresponding set of equations (Theorem 1).

The function s_α depends clearly on the argumentation graph $\mathcal{F}$. However, for the sake of simplicity of notations, we do not explicitly write $s_\alpha^{\mathcal{F}}$.

According to s_α, arguments with small burden numbers are more acceptable than arguments with greater numbers.

Definition 7 **($\alpha-\mathtt{BBS}$)** *Let $\alpha \in (0, +\infty)$. The ranking semantics $\alpha-\mathtt{BBS}$ maps any argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ into a ranking $\succeq_\alpha^{\mathcal{F}}$ on $\mathcal{A}$ such that for all $a, b \in \mathcal{A}$,*

$$a \succeq_\alpha^{\mathcal{F}} b \text{ iff } s_\alpha(a) \leq s_\alpha(b).$$

The two relations $\succ_\alpha^{\mathcal{F}}$ and $\equiv_\alpha^{\mathcal{F}}$ are defined as in Definition 2. When $\mathcal{F}$ is clear from the context, we write $\succeq_\alpha$ (resp. $\succ_\alpha$, $\equiv_\alpha$) instead of $\succeq_\alpha^{\mathcal{F}}$ (resp. $\succ_\alpha^{\mathcal{F}}$, $\equiv_\alpha^{\mathcal{F}}$). Let us illustrate $\alpha-\mathtt{BBS}$ on the running example.

Example 1 *Let $\alpha = 1$. Consider graphs $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ from the introduction. Since there are no cycles, we can start by calculating the values of non-attacked arguments, then consider the arguments attacked by them, and so on.*

- $\mathcal{F}_1$*:* $s_\alpha(p) = s_\alpha(q) = s_\alpha(r) = s_\alpha(s) = 1$, $s_\alpha(a) = 2$, $s_\alpha(b) = 4$
- $\mathcal{F}_2$*:* $s_\alpha(p) = s_\alpha(b) = 1$, $s_\alpha(q) = 2$, $s_\alpha(a) = 1.5$
- $\mathcal{F}_3$*:* $s_\alpha(t) = s_\alpha(v) = s_\alpha(x) = s_\alpha(y) = s_\alpha(z) = 1$, $s_\alpha(p) = s_\alpha(q) = s_\alpha(a) = 2$, $s_\alpha(r) = 3$, $s_\alpha(s) = 4$, $s_\alpha(b) = \frac{19}{12}$.

Note that for every $\alpha > 0$, a is more acceptable than b in $\mathcal{F}_1$, whereas b is better than a in both $\mathcal{F}_2$ and $\mathcal{F}_3$. The situation is different for $\mathcal{F}_4$. The parameter α plays an important role in this example as shown in Figure 2. There exists $\alpha' \approx 1.585$ such that for $\alpha < \alpha'$, argument a is more acceptable than b and for $\alpha > \alpha'$, argument b is more acceptable than a. If $\alpha = \alpha'$ then a and b are equally acceptable.

Now we turn to the question: does s_α from Definition 6 exist for every argumentation graph $\mathcal{F}$? First note that in Definition 6, the burden number of an argument depends on the burden numbers of other arguments. This raises the question whether the system of equations of Definition 6 (one equation per argument) has a solution. Clearly, for argumentation graphs that do not contain cycles such a solution exists; moreover, it is unique (i.e. each argument has a single score). It is sufficient to start by non attacked arguments, continue by calculating burden numbers of arguments attacked by them, and so on. The situation is not obvious when the graph contains cycles. In what follows, we show that, even in this general case, the system of equations has a *unique* solution. Moreover, the next theorem provides an algorithm for computing it. In order to prove this result, we need to write the system of equations from Definition 6 in a fixed-point form[1]. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation graph with $\mathcal{A} = \{a_1, \ldots, a_n\}$. We denote by $\mathbf{v} = (v_1, \ldots, v_n)$ the vector of n burden numbers of arguments of $\mathcal{A}$. For each argument a_i, the fixed-point form of its equation can be written as

$$f_i^\alpha(\mathbf{v}) = 1 + \left(\sum_{j \text{ s.t. } a_j \in \mathtt{Att}(a_i)} \frac{1}{(v_j)^\alpha} \right)^{1/\alpha}$$

Note that if $\mathtt{Att}(a_i) = \emptyset$, $f_i^\alpha(\mathbf{v}) = 1$. The fixed-point form of the system of equations is represented by $\mathbf{v} = F_\alpha(\mathbf{v}) = [f_1^\alpha(\mathbf{v}) \ldots f_n^\alpha(\mathbf{v})]$. The proofs can be found in the appendix.

Theorem 1 *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation graph with $\mathcal{A} = \{a_1, \ldots, a_n\}$. Let $\mathbf{v} = F_\alpha(\mathbf{v}) = [f_1^\alpha(\mathbf{v}) \ldots f_n^\alpha(\mathbf{v})]$ be the fixed-point form of n equations from Definition 6. There exists a unique solution $\mathbf{v}^* = (v_1^*, \ldots, v_n^*) \in [1, +\infty)^n$ of this system of equations, which is exactly the limit of the sequence $\{\mathbf{v}^{(k)}\}_{k=0}^{+\infty}$ defined from an arbitrarily selected $\mathbf{v}^0 = (v_1^{(0)}, \ldots, v_n^{(0)}) \in [1, +\infty)^n$ and generated by: for each $k \geq 1$,*

$$\mathbf{v}^{(k)} = (v_1^{(k)}, \ldots, v_n^{(k)}) = F_\alpha(\mathbf{v}^{k-1})$$

To sum up, the previous theorem ensures that each argument in an argumentation graph has a burden number, furthermore it is unique. This result is very important for ranking arguments.

Note that the elements of a cycle have the same strength. Formally: for every argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ with

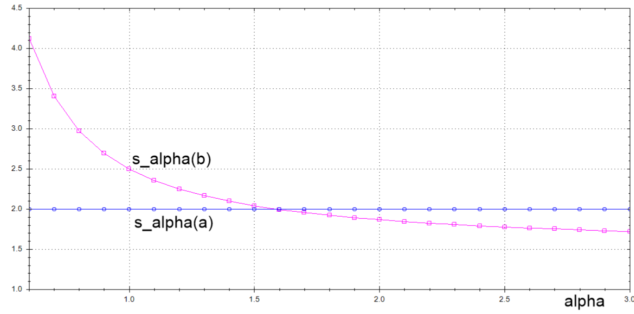

Figure 2: $s_\alpha(a)$ and $s_\alpha(b)$ in $\mathcal{F}_4$. x-axis shows the value of α. y-axis shows the values $s_\alpha(a)$ and $s_\alpha(b)$.

[1] The notion of fixed-point form is a well-know concept in numerical analysis, see e.g. the book by Burden and Faires (2001).

