How to Decrease and Resolve Inconsistency of a Knowledge Base?

Dragan Doder[1] and Srdjan Vesic[2]
[1]*Computer Science and Communication, University of Luxembourg, Luxembourg*
[2]*CRIL, CNRS – Univ. Artois, Lens, France*
{*dragan.doder@uni.lu, vesic@cril.fr*}

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Abstract: This paper studies different techniques for measuring and decreasing inconsistency of a knowledge base. We define an operation that allows to decrease inconsistency of a knowledge base while losing a minimal amount of information. We also propose two different ways to compare knowledge bases. The first is a partial order that we define on the set of knowledge bases. We study this relation and identify its link with a particular class of inconsistency measures. We also study the links between the partial order we introduce and information measures. The second way we propose to compare knowledge bases is to define a class of metrics that give us a distance between knowledge bases. They are based on symmetric set difference of models of pairs of formulae from the two sets in question.

1 INTRODUCTION

Reasoning under inconsistency is one of the keywords of Artificial Intelligence. Some authors propose to get rid of inconsistency, some to reason with it. Convincing arguments can be constructed even to support the fact that inconsistency is not always a bad thing (Gabbay and Hunter, 1991; Gabbay and Hunter, 1993). The question how to *measure* inconsistency of a knowledge base has attracted attention of researchers in recent years (Knight, 2002; Hunter and Konieczny, 2005; Hunter and Konieczny, 2008; Hunter and Konieczny, 2010; Mu et al., 2011; Grant and Hunter, 2011b; Grant and Hunter, 2013). In some papers, the measure of inconsistency depends on the proportion of the language that is affected by the inconsistency in a theory (Konieczny et al., 2003; Grant and Hunter, 2006; Hunter and Konieczny, 2010). Another approach consists in considering the number of formulae needed to produce a contradiction (Knight, 2002; Doder et al., 2010). According to Sorensen, an inconsistent set S is better than an inconsistent set S', if the shortest derivation of a contradiction requires more members of S than S' (Sorensen, 1988). A related question is how to decrease its inconsistency while keeping as much information as possible (Hunter and Konieczny, 2005; Grant and Hunter, 2011a; Grant and Hunter, 2011b). For example, changing $\Sigma = \{x \wedge z, \neg x\}$ to $\Sigma' = \{x \wedge z, \neg x \vee y\}$ allows us to obtain a consistent knowledge base. Grant and Hunter propose three operations in order to decrease inconsistency of a knowledge base: deletion, weakening and splitting (Grant and Hunter, 2011b; Grant and Hunter, 2011a). Deleting a formula is the most drastic operation, but it allows to easily get rid of the conflicts. Weakening consists in changing a formula by another formula logically implied by it. Splitting a formula into its conjuncts may isolate the really problematic conjuncts. It also allows at a later step to delete just the portion of the conjunction involved in the inconsistency.

In this paper, we aim at studying different ways to compare knowledge bases in terms of inconsistency and at proposing techniques for decreasing inconsistency. We use just one operation to decrease the inconsistency of the knowledge base: minimal weakening. More particularly, we are guided by the following research objectives:

1. Define a partial order $\succeq$ on the set of knowledge bases based on weakening. Identify the necessary and sufficient conditions that an inconsistency measure must satisfy so that $\Sigma \succeq \Sigma'$ implies that Σ' is less inconsistent than Σ. Study the link between our partial order and measures of information: does $\Sigma \succeq \Sigma'$ imply that Σ has more information than Σ'?
2. Since $\succeq$ is a partial order, explore the possibilities of defining a distance between knowledge bases Σ and Σ'. Is it possible to do this by considering distances based on symmetric set difference of mod-

els of pairs of formulae (φ, φ') where $\varphi \in \Sigma$ and $\varphi' \in \Sigma'$? Can we prove that such a function is a metric? Can we use this function to measure the distance of a knowledge base to the closest consistent knowledge base? What is the link between such a function and the partial order $\succeq$?