$\mathcal{A} = \{a_1, \dots, a_n\}$ and $\mathcal{R} = \{(a_i, a_{i+1}) \mid i \in \{1, 2, \dots, n-1\}\} \cup \{(a_n, a_1)\}$, for every $\alpha \in [1, +\infty)$, for every $i, j \in \{1, 2, \dots, n\}$, we have $s_\alpha(a_i) = s_\alpha(a_j)$.

Properties of α-BBS

This section investigates the properties of the new semantics. We start with a result showing how α is related to the two parameters of compensation.

Theorem 2 *For every $\alpha \in (0, +\infty)$, $\alpha-$*BBS *satisfies compensation at degree (n, i) if and only if $n^{1/\alpha} = 1 + i^{1/\alpha}$.*

The previous result does not answer the question: given a pair (n, i), is there a value of α such that $\alpha-$BBS satisfies compensation at degree (n, i)? The next theorem solves that issue. Namely, we show that whenever $1 \leq i < n$, there exists a unique value of α such that $\alpha-$BBS satisfies compensation at degree (n, i).

Theorem 3 *For every $n, i \in \mathbb{N}$ such that $n > i$ there exists a unique $\alpha \in (0, +\infty)$ s.t. $\alpha-$*BBS *satisfies compensation at degree (n, i).*

Recently, Amgoud and Nen-Naim (2013) proposed postulates for ranking-based semantics. Each postulate is seen as a desirable property that might be satisfied by a semantics. Some of those postulates are mandatory while others are optional. The remainder of this section shows that $\alpha-$BBS satisfy all the mandatory postulates, for every $\alpha \in (0, +\infty)$.

We start by noticing that the scores of arguments do not depend on their names. Hence, $\alpha-$BBS satisfy the *abstraction* postulate, called *language independence* in (Baroni and Giacomin 2007).

We continue by showing that the question whether an argument a is ranked above or tied with another argument b is independent of any argument or attack that is neither connected to a nor b (i.e., there is no path from that argument or attack to a or b, ignoring the direction of the edges).

Theorem 4 *Let $\alpha \in (0, +\infty)$, and let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}_1 = (\mathcal{A}_1, \mathcal{R}_1)$ be two argumentation graphs such that $\mathcal{A} \cap \mathcal{A}_1 = \emptyset$. Let $\mathcal{F}' = (\mathcal{A} \cup \mathcal{A}_1, \mathcal{R} \cup \mathcal{R}_1)$. Then $\forall a, b \in \mathcal{A}$, $a \succeq^{\mathcal{F}}_\alpha b$ iff $a \succeq^{\mathcal{F}'}_\alpha b$.*

According to $\alpha-$BBS semantics, non-attacked arguments are strictly more acceptable than attacked ones.

Theorem 5 *For every $\alpha \in (0, +\infty)$, for any argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\forall\, a, b \in \mathcal{A}$, the following holds: if $\mathtt{Att}(a) = \emptyset$ and $\mathtt{Att}(b) \neq \emptyset$, then $a \succ^{\mathcal{F}}_\alpha b$.*

Similarly, being defended is better than not.

Theorem 6 *For every $\alpha \in (0, +\infty)$, for any argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\forall\, a, b \in \mathcal{A}$, the following holds: if $|\mathtt{Att}(a)| = |\mathtt{Att}(b)|, \mathtt{Def}(a) \neq \emptyset$, and $\mathtt{Def}(b) = \emptyset$, then $a \succ^{\mathcal{F}}_\alpha b$.*

$\alpha-$BBS ensure that an argument a is at least as acceptable as b in case the attackers of b are at least as numerous and well-ranked as those of a.

Theorem 7 *For every $\alpha \in (0, +\infty)$, for any argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\forall\, a, b \in \mathcal{A}$, the following holds: if there exists an injective function f from $\mathtt{Att}(a)$ to $\mathtt{Att}(b)$ such that for all $c \in \mathtt{Att}(a)$, $f(c) \succeq^{\mathcal{F}}_\alpha c$, then $a \succeq^{\mathcal{F}}_\alpha b$.*

The next result shows that in case the attackers of an argument b dominate the attackers of a in terms of quantity and/or quality, then a is more acceptable than b.

Theorem 8 *For every $\alpha \in (0, +\infty)$, for any argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\forall\, a, b \in \mathcal{A}$, if*

- *there exists an injective function f from $\mathtt{Att}(a)$ to $\mathtt{Att}(b)$ such that for all $c \in \mathtt{Att}(a)$, $f(c) \succeq^{\mathcal{F}}_\alpha c$, and*
- *$|\mathtt{Att}(a)| < |\mathtt{Att}(b)|$ or $\exists c' \in \mathtt{Att}(a)$ with $f(c') \succ^{\mathcal{F}}_\alpha c'$,*

then $a \succ^{\mathcal{F}}_\alpha b$.

Finally, we show that $\alpha-$BBS treat equally the arguments in an elementary cycle. Indeed, they assign them the same burden number.

Theorem 9 *For every $\alpha \in (0, +\infty)$, for any argumentation graph $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ such that $\mathcal{A} = \{a_1, a_2, \dots, a_n\}$, $\mathtt{Att}(a_n) = \{a_1\}$ and for all $i \in \{1, 2, \dots, n-1\}$, $\mathtt{Att}(a_i) = \{a_{i+1}\}$, we have that $\forall a, b \in \mathcal{A}, a \equiv^{\mathcal{F}}_\alpha b$.*

Experimental Results

Theorem 1 not only shows the uniqueness of arguments' burden numbers but also suggests an algorithm for computing those numbers. It proceeds as follows. At step 1: set the burden number of every argument to 1, i.e. for every $a \in \mathcal{A}$, ${s_\alpha}^1(a) = 1$. At step $k \in \mathbb{N}$, we update the burden number of each argument as follows: ${s_\alpha}^k(a) = 1 + \left(\sum_{b \in \mathtt{Att}(a)} \frac{1}{({s_\alpha}^{k-1}(b))^\alpha}\right)^{1/\alpha}$. We stop when the norm of the differences of the two subsequent burden numbers is sufficiently small, i.e. when $||({s_\alpha}^k(a_1), \dots, {s_\alpha}^k(a_n)) - ({s_\alpha}^{k-1}(a_1), \dots, {s_\alpha}^{k-1}(a_n))|| < \epsilon$, where $\epsilon > 0$ is a small, fixed number (called precision). In our implementation we simply use the l^2-norm, defined as follows: for a vector $\mathbf{v} = (v_1, \dots, v_n)$, $||\mathbf{v}|| = \sqrt{v_1^2 + \dots + v_n^2}$.

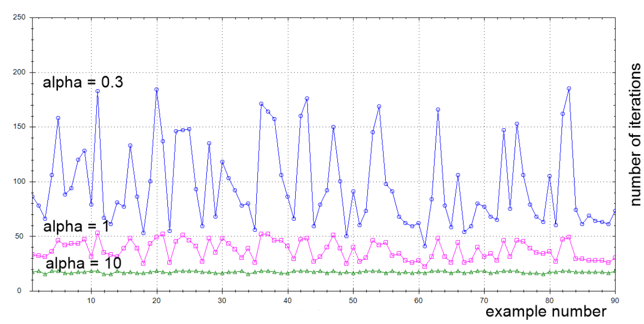

Figure 3: Convergence speed in function of parameter α. x-axis shows example number (benchmark contains 90 examples, examples are treated in the alphabetic order). y-axis shows the number of iterations needed to obtain precision $\epsilon = 0.00001$.

We implemented the previous algorithm in Java and ran it on a PC (Intel Core i7-3740QM CPU @ 2.70 GHz, 16 GB RAM, 64-bit Windows 8 Pro). The goal was to check the efficiency of our approach, i.e. the number of iterations needed to calculate the ranking of arguments of an argumentation graph. We used the publicly available benchmark provided at the website of the ICCMA (International Competition on Computational Models of Argumentation). It consists of 90 argumentation graphs, having different numbers of arguments and different graph structures and can be downloaded at `http://argumentationcompetition.org/2015/iccma15_testcases2.zip`. We ran the examples for 100 different values of parameter α. The results are very promising. Indeed, the algorithm converges very quickly. Figure 3 shows the results for three values of α, namely $0.3, 1$ and 10. The number of iterations is the greatest for $\alpha = 0.3$ and the least for $\alpha = 10$. This is the general behaviour we noticed (e.g. for other values of α) but for readability we included the graphs for three values of α only. In average, the algorithm performed 97 iterations for $\alpha = 0.3$, 37 iterations for $\alpha = 1$ and 17 iterations for $\alpha = 10$. All the examples from the benchmark for three different values of parameter α, namely $0.3, 1$ and 10, were calculated in less than two minutes with precision $\epsilon = 0.00001$.

Related Work

Evaluation of arguments is a crucial task in argumentation-based reasoning. The main goal is to define the acceptability status of each argument. For that purpose, several semantics are proposed in the literature. They are partitioned in two families: extension semantics and non-extension semantics.

Extension semantics, initiated by Dung in (1995), start by identifying sets of acceptable arguments called extensions, then assign a status to each argument. An argument is sceptically accepted if it belongs to all extensions, credulously accepted if it belongs only to some extensions, and rejected if it does not belong to any extension. Arguments can thus be rank-ordered as follows: sceptically accepted arguments are more acceptable than credulously accepted arguments, which are themselves more acceptable than rejected arguments. Such semantics violate compensation. The reason is that an attack emanating from a non-attacked argument destroys its target, i.e., the targeted argument gets necessarily the status "rejected". Consequently, in the compensation postulate, the argument which gets the strong attack is rejected and the attackers of the second argument are all rejected. Consequently, the second argument may be reinstated leading thus to the violation of compensation. Consider graph $\mathcal{F}_4$ from the introduction. It has a single extension $\{p, x, y, z, b\}$ which is grounded, preferred, stable and semi-stable. The argument b is sceptically accepted while a is rejected. Hence, b is ranked higher than a.
Recently, extension semantics are extended by Grossi and Modgil (2015) by taking into account the number of attackers and the number of defenders when computing extensions. The new semantics are based on two parameters and return a ranking on the set of arguments. That work is very different from ours. For example, in their framework, it might be the case that some arguments cannot be compared (see Definition 10 of their paper) whereas $\alpha-\texttt{BBS}$ always provides a unique and complete ranking.

The second family of non-extension semantics, called also *ranking semantics*, covers those semantics that do not compute extensions in order to produce a ranking on the set of arguments. The first ranking semantics in the literature was proposed by Besnard and Hunter (2001) for *acyclic* graphs. It assigns numerical scores to arguments using the so-called *h-categoriser* function defined as follows: for every argumentation graph $(\mathcal{A}, \mathcal{R})$, $c : \mathcal{A} \to (0, 1]$ such that for every $a \in \mathcal{A}$,

$$c(a) = \frac{1}{1 + \sum_{b \in \texttt{Att}(a)} c(b)}$$

with $c(a) = 1$ if $\texttt{Att}(a) = \emptyset$. According to *h-categoriser*, the greater the score, the better the argument[2]. The same function was used by Cayrol and Lagasquie-Schiex (2005) to assign scores to arguments of any graph. However, they did not prove that the function is still well-defined when graphs contain cycles. This was proved recently by Pu et al. (2014). We show next that this semantics satisfies compensation, but only for pairs $(k + 1, k)$ with $k \in \{0, 1, 2, \ldots\}$.

Proposition 1 *For every $k \in \{0, 1, 2, \ldots\}$, h-categoriser semantics satisfies compensation at degree $(k+1, k)$. It does not satisfy compensation for any other pair of values.*

Thus, $\alpha-\texttt{BBS}$ has much better compensation capacity than *h-categoriser*, which allows to compensate if and only if $n = k + 1$. To illustrate, recall that $\alpha-\texttt{BBS}$ can compensate in case of graph $\mathcal{F}_4$ from the introduction, whereas *h-categoriser* cannot (it gives precedence to a over b).

Another ranking semantics is that proposed by Amgoud and Ben-Naim (2013). This semantics, called Bbs, sets the burden number of an argument a at step 1 to be 1; at step k it is defined as

$$Bur_k(a) = 1 + \sum_{b \in \texttt{Att}(a)} \frac{1}{Bur_{k-1}(b)}$$

with $Bur_k(a) = 1$ if $\texttt{Att}(a) = \emptyset$. Whether $\lim_{k \to +\infty} Bur_k(a)$ exists was an open problem until now. Our next result answers this question.

Proposition 2 *For every argumentation graph $(\mathcal{A}, \mathcal{R})$, for every $a \in \mathcal{A}$,*

$$\lim_{k \to +\infty} Bur_k(a) = s_1(a).$$

It is worth pointing out that Bbs is different from $\alpha-\texttt{BBS}$ semantics including the one with $\alpha = 1$. The reason is that Bbs compares arguments using lexicographic order of sequences $(Bur_1(a_i), Bur_2(a_i), \ldots)$ while $\alpha-\texttt{BBS}$ compare the limits of those sequences. Consequently, Bbs does not allow for compensation. It declares a better than b as soon as a has strictly less attackers than b. For instance, in graph $\mathcal{F}_4$ of Figure 1, Bbs declares a as more acceptable than b.

Thimm (2012) proposed several ways for assigning numerical scores to arguments. However, none is based on the principle "the greater the number of attacks on an argument, the weaker is its rank". Consider for instance his Centroid function. In graph $\mathcal{F}_1$, it assigns score 1 for arguments p, q, r and s and score 0 for a and b. Note that in $\mathcal{F}_1$ the attackers of

[2]note the contrast with $\alpha-\texttt{BBS}$, where arguments' scores are their burdens

a and b are all strong (i.e. non-attacked), thus no compensation is possible. $\alpha-$BBS semantics give precedence to a over b since it obeys to the above principle.