We study the above research objectives and provide answers to those questions. The rest of the paper is organised as follows: the next section introduces the setting, Section 3 studies and compares some inconsistency measures from the literature. We continue by defining a partial order $\succeq$ on the set of knowledge bases and studying its properties in Section 4. In Section 5, we apply our minimal weakening to stepwise inconsistency resolution and analyse the obtained result. Section 6 is devoted to defining a family of metrics to measure distance between two knowledge bases. The last section concludes.

2 FORMAL SETTING AND NOTATION

We suppose a finite set $\mathtt{Var}$ of propositional letters is used. We denote by $\mathcal{L}$ the set of well-formed classical propositional logic formulae made from $\mathtt{Var}$, $\vdash$ stands for classical entailment, and $\equiv$ for logical equivalence. For two sets S and S', we write $S \equiv S'$ if and only if for every $\varphi \in S'$, $S \vdash \varphi$ and for every $\varphi \in S$, $S' \vdash \varphi$. We say that a set S is complete if and only if S has exactly one model.

Our goal is to measure and decrease inconsistency of a finite knowledge base. In doing so, we consider changing its formulae. Our ambition in this paper is to compare the inconsistency of knowledge bases obtained in this process. Note that it may happen that a knowledge base $\Sigma = \{x \vee y, x, \neg x\}$ is weakened to $\Sigma' = \{x \vee y, x \vee y, \neg x\}$; observe that Σ' contains the formula $x \vee y$ twice. We do not want to lose this information as we want to compare the "evolution" of Σ towards a consistent knowledge base. This is why, in this paper, we suppose that Σ, Σ', Σ''... are multi-sets rather than sets. So when we write $\{x \vee y, x \vee y, \neg x\}$ we consider a multi-set where $x \vee y$ appears two times and $\neg x$ appears once. To simplify the presentation, in the rest of the paper, we sometimes use the word *set* instead of *multi-set* when there is no danger of confusion. For a multi-set S, we denote by $\mathtt{set}(S)$ the set of formulae that appear at least once in S. We draw the reader's attention to the following detail. Throughout the paper, we study different functions that are defined on sets. If not stated otherwise, we suppose that applying such a function to a multi-set yields (by convention) a result that would be obtained by applying a given function on a set obtained from the given multi-set by deleting duplicates. Formally, given a function f defined on sets, if S is a multi-set, we define $f(S) = f(\mathtt{set}(S))$.

We suppose that a knowledge base is a finite multi-set of classical propositional logic formulae. For a (multi-)set of formulae S, we denote by $\mathcal{P}(S)$ the set of all propositional variables appearing in S. For example, $\mathcal{P}(\{x \wedge y, x \vee \neg(y \rightarrow \neg z)\}) = \{x, y, z\}$.

We use the notation $\mathtt{MinConf}(\Sigma)$ for the set of minimal (w.r.t. set inclusion) inconsistent subsets of $\mathtt{set}(\Sigma)$. A formula φ is called a free formula of a knowledge base Σ if and only if φ does not belong to any minimal (w.r.t. set inclusion) inconsistent subset of $\mathtt{set}(\Sigma)$.

3 INCONSISTENCY MEASURES

This section formally defines the notion of an inconsistency measure and presents related work. We start by briefly recalling the definition of an inconsistency measure (Grant and Hunter, 2011b).