This principle is also not followed by a semantics proposed by da Costa Pereira et al. (2011). The semantics assigns numerical scores to arguments by taking into account degrees of trust in sources. In graph $\mathcal{F}_1$, if all sources are trusted at degree 1, then both a and b will get score 0.

Matt and Toni (2008) proposed another function which assigns scores for arguments. To illustrate the difference between that function and $\alpha-$BBS, consider the fourth graph of Table 1 in their paper (Matt and Toni 2008, page 291). The arguments b and c have the same score (0.25) despite the fact that c is attacked by one argument having score 1 and b is attacked by two arguments having respectively scores 1 and 0.25. According to Theorem 8, $\alpha-$BBS give precedence to c over b.

Leite and Martins (2011) defined a family of semantics that take as input an argumentation graph (i.e., a set of arguments and attacks between them) and (positive and/or negative) votes on arguments. Like $\alpha-$BBS, those semantics are sensitive to the number and strength of attackers, assign numerical scores to arguments, and rank the arguments. However, unlike $\alpha-$BBS, there is no guarantee that their semantics assign a single score to each argument. The property holds only under some conditions, see Theorem 13 in their paper (Leite and Martins 2011). Note the existence of a parameter (called ϵ) in their framework. By changing its value, we can, in some situations, make an argument become stronger or weaker than another argument. In particular, by picking the "right" values of votes and the "right" value of ϵ, we can make arguments a and b in our figure $\mathcal{F}_4$ have the same scores. However, there is no universal rule or theoretical result that shows how to do this in the general case. We think that our parameter α represents a simple and transparent ("user friendly") way to compensate between strength and number of attacks. Whereas in the work by Leite and Martins (2011) it is not completely clear how to change parameter ϵ in order to obtain compensation at given degree.

It is worth mentioning that the notion of compensation is different from that of *accrual* (Lucero, Chesñevar, and Simari 2009). Compensation shows how the attacks on a given argument can be aggregated in order to evaluate its acceptability degree. Accrual focuses on claims, and shows how different reasons (i.e., arguments) supporting the same claim may be aggregated into a single argument.

There are other approaches that might seem related at the first sight since they attach scores to arguments. However, they are very different from our work since those scores do not reflect acceptability of arguments. For instance, the score of an argument can be its degree of *controversiality* (Thimm and Kern-Isberner 2014) or its degree of *relevance* at a given step of a dialogue (Bonzon, Maudet, and Moretti 2014).

Conclusion

Starting from the observation that in some situations (like dialogues) an attack does not completely destroy its target (but only weakens it) we argued that the effect of several attacks on an argument is not necessarily the same as the effect of one attack. In this context, we considered the possibility where several weak attacks can compensate a strong one.

We started by formalizing the notion of compensation and highlighting its two parameters. Then we proposed a family of semantics, $\alpha-$BBS, that satisfy compensation at different degrees as well as other rationality postulates from the literature. These semantics take as input an argumentation graph and return a ranking on the set of arguments. We implemented an algorithm that computes this ranking and ran several experiments. The results show that the ranking is calculated very quickly. Furthermore, at any time during the execution of the algorithm, one can have a very good approximation of arguments' burden numbers, contrary to extension semantics where no information about the status of a given argument is available until the extensions are computed.

Observe that $\alpha-$BBS does not compute extensions; instead, it attaches a score to every argument. However, it is possible to use those scores in order to define extensions. For instance, we can take conflict-free sets that maximise the sum of arguments scores. Another possibility would be to define a stratification of the set of arguments, based on their scores (each stratum contains the arguments having the same score) and proceed like in the case of preferred sub-theories (Brewka 1989). The idea is to start with the strongest non-conflicting arguments, and to add more arguments while staying conflict-free.

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Appendix

Proof ***of Theorem 1***. *Notation: for two vectors* $\mathbf{u} = (u_1, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, \ldots, v_n)$*, we write* $\mathbf{u} \leq \mathbf{v}$ *if and only if for every* $k \in \{1, \ldots, n\}$ *it holds that* $u_k \leq v_k$. *Denote* $\mathbf{u}' = (u'_1, \ldots, u'_n) = F_\alpha((1, \ldots, 1))$ *and observe that for every* $i \in \{1, \ldots, n\}$, $u'_i = 1 + |\mathtt{Att}(a_i)|^{1/\alpha}$. *Note that* $F_\alpha : [1, +\infty)^n \to [1, +\infty)^n$ *is a non-increasing function by definition, i.e. for every* $\mathbf{u} = (u_1, \ldots, u_n)$ *and* $\mathbf{v} = (v_1, \ldots, v_n)$, $\mathbf{u} \leq \mathbf{v}$ *if and only if* $F_\alpha(\mathbf{u}) \geq F_\alpha(\mathbf{v})$. *Hence for every* $\mathbf{u} = (u_1, \ldots, u_n)$ *such that* $(1, \ldots, 1) \leq \mathbf{u} \leq \mathbf{u}'$, *we have* $(1, \ldots, 1) \leq F_\alpha(\mathbf{u}) \leq \mathbf{u}'$. *Note that* F_α *is continuous and let us consider the restriction of* F_α *on the compact and convex set* $\mathcal{D} = [1, u'_1] \times \ldots \times [1, u'_n]$. *Thus, Brouwer fixed-point theorem implies that* F_α *has at least one fixed point on* $\mathcal{D}$. *Observe that* $\mathbf{u}^*$ *is a fixed point of* F_α *if and only if* $\mathbf{u}^*$ *is the solution of* n *equations from Definition 7 corresponding to arguments* $a_1, \ldots, a_n$ *and the attacks from* $\mathcal{F}$. *Thus, there exists* at least one *solution of* n *equations from Definition 7. Our next goal is to show that the solution is unique and that it is exactly the limit of the sequence* $\mathbf{v}^{(k)}$.

We first prove that that sequence $\{\mathbf{v}^{(k)}\}_{k=0}^{+\infty}$ *converges towards the unique fixed point* $\mathbf{v}^* = (v^*_1, \ldots, v^*_n)$ *un-*

der the hypothesis that $\mathbf{v}^{(0)} = (1, \ldots, 1)$. We later generalize this result, i.e. show that it is valid for every $\mathbf{v}^0 = (v_1^{(0)}, \ldots, v_n^{(0)}) \in [1, +\infty)^n$. Let $\mathbf{u}^{(0)} = (1, \ldots, 1)$. Note that $\mathbf{u}^{(1)} = (1 + |\mathtt{Att}(a_1)|^{\frac{1}{\alpha}}, \ldots, 1 + |\mathtt{Att}(a_n)|^{\frac{1}{\alpha}})$. Let $\mathbf{u}^{(k)} = F_\alpha(\mathbf{u}^{(k-1)})$ for each $k \geq 2$. Let $\mathbf{u}^{(i)} = (u_1^{(i)}, \ldots, u_n^{(i)})$ and $\mathbf{u}^{(j)} = (u_1^{(j)}, \ldots, u_n^{(j)})$ be two vectors and π be a scalar. We write $\mathbf{u}^{(i)} \leq \mathbf{u}^{(j)}$ if and only if for every $k \in \{1, \ldots, n\}$ it holds that $u_k^{(i)} \leq u_k^{(j)}$. We write $\mathbf{u}^{(i)} + \pi$ as a shorthand for $(u_1^{(i)} + \pi, \ldots, u_n^{(i)} + \pi)$.

Denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of natural numbers. Our first goal is to prove that for every $k \in \mathbb{N}$ we have

$$\mathbf{u}^{(0)} \leq \mathbf{u}^{(2)} \leq \mathbf{u}^{(4)} \leq \ldots \leq \mathbf{u}^{(2k)} \leq \mathbf{u}^{(2k+1)} \leq \ldots \\ \ldots \leq \mathbf{u}^{(5)} \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}. \tag{1}$$

Let us prove (1) by induction. Inductive hypothesis is:

$$H_i : \mathbf{u}^{(0)} \leq \mathbf{u}^{(2)} \leq \mathbf{u}^{(4)} \leq \ldots \leq \mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+1)} \leq \ldots \\ \ldots \leq \mathbf{u}^{(5)} \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}$$

Base: Note that $\mathbf{u}^{(0)} \leq \mathbf{u}^{(1)}$ follows directly from the definition of the sequence. Let us show that

$$H_1 : \mathbf{u}^{(0)} \leq \mathbf{u}^{(2)} \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}$$

holds. Observe that F_α is non-increasing, hence for every $\mathbf{u}^{(i)}, \mathbf{u}^{(j)}$ we have that $\mathbf{u}^{(i)} \leq \mathbf{u}^{(j)}$ if and only if $F_\alpha(\mathbf{u}^{(i)}) \geq F_\alpha(\mathbf{u}^{(j)})$. Note that for every $\mathbf{u}^{(i)}$ we have $\mathbf{u}^{(0)} \leq \mathbf{u}^{(i)}$ directly from the definition, thus $\mathbf{u}^{(0)} \leq \mathbf{u}^{(2)}$. Also, since $\mathbf{u}^{(0)} \leq \mathbf{u}^{(1)}$ and F_α is non-increasing

$$\mathbf{u}^{(2)} = F_\alpha(\mathbf{u}^{(1)}) \leq F_\alpha(\mathbf{u}^{(0)}) = \mathbf{u}^{(1)}.$$

Hence $\mathbf{u}^{(0)} \leq \mathbf{u}^{(2)} \leq \mathbf{u}^{(1)}$. Again by applying non-increasingness pf F_α, this time on $\mathbf{u}^{(2)} \leq \mathbf{u}^{(1)}$ and $\mathbf{u}^{(0)} \leq \mathbf{u}^{(2)}$, we obtain $\mathbf{u}^{(2)} \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}$. By combining this with $\mathbf{u}^{(0)} \leq \mathbf{u}^{(2)}$, we obtain H_1.

Step: Let $i \in \{1, \ldots, n\}$; suppose that H_i holds and let us show that H_{i+1} also holds. We again base our reasoning on non-increasingness of F_α. From $\mathbf{u}^{(2i+1)} \leq \mathbf{u}^{(2i-1)}$ we have $F_\alpha(\mathbf{u}^{(2i-1)}) \leq F_\alpha(\mathbf{u}^{(2i+1)})$ i.e. $\mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+2)}$. Similarly, $\mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+1)}$ implies $\mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+1)}$. Hence

$$\mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+1)}. \tag{2}$$

Now $\mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+1)}$ implies $F_\alpha(\mathbf{u}^{(2i+1)}) \leq F_\alpha(\mathbf{u}^{(2i+2)}) \leq F_\alpha(\mathbf{u}^{(2i)})$ i.e.

$$\mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+3)} \leq \mathbf{u}^{(2i+1)}. \tag{3}$$

From (2) and (3) we obtain

$$\mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+3)} \leq \mathbf{u}^{(2i+1)}.$$

By combining this with H_i we obtain H_{i+1}. This ends the proof by induction. We conclude that for every $k \in \mathbb{N}$, (1) holds.

From (3) we see that for every $k \in \mathbb{N}$, there exists $0 < \varphi \leq 1$ such that

$$\varphi \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}.$$

Let us define

$$\pi_k = \sup\{\pi \mid \pi \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}\}.$$

Observe that for every $k \geq 1$, $\pi_k \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}$. Also, we have $0 < \pi_1 \leq \pi_2 \leq \ldots \leq \pi_k \leq \ldots \leq 1$. Our goal is to prove that $\lim_{k \to +\infty} \pi_k = 1$.

Recall our notation

$$F_\alpha(\mathbf{v}) = [f_1^\alpha(\mathbf{v}) \ldots f_n^\alpha(\mathbf{v})].$$

Thus for a vector $\mathbf{x} = (x_1, \ldots, x_n)$ we have that for every $i \in \{1, \ldots, n\}$ it holds that

$$f_i^\alpha(\mathbf{x}) = 1 + \left(\sum_{j \text{ s.t. } a_j \in \mathtt{Att}(a_i)} \frac{1}{(x_j)^\alpha} \right)^{1/\alpha}$$

and for a scalar $0 < \pi \leq 1$ we obtain

$$\begin{aligned} f_i^\alpha(\pi \mathbf{x}) &= 1 + \left(\sum_{j \text{ s.t. } a_j \in \mathtt{Att}(a_i)} \frac{1}{(\pi x_j)^\alpha} \right)^{1/\alpha} \\ &= 1 + \left(\frac{1}{\pi^\alpha} \sum_{j \text{ s.t. } a_j \in \mathtt{Att}(a_i)} \frac{1}{(x_j)^\alpha} \right)^{1/\alpha} \\ &= 1 + \frac{1}{\pi} \left(\sum_{j \text{ s.t. } a_j \in \mathtt{Att}(a_i)} \frac{1}{(x_j)^\alpha} \right)^{1/\alpha} \\ &= 1 + \frac{1}{\pi} \left(\left(\sum_{j \text{ s.t. } a_j \in \mathtt{Att}(a_i)} \frac{1}{(x_j)^\alpha} \right)^{1/\alpha} + 1 - 1 \right) \\ &= 1 + \frac{1}{\pi} \left(f_i^\alpha(\mathbf{x}) - 1 \right) \end{aligned}$$

We conclude that

$$f_i^\alpha(\pi \mathbf{x}) = \frac{\pi + f_i^\alpha(\mathbf{x}) - 1}{\pi} \tag{4}$$