Definition 1 (Inconsistency measure)**.** *An inconsistency measure* $\mathtt{Inc}$ *is a function that for every finite set of formulae returns a non-negative real number and satisfies the following properties for all finite sets* $\Sigma, \Sigma' \subseteq \mathcal{L}$ *and all formulae* $\varphi, \psi \in \mathcal{L}$*:*

- $\mathtt{Inc}(\Sigma) = 0$ *iff* Σ *is a consistent set* *(consistency)*
- $\mathtt{Inc}(\Sigma \cup \Sigma') \geq \mathtt{Inc}(\Sigma)$ *(monotony)*
- *If* φ *is a free formula of* $\Sigma \cup \{\varphi\}$*, then* $\mathtt{Inc}(\Sigma \cup \{\varphi\}) = \mathtt{Inc}(\Sigma)$ *(free formula independence)*

An inconsistency measure gives a number indicating how conflicting a knowledge base is. For example, the function $\mathtt{Inc}_{MI}$ (Hunter and Konieczny, 2010) is defined as the number of minimal inconsistent subsets of Σ. (Recall that for a set X, $|X|$ stands for its cardinality.)

Definition 2 (MI inconsistency measure)**.**

$$\mathtt{Inc}_{\mathtt{MI}}(\Sigma) = |\ \mathtt{MinConf}(\Sigma)\ |$$

Example 1. *Let* $\Sigma = \{\varphi, \neg\varphi, \varphi \rightarrow \psi, \neg\psi, \omega\}$*. Then,* $\mathtt{MinConf}(\Sigma) = \{C_1, C_2\}$*, with* $C_1 = \{\varphi, \neg\varphi\}$ *and* $C_2 = \{\varphi, \varphi \rightarrow \psi, \neg\psi\}$*. Thus,* $\mathtt{Inc}_{\mathtt{MI}}(\Sigma) = 2$.

Another property that can be satisfied by an inconsistency measure is dominance (Hunter and Konieczny, 2010). The basic idea behind it is that if a formula is weakened, the knowledge base becomes less inconsistent.

Definition 3 (Dominance). *An inconsistency measure* $\texttt{Inc}$ *satisfies dominance if for all finite sets* $\Sigma \subseteq \mathcal{L}$ *and all formulae* $\varphi, \psi \in \mathcal{L}$ *such that* $\varphi \vdash \psi$ *and* $\varphi \nvdash \perp$ *we have that* $\texttt{Inc}(\Sigma \cup \{\varphi\}) \geq \texttt{Inc}(\Sigma \cup \{\psi\})$.

Mu et al. claim that dominance is not appropriate for characterizing variable-based measures for the degree of inconsistency (Mu et al., 2011). They argue that $\{x, \neg x\}$ might be seen as more inconsistent than $\{x \wedge y, \neg x\}$, since the latter set contains a non-contradictory information that y holds. However, *dominance* implies the contrary. They also show that $\texttt{Inc}_{MI}$ does not satisfy dominance.

An optional but not essential property is normalisation (Hunter and Konieczny, 2008). It says for every knowledge base Σ we have $0 \leq \texttt{Inc}(\Sigma) \leq 1$.

4 DEFINING A PARTIAL ORDER ON KNOWLEDGE BASES

Our goal is to decrease inconsistency of a knowledge base Σ by changing it (possibly several times). To be able to compare the knowledge bases Σ', Σ'' obtained from Σ in this process, we define the relation $\succeq$ which compares two multi-sets of the same cardinality.

Definition 4. *Let* Σ *and* Σ' *be two knowledge bases. We say that* $\Sigma \succeq \Sigma'$ *if and only if* Σ *and* Σ' *can be written as* $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ *and* $\Sigma' = \{\varphi'_1, \ldots, \varphi'_n\}$ *such that for every i, it holds that* $\vdash \varphi_i \rightarrow \varphi'_i$.

The reader can easily note that $\succeq$ is reflexive and transitive. We also write $\Sigma \succ \Sigma'$, if $\Sigma \succeq \Sigma'$ and not $\Sigma' \succeq \Sigma$. Relation $\succeq$ corresponds to (one or several) weakening(s).

For some results[1], mostly in Sections 4 and 5, we use the notion of complete conjunctive normal form (Whitesitt, 1995). This allows us to identify minimal changes that one can apply to a knowledge base, as will be shown later. For a formula φ, we define its *complete conjunctive normal form* (CCNF) denoted $\texttt{CCNF}(\varphi)$ as the formula φ' such that $\varphi' \equiv \varphi$ is in conjunctive normal form (CNF) and it has the form $\varphi' = \alpha_1 \wedge \ldots \wedge \alpha_n$ where every α_i is a disjunction of literals and for every α_i we have $\mathcal{P}(\{\alpha_i\}) = \texttt{Var}$. We call $\alpha_1, \ldots, \alpha_n$ the *co-atoms* of φ and we write $\texttt{Coatoms}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$[2]. For example if $\texttt{Var} = \{x, y, z\}$ and $\varphi = \neg x \vee y$ then $\texttt{CCNF}(\varphi) = (\neg x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z)$ and $\texttt{Coatoms}(\varphi) = \{\neg x \vee y \vee z, \neg x \vee y \vee \neg z\}$.

For a set S, we define $\texttt{Coatoms}(S) = \bigcup_{\varphi \in S} \texttt{Coatoms}(\varphi)$, i.e. $\texttt{Coatoms}(S)$ is the union of all co-atoms of all formulae in S. Note that both functions $\texttt{CCNF}$ and $\texttt{Coatoms}$ depend on $\texttt{Var}$, but we do not write $\texttt{CCNF}_{\texttt{Var}}$ nor $\texttt{Coatoms}_{\texttt{Var}}$ since $\texttt{Var}$ is fixed, thus there is no danger of confusion.

By convention, we define $\texttt{CCNF}(\varphi) = \top$ if and only if φ is a tautology. We do not distinguish between logically equivalent formulae. For example, we consider $(x \vee y) \wedge (x \vee \neg y)$ and $(x \vee \neg y) \wedge (y \vee x)$ to be the same formula.

The idea of using CCNF is to make all propositional variables explicit in order to allow for some basic operations on formulae we define later to be executed more easily. Of course, most of the time users might see using CCNF too restrictive. Note, however, that this question is not in the focus of our paper. Every formula can be represented in CCNF, thus we can allow users to insert formulae in any form in the knowledge base and suppose that they will be automatically translated into CCNF. There are also concerns about computational complexity regarding translating formulae into CCNF and back (for example for the purpose of shortening them before displaying them to a user). While we are aware that this question is of great interest, it is out of the scope of the present paper.

Note that $\succeq$ is a partial order if the formulae are in CCNF.

Proposition 1. *Relation* $\succeq$ *is a partial order, i.e. a reflexive, antisymmetric and transitive relation if all the formulae are in CCNF.*

Proof. It is easy to see that $\succeq$ is reflexive and transitive. Let us show that it is antisymmetric. Suppose that $\Sigma \succeq \Sigma'$ and $\Sigma' \succeq \Sigma$. Then Σ and Σ' can be written as $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ and $\Sigma' = \{\varphi'_1, \ldots, \varphi'_n\}$ such that

$$\vdash \varphi_i \rightarrow \varphi'_i, \text{ for every i} \in \{1, 2, \ldots, \text{n}\} \tag{1}$$

and there exist a permutation π of the set $\{1, 2, \ldots, n\}$ such that

$$\vdash \varphi'_i \rightarrow \varphi_{\pi(i)}, \text{ for every i} \in \{1, 2, \ldots, \text{n}\}. \tag{2}$$

Let $i \in \{1, 2, \ldots, n\}$. From (1), we conclude $\vdash \varphi_{\pi(i)} \rightarrow \varphi'_{\pi(i)}$. From (2) we obtain $\vdash \varphi'_{\pi(i)} \rightarrow \varphi_{\pi^2(i)}$. By transitivity of implication, we have that $\vdash \varphi'_i \rightarrow \varphi_{\pi^2(i)}$. Repeating the same argument, we can prove $\vdash \varphi'_i \rightarrow$

[1] to be completely precise, for Definition 5, Propositions 1, 2, 3, 4, 5, 8, 9, 10 and 13, Lemma 2 and Corollary 1

[2] The term "atom" comes from Boolean algebras where it denotes an element whose intersection with any other element gives 0 or itself. In the Lindenbaum algebra of formulae over $\texttt{Var}$, considered as a Boolean algebra, formulae with a unique model are exactly the atoms, and the formulae α_i defined above are their complements, hereof called co-atoms.