Recall that for every $k \in \mathbb{N}$, $\pi_k \mathbf{u}^{(2k-1)} \leq \mathbf{u}^{(2k)}$. By applying F_α on this inequality and from non-increasingness of F_α, we have $F_\alpha(\mathbf{u}^{(2k)}) \leq F_\alpha(\pi_k \mathbf{u}^{(2k-1)})$. That is, for every $i \in \{1, \ldots, n\}$

$$f_i^\alpha(\mathbf{u}^{(2k)}) \leq f_i^\alpha(\pi_k \mathbf{u}^{(2k-1)}). \tag{5}$$

Thus, by using (4) on (5) we obtain:

$$f_i^\alpha(\mathbf{u}^{(2k)}) \leq \frac{\pi_k + f_i^\alpha(\mathbf{u}^{(2k-1)}) - 1}{\pi_k}$$

that is

$$u_i^{(2k+1)} \leq \frac{\pi_k + u_i^{(2k)} - 1}{\pi_k} \tag{6}$$

Recall the definition of π_k:

$$\pi_k = \sup\{\pi \mid \pi\mathbf{u}^{(2k-1)} \le \mathbf{u}^{(2k)}\} \tag{7}$$

Our goal is to combine (6) *and* (7). *So let us transform* (6) *to*

$$\frac{\pi_k}{\pi_k + u_i^{(2k)} - 1}\, u_i^{(2k+1)} \le 1$$

and multiply both sides by $u_i^{(2k+2)}$ to obtain

$$\left(\frac{\pi_k u_i^{(2k+2)}}{\pi_k + u_i^{(2k)} - 1}\right) u_i^{(2k+1)} \le u_i^{(2k+2)} \tag{8}$$

Note that (8) *holds for every $i \in \{1, \ldots, n\}$. Hence, for every $i \in \{1, \ldots, n\}$ we have*

$$\min_{j\in\{1,\ldots,n\}} \left(\frac{\pi_k u_j^{(2k+2)}}{\pi_k + u_j^{(2k)} - 1}\right) u_i^{(2k+1)} \le u_i^{(2k+2)} \tag{9}$$

Let us re-write (7) *as follows to be able to combine it with* (9)

$$\pi_{k+1} = \sup\{\pi \mid \forall i \in \{1, \ldots, n\}, \pi u_i^{(2k+1)} \le u_i^{(2k+2)}\} \tag{10}$$

Now we are ready to combine (9) *and* (10) *and we obtain that for every $k \in \mathbb{N}$, there exists $i_k \in \{1, \ldots, n\}$ such that*

$$\pi_{k+1} \ge \frac{\pi_k u_{i_k}^{(2k)}}{\pi_k + u_{i_k}^{(2k)} - 1} \tag{11}$$

To understand why the previous holds, it is sufficient to observe that we can, for every $k \in \mathbb{N}$, take $i_k \in \{1, \ldots, n\}$ such that

$$i_k = \arg\min_j \frac{\pi_k u_j^{(2k+2)}}{\pi_k + u_j^{(2k)} - 1}$$

For a given k, there might exist even several different values $i_k^1, i_k^2, \ldots, i_k^{(n)} \in \{1, \ldots, n\}$ satisfying the previous condition. Observe the sequence $\{i_k\}_{k=1}^{+\infty}$ formed by those numbers (in case when several numbers $i_k^1, i_k^2, \ldots, i_k^{(n)}$ exist, we choose the smallest one, for the sake of determinism). Since i_k is an infinite sequence, there exists $l \in \{1, \ldots, n\}$ such that l appears infinitely many times in i_k. Choose this element. (Again, it might not be unique, but we can again choose the smallest one.)

Let us define a sub-sequence $\{\pi'_k\}_{k=1}^{+\infty}$ of sequence $\{\pi_k\}_{k=1}^{+\infty}$ obtained by taking exactly the elements of $\{\pi_k\}$ such that $i_k = l$. From (9) *we have that for every $j \in \mathbb{N}$*

$$\pi'_{j+1} \ge \frac{\pi'_j u_l^{(2j+2)}}{\pi'_j + u_l^{2j} - 1} \tag{12}$$

Note that $\{\pi_k\}_{k=1}^{+\infty}$ converges since it is non-decreasing and bounded. Denote $\lim_{k\to+\infty} \pi_k = \pi$ and observe that $0 < \pi \le 1$. The sequence $\{u_l^{(2k)}\}_{k=1}^{+\infty}$ converges for the same reasons. Let us write $\lim_{k\to+\infty} u_l^{(2k)} = u$. Sequence $\{\pi'_k\}$ converges since it is a sub-sequence of a converging sequence, furthermore, $\lim_{k\to+\infty} \pi'_k = \pi$. From (12) *we have*

$$\pi \ge \frac{\pi u}{u + \pi - 1}$$

Which is equivalent to

$$\pi \ge 1.$$

Since $\pi \le 1$ and $\pi \ge 1$ then $\pi = 1$. From $\pi_k\mathbf{u}^{(2k-1)} \le \mathbf{u}^{(2k)}$ we have $-\mathbf{u}^{(2k)} \le -\pi_k\mathbf{u}^{(2k-1)}$. Thus,

$$0 \le \mathbf{u}^{(2k+1)} - \mathbf{u}^{(2k)} \le \mathbf{u}^{(2k+1)} - \pi_k\mathbf{u}^{(2k-1)} \le \mathbf{u}^{(2k-1)} - \pi_k\mathbf{u}^{(2k-1)} \le \mathbf{u}^{(2k-1)}(1 - \pi_k) \le \mathbf{u}^{(1)}(1 - \pi_k).$$

By letting $k \to +\infty$ we have

$$0 \le \lim_{k\to+\infty} \mathbf{u}^{(2k+1)} - \lim_{k\to+\infty} \mathbf{u}^{(2k)} \le 0.$$

Thus there exists $\mathbf{u}^* = (u_1^*, \ldots, u_n^*)$ such that

$$\lim_{k\to+\infty} \mathbf{u}^{(2k+1)} = \lim_{k\to+\infty} \mathbf{u}^{(2k)} = \mathbf{u}^* \tag{13}$$

Observe that for every $k \ge 1$

$$\mathbf{u}^{(2k)} \le \mathbf{u}^* \le \mathbf{u}^{(2k+1)}. \tag{14}$$

From this fact, we have that for every $k \ge 1$

$$\mathbf{u}^{(2k)} \le F_\alpha(\mathbf{u}^*) \le \mathbf{u}^{(2k+1)}. \tag{15}$$