$\varphi_{\pi^k(i)}$, for every positive integer k. Note that there is ℓ such that π^ℓ is identity on $\{1,2,\ldots,n\}$, so

$$\vdash \varphi'_i \to \varphi_i, \text{ for every i} \in \{1,2,\ldots,\text{n}\}. \qquad (3)$$

From (1) and (3) we have $\vdash \varphi'_i \leftrightarrow \varphi_i$ for each i. If the formulas are in CCNF, then $\varphi = \varphi'$ for each i, and, consequently, $\Sigma = \Sigma'$, so the relation $\succeq$ is antisymmetric. □

We now present a useful characterization of consistency for knowledge bases in CCNF. Namely, a knowledge base Σ is consistent if and only if there exists a co-atom that is absent from all formulae in Σ.

Proposition 2. *Let Σ be a knowledge base in* `CCNF`*. Σ is consistent if and only if there exists a co-atom α_i (i.e. a disjunction of literals such that $\mathcal{P}(\alpha_i) = \mathtt{Var}$) such that there exists no $\varphi \in \Sigma$ with $\alpha_i \in \mathtt{Coatoms}(\varphi)$.*

Proof. We first show that if every co-atom appears in at least one formula then $\Sigma \vdash \bot$. Suppose that for every co-atom α, there exists $\varphi \in \Sigma$ such that $\alpha \in \mathtt{Coatoms}(\varphi)$. Since every co-atom of Σ is in at least one formula of Σ, then $\Sigma \vdash \mathtt{Coatoms}(\Sigma)$. Observe that for every valuation, at least one co-atom α_j is false. Thus, $\mathtt{Coatoms}(\Sigma) \vdash \bot$. Consequently, Σ is inconsistent.

We now show that if there exists a co-atom α_i that is absent from all the formulae of Σ, we can construct a model of Σ. Define a valuation $v_{\alpha_i} : \mathtt{Var} \longrightarrow \{\mathtt{true}, \mathtt{false}\}$ in the following manner: for every $x_i \in \mathtt{Var}$, let $v_{\alpha_i}(x_i) = \mathtt{true}$ if and only if $\neg x_i$ appears in α_i. (We have thus that for every x_i, $v_{\alpha_i}(x_i) = \mathtt{false}$ if and only if x_i appears in α_i.) Let us show that all the formulae of Σ are true in this valuation. Let $\varphi \in \Sigma$ with $\varphi = \alpha_{i_1} \wedge \ldots \wedge \alpha_{i_m}$. Let us show that every co-atom of φ is true in valuation v_{α_i}. Consider α_{i_j} from φ. We have $\alpha_i \neq \alpha_{i_j}$; thus there exists a variable x_k such that either (x_k appears in α_{i_j} and $\neg x_k$ appears in α_i) or ($\neg x_k$ appears in α_{i_j} and x_k appears in α_i). In both cases, α_{i_j} is true. Since this holds for all co-atoms of all formulae, Σ is consistent. □

Using the notion of co-atom allows to define the minimal operations for decreasing inconsistency of a formula or a (multi)-set of formulae. The first operation is deleting a co-atom from a formula, the second one consists in deleting a co-atom from all formulae of a set, and the third specifies the notation for deleting a co-atom from a formula of a set without changing other formulae in that set.

Definition 5. *Let φ be a formula in* `CCNF`*. We define*

$$\varphi_{del(\alpha_i)} = \bigwedge_{\alpha_k \in \mathtt{Coatoms}(\varphi) \setminus \{\alpha_i\}} \alpha_k.$$

For a (multi-)set S of formulae in CCNF, we define

$$S_{del(\alpha_i)} = \{\varphi_{del(\alpha_i)} \mid \varphi \in S\}.$$

$$S(\varphi, del(\alpha_i)) = S \setminus \{\varphi\} \cup \{\varphi_{del(\alpha_i)}\}.$$

S can be a set or a multi-set; in both cases, $S_{del(\alpha_i)}$ is obtained from S by deleting α_i from all the formulae of S. In case when S is a set, $S(\varphi, del(\alpha_i))$ is also a set. In case when S is a multi-set, $S(\varphi, del(\alpha_i))$ is also a multi-set (where α_i was deleted from all occurrences of formula φ).

Recall that, in `CCNF`, we identify the empty formula with the tautology.

Example 2. *Let $\varphi_1 = (\neg x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z)$ and $\varphi_2 = (\neg x \vee y \vee z) \wedge (x \vee y \vee z)$. Denote $S = \{\varphi_1, \varphi_2\}$ and $\alpha_1 = (\neg x \vee y \vee z)$. Then, $\varphi_{del(\alpha_1)} = \neg x \vee y \vee \neg z$, $S_{del(\alpha_1)} = \{\neg x \vee y \vee \neg z, x \vee y \vee z\}$ and $S(\varphi_1, del(\alpha_1)) = \{\neg x \vee y \vee \neg z, (\neg x \vee y \vee z) \wedge (x \vee y \vee z)\}$.*

An important advantage of representing a knowledge base in CCNF is that removing a co-atom from a formula is a minimal weakening in the semantical sense, since it enlarges the set of models of the formula for exactly one valuation.

In what follows, $Mod(\varphi)$ denotes the set of models of φ.

Proposition 3. *Let φ be a formula in CCNF and α_i be a co-atom of φ. Let $v_{\alpha_i} : \mathtt{Var} \longrightarrow \{\mathtt{true}, \mathtt{false}\}$ be the valuation such that for every $x_i \in \mathtt{Var}$, $v_{\alpha_i}(x_i) = \mathtt{true}$ if and only if $\neg x_i$ appears in α_i. Then*

$$Mod(\varphi_{del(\alpha_i)}) = Mod(\varphi) \cup \{v_{\alpha_i}\}.$$

Proof. The proof follows directly from the fact that φ is in CCNF. □

We now show that the basic operation we propose in Definition 5 is a minimal change, in the sense that for every two knowledge bases such that $\Sigma \succ \Sigma'$, there exists a minimal change of Σ resulting in a knowledge base between Σ and Σ'.

Proposition 4. *For every two knowledge bases Σ and Σ' in CCNF, if $\Sigma \succ \Sigma'$ then there exists $\varphi \in \Sigma$ and $\alpha_i \in \mathtt{Coatoms}(\varphi)$ such that $\Sigma \succ \Sigma(\varphi, del(\alpha_i)) \succeq \Sigma'$.*

Proof. If $\Sigma \succ \Sigma'$, then $\Sigma \succeq \Sigma'$, so $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ and $\Sigma' = \{\varphi'_1, \ldots, \varphi'_n\}$ where $\vdash \varphi_i \to \varphi'_i$ for every i. From $\Sigma \neq \Sigma'$ we obtain that there is j such that $\varphi_j \neq \varphi'_j$. Since φ_j and φ'_j are in CCNF, we conclude that $\mathtt{Coatoms}(\varphi'_j) \subset \mathtt{Coatoms}(\varphi_j)$. Let us choose a co-atom $\alpha_k \in \mathtt{Coatoms}(\varphi_j) \setminus \mathtt{Coatoms}(\varphi'_j)$. Obviously, $\vdash \varphi_j \to \varphi_{j_{del(\alpha_k)}}$ and $\vdash \varphi_{j_{del(\alpha_k)}} \to \varphi'_j$. Consequently, $\Sigma \succ \Sigma(\varphi_j, del(\alpha_k)) \succeq \Sigma'$. □

Also, whenever $\Sigma \succ \Sigma'$ then Σ' can be obtained from Σ by minimal changes.