From (13), (14) *and* (15), *by letting $k \to +\infty$, we have*

$$F_\alpha(\mathbf{u}^*) = \mathbf{u}^*.$$

Thus we showed that F_α has a fixed point $\mathbf{u}^*$ that is exactly the limit of the sequence $\{\mathbf{u}^{(k)}\}_{k=0}^{+\infty}$ where $\mathbf{u}^{(0)} = (1, \ldots, 1)$ and for every $k \in \mathbb{N}$, $\mathbf{u}^{(k)} = F_\alpha(\mathbf{u}^{(k-1)})$. Let us now show that $\{\mathbf{v}^{(k)}\}_{k=0}^{+\infty}$ such that for every $k \in \mathbb{N}$, $\mathbf{v}^{(k)} = F_\alpha(\mathbf{v}^{(k-1)})$ converges towards $\mathbf{u}^*$ independently of the choice of element $\mathbf{v}^{(0)} \in [1, +\infty)^n$. Let $\mathbf{v}^{(0)} = (v_1^{(0)}, \ldots, v_n^{(0)}) \in [1, +\infty)^n$ and let for every $k \in \mathbb{N}$ $\mathbf{v}^{(k)} = F_\alpha(\mathbf{v}^{(k-1)})$. Informally speaking, we show that the sequence $\mathbf{v}^{(k)}$ is between $\mathbf{u}^{(2k)}$ and $\mathbf{u}^{(2k+1)}$. Formally, let us show by induction that for every $k \in \mathbb{N}$

$$\mathbf{u}^{(2k)} \le \mathbf{v}^{(2k)} \le \mathbf{u}^{(2k-1)} \text{ and } \mathbf{u}^{(2k)} \le \mathbf{v}^{(2k+1)} \le \mathbf{u}^{(2k+1)} \tag{16}$$

Base. Note that $\mathbf{u}^{(0)} \le \mathbf{v}^{(0)}$. From this fact and since F_α is non-increasing, we have $\mathbf{v}^{(1)} \le \mathbf{u}^{(1)}$. Observe also that $\mathbf{u}^{(0)} \le \mathbf{v}^{(1)}$. Thus, $\mathbf{u}^{(0)}\mathbf{v}^{(1)} \le \mathbf{u}^{(1)}$. From this fact and again since F_α is non-increasing, we obtain $\mathbf{u}^{(2)} \le \mathbf{v}^{(2)} \le \mathbf{u}^{(3)}$.

Step. Suppose that $\mathbf{u}^{(2i)} \le \mathbf{v}^{(2i)} \le \mathbf{u}^{(2i-1)}$ and $\mathbf{u}^{(2i)} \le \mathbf{v}^{(2i+1)} \le \mathbf{u}^{(2i+1)}$ hold. From the second part of the inductive hypothesis we have $\mathbf{u}^{(2i+2)} \le \mathbf{v}^{(2i+2)} \le \mathbf{u}^{(2i+1)}$. By using the same reasoning we obtain $\mathbf{u}^{(2i+2)} \le \mathbf{v}^{(2i+3)} \le \mathbf{u}^{(2i+3)}$. This ends the inductive proof.

From (16) *we have*

$$\mathbf{u}^* = \lim_{k\to+\infty} \mathbf{u}^{(2k)} \le \lim_{k\to+\infty} \mathbf{v}^{(2k)} \le \lim_{k\to+\infty} \mathbf{u}^{(2k-1)} = \mathbf{u}^*$$

and

$$\mathbf{u}^* = \lim_{k\to+\infty} \mathbf{u}^{(2k)} \leq \lim_{k\to+\infty} \mathbf{v}^{(2k+1)} \leq \lim_{k\to+\infty} \mathbf{u}^{(2k+1)} = \mathbf{u}^*$$

In other words, $\lim_{k\to+\infty} \mathbf{v}^{(2k)} = \lim_{k\to+\infty} \mathbf{v}^{(2k+1)} = \mathbf{u}^*$.

We have seen that the sequence $\{\mathbf{u}^{(i)}\}_{i=1}^{+\infty}$ *converges towards* $\mathbf{u}^*$ *independently of the choice of the first element* $\mathbf{u}^{(0)}$. *Let us now show that* $\mathbf{u}^*$ *is the unique fixed point of* F_α. *Suppose* $\mathbf{w}^* = (w_1^{(0)}, \ldots, w_n^{(n)}) \in [1, +\infty)^n$ *is a fixed point of* F_α, *i.e.* $F_\alpha(\mathbf{w}^*) = F_\alpha$. *Define the sequence* $\{\mathbf{w}^{(k)}\}_{k=1}^{+\infty}$ *as* $\mathbf{w}^{(0)} = \mathbf{w}^*$ *and for every* $k \in \mathbb{N}$, $\mathbf{w}^{(k)} = F_\alpha(\mathbf{w}^{(k-1)})$. *At one hand,* $\mathbf{w}^{(1)} = \mathbf{w}^{(2)} = \ldots = \mathbf{w}^*$ *thus* $\lim_{k\to+\infty} \mathbf{w}^{(k)} = \mathbf{w}^*$. *At the other hand, we know that independently of the starting value, every sequence generated by* F_α *converges to* $\mathbf{u}^*$, *i.e.* $\lim_{k\to+\infty} \mathbf{w}^{(k)} = \mathbf{u}^*$. *Hence,* $\mathbf{w}^* = \mathbf{u}^*$. *So* F_α *has a unique fixed point.* ■

Note that the previous proof uses the ideas from the proof of Lemma 2.1 by Li et al. (2005), which were also recently used by Pu et al. (2014).

Lemma 1 *For all argumentation graphs* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, *for all* $a \in \mathcal{C}^i(\mathcal{F})$, $s_\alpha(a) = \begin{cases} 1 + i^{1/\alpha} & \textit{for } i \geq 1 \\ 1 & \textit{for } i = 0 \end{cases}$

We drop the proof of the lemma due to space restriction.

Proof ***of Theorem 2***. *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation graph and* $a, b \in \mathcal{A}$. *Assume that* $|\mathtt{Att}(a)| = n$ *and* $\mathtt{Att}(a) \subseteq \mathcal{C}^i(\mathcal{F})$. *Let* $\mathtt{Att}(a) = \{a_1, \ldots, a_n\}$. *From Lemma 1,* $s_\alpha(a_1) = \ldots = s_\alpha(a_n) = 1 + i^{1/\alpha}$. *Thus,*

$$s_\alpha(a) = 1 + \left(\frac{n}{(1 + i^{1/\alpha})^\alpha}\right)^{1/\alpha}.$$

Then,

$$s_\alpha(a) = 1 + \frac{n^{1/\alpha}}{1 + i^{1/\alpha}}.$$

Assume now that $|\mathtt{Att}(b)| = 1$ *and* $\mathtt{Att}(b) \subseteq \mathcal{C}^0(\mathcal{F})$. *Thus,* $s_\alpha(b) = 2$. *If* $s_\alpha(a) = s_\alpha(b)$, *then* $n^{1/\alpha} = 1 + i^{1/\alpha}$. ■

Proofs of theorems 3, 4 and 5 are relatively easy and are dropped due to space constraints.

Throughout the remainder of the document, we use the following notation. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation graph with $\mathcal{A} = \{a_1, \ldots, a_n\}$, let $a_j \in \mathcal{A}$ and $\alpha > 0$. Let $\mathbf{u}^{(0)} = (u_1^{(0)}, \ldots, u_n^{(0)}) = (1, \ldots, 1)$ and $\mathbf{u}^{(k)} = F_\alpha(\mathbf{u}^{(k-1)})$ for every $k \in \mathbb{N}$, where F_α is defined as in the proof of Theorem 1. We denote by $s_{\alpha,i}^{\mathcal{F}}(a_j)$ the coordinate j of vector $\mathbf{u}^{(i)}$, that is, roughly speaking, the "score" of argument a_j after i iterations of applying F_α.

Proof ***of Theorem 6***. *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation graph, and let* $a, b \in \mathcal{A}$ *be such that* $|\mathtt{Att}(a)| = |\mathtt{Att}(b)|$, $\mathtt{Def}(a) \neq \emptyset$ *and* $\mathtt{Def}(b) = \emptyset$. *Since* $|\mathtt{Att}(a)| = |\mathtt{Att}(b)|$ *and* $\mathtt{Def}(b) = \emptyset$, *then there exists a bijective function* f *from* $\mathtt{Att}(a)$ *to* $\mathtt{Att}(b)$ *such that for all* $a_i \in \mathtt{Att}(a)$, $s_\alpha(a_i) \geq s_\alpha(f(a_i))$. *Consequently,* $(s_\alpha(a_i))^\alpha \geq (s_\alpha(f(a_i)))^\alpha$. *Furthermore, for every* $a_i \in \mathtt{Att}(a)$,

$$\frac{1}{(s_\alpha(a_i))^\alpha} \leq \frac{1}{(s_\alpha(f(a_i)))^\alpha}$$

Since $\mathtt{Def}(a) \neq \emptyset$, *there exists* $a^* \in \mathtt{Att}(a)$ *such that* $s_\alpha(a^*) > s_\alpha(f(a^*))$. *Thus,*

$$\left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(a_i))^\alpha}\right)^{1/\alpha} \leq \left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(f(a_i))^\alpha}\right)^{1/\alpha}$$

It follows that $s_\alpha(a) < s_\alpha(b)$ *and so* $a \succ b$. ■

Proof ***of Theorem 7***. *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation graph and* $a, b \in \mathcal{A}$. *Let* f *be an injective function from* $\mathtt{Att}(a)$ *to* $\mathtt{Att}(b)$ *such that for all* $a_i \in \mathtt{Att}(a)$, $s_\alpha(a_i) \geq s_\alpha(f(a_i))$. *Consequently, for every* $a_i \in \mathtt{Att}(a)$,

$$\frac{1}{(s_\alpha(a_i))^\alpha} \leq \frac{1}{(s_\alpha(f(a_i)))^\alpha}$$

So we have

$$\left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(a_i))^\alpha}\right)^{1/\alpha} \leq \left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(f(a_i))^\alpha}\right)^{1/\alpha}$$

It follows that $s_\alpha(a) \leq s_\alpha(b)$ *and so* $a \succeq b$. ■

Proof ***of Theorem 8***. *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation graph and* $a, b \in \mathcal{A}$. *Let* f *be an injective function from* $\mathtt{Att}(a)$ *to* $\mathtt{Att}(b)$ *such that for all* $a_i \in \mathtt{Att}(a)$, $s_\alpha(a_i) \geq s_\alpha(f(a_i))$. *Consequently, for every* $a_i \in \mathtt{Att}(a)$, *we have:*

$$\frac{1}{(s_\alpha(a_i))^\alpha} \leq \frac{1}{(s_\alpha(f(a_i)))^\alpha} \tag{17}$$

There are two cases:

- *There exists* $a' \in \mathtt{Att}(a)$ *such that* $s_\alpha(a') > s_\alpha(f(a'))$. *Then,*

  $$\frac{1}{(s_\alpha(a'))^\alpha} < \frac{1}{(s_\alpha(f(a')))^\alpha}$$

  From this fact and the fact that (17) *holds for every* $a_i \in \mathtt{Att}(a)$, *we obtain*

  $$s_\alpha(a) < 1 + \left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(f(a_i))^\alpha}\right)^{1/\alpha}$$

  which means that $s_\alpha(a) < s_\alpha(b)$, *i.e.* $a \succ b$.

- *Suppose* $|\mathtt{Att}(a)| < |\mathtt{Att}(b)|$. *Thus,* $\mathtt{Att}(b) = \{f(a_i) \mid a_i \in \mathtt{Att}(a)\} \cup X$ *with* $X \neq \emptyset$. *Since* (17) *holds for every* $a_i \in \mathtt{Att}(a)$, *we have*

$$s_\alpha(a) = 1 + \left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(a_i))^\alpha} \right)^{1/\alpha}$$
$$\leq 1 + \left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{(s_\alpha(f(a_i))^\alpha} \right)^{1/\alpha}$$

Note that

$$s_\alpha(b) = 1 + \left(\sum_{a_i \in \mathtt{Att}(a)} \frac{1}{s_\alpha(f(a_i))^\alpha} + \sum_{x_i \in X} \frac{1}{s_\alpha(x_i)^\alpha} \right)^{1/\alpha}$$

hence $s_\alpha(b) > s_\alpha(a)$, *i.e.* $a \succ b$.

■

Proof ***of Theorem 9***. *Let us consider the algorithm from the proof of Theorem 1 for calculating the arguments' scores. We show, using the induction, that for every* $k \in \mathbb{N}$, *there exists* $\beta_k \in [1, +\infty)$ *such that* $\mathbf{u}^{(k)} = (\beta_k, \ldots, \beta_k)$.
Base: For $k = 0$, *obviously* $\beta_k = 1$.
Step: Now let $k > 0$ *and suppose that* $\mathbf{u}^{(k-1)} = (\beta_{k-1}, \ldots, \beta_{k-1})$ *for some* $\beta_{k-1} \in [1, +\infty)$. *Then for each argument* a_i *we have* $f_i^\alpha(u^{(k-1)}) = 1 + \frac{1}{\beta_{k-1}}$. *In other words,* $\beta_k = 1 + \frac{1}{\beta_{k-1}}$.

The convergence result (Theorem 1) now guarantees that the scores of all arguments coincide. ■

Proof ***of Proposition 1***. *Due to space restriction, we give only the proof sketch. First, show that when* $\alpha = 1$, *the score returned by* h*-categoriser for a given argument is exactly the inverse of the burden number returned by* s_α *for that argument. Then, use Theorem 2 with* $\alpha = 1$. ■

Proof ***of Proposition 2***. *Let* $\alpha = 1$. *Like in Theorem 1, denote* $\mathcal{A} = \{a_1, \ldots, a_n\}$. *Observe the sequence* $\{\mathbf{u}^{(k)}\}_{k=1}^{+\infty}$ *from the proof of that theorem. We have that for every* $k \in \mathbb{N}$, $\mathbf{u}^{(k)} = (u_1^{(k)}, \ldots, u_n^{(k)})$ *with* $u_i^{(k)} = Bur_k(a_i)$. *Thus* $\lim_{k \to +\infty} Bur_k(a) = s_1(a)$. ■

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