Proposition 5. *For every two knowledge bases Σ and Σ' in CCNF, if $\Sigma \succ \Sigma'$ then there exist $n \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_n$ and $\alpha_{k_1} \in \mathtt{Coatoms}(\varphi_1), \ldots, \alpha_{k_n} \in \mathtt{Coatoms}(\varphi_n)$ such that $\Sigma = \Sigma_0 \succ \Sigma_1 \succ \cdots \succ \Sigma_n = \Sigma'$, where $\Sigma_i = \Sigma_{i-1}(\varphi_i, del(\alpha_{k_i}))$ for $i \in \{1, \ldots, n\}$.*

Proof. First, note that, by Proposition 4, if $\Sigma \succ \Sigma'$ then there exist $\varphi_1 \in \Sigma$ and $\alpha_{k_1} \in \mathtt{Coatoms}(\varphi_1)$ such that $\Sigma \succ \Sigma(\varphi_1, del(\alpha_{k_1})) \succeq \Sigma'$. If $\Sigma(\varphi_1, del(\alpha_{k_1})) = \Sigma'$, we are done. Otherwise, we can apply Proposition 4 to $\Sigma_1 = \Sigma(\varphi_1, del(\alpha_{k_1}))$ and Σ'. More generally, if $\Sigma \succ \Sigma_1 \succ \cdots \succ \Sigma_i \succ \Sigma'$, by Proposition 4 there are $\varphi_{i+1} \in \Sigma_i$ and $\alpha_{k_{i+1}} \in \mathtt{Coatoms}(\varphi_{i+1})$ such that $\Sigma \succ \Sigma_1 \succ \cdots \succ \Sigma_i \succ \Sigma_{i+1} \succeq \Sigma'$. Since Var is finite, and all the formulae are in CCNF, there are only finitely many different knowledge bases of given cardinality. Thus, there is $n \in \mathbb{N}$ such that $\Sigma_n = \Sigma'$. $\square$

According to the previous result, every weakening of a knowledge base in CCNF may be obtained as a sequence of consecutive minimal weakenings. We call this procedure (leading to a consistent knowledge base) *stepwise inconsistency resolution based on minimal weakening*. The next section shows that those changes lead to the most informative theories.

We also prove for a number of inconsistency measures Inc that the partial order $\succeq$ we introduced on the knowledge bases is strongly related to the inconsistency of a knowledge base. We characterize the case when $\Sigma \succeq \Sigma'$ implies that Σ is more inconsistent than Σ'. Namely, the previous statement holds if and only if the corresponding inconsistency measure enjoys *strong dominance*.

Definition 6 (Strong dominance)**.** *An inconsistency measure* Inc *satisfies strong dominance if and only if for every knowledge base Σ and for every two formulae φ, ψ*

$$\varphi \vdash \psi \text{ implies } \mathtt{Inc}(\Sigma \cup \{\varphi\}) \geq \mathtt{Inc}(\Sigma \cup \{\psi\}).$$

Note that the only difference between dominance and strong dominance is that the former also stipulates that the inconsistency of a knowledge base cannot increase if a contradiction from Σ is exchanged with another formula. This condition is naturally adopted in a semantic-based approach, like the one based on CCNF, where two logically equivalent formulae are considered equal. Namely, in CCNF, contradiction contains all the possible co-atoms. Thus, deleting one co-atom cannot increase inconsistency.

Obviously, strong dominance implies dominance. Also, we are not aware of any inconsistency measure proposed in literature which satisfies dominance, but does not satisfy strong dominance. Note that Mu et al. (2011) show that $\mathtt{Inc}_{\mathtt{MI}}$ does not satisfy dominance, hence it does not satisfy strong dominance.

The next result shows that the partial order of sets based on $\succeq$ is also an order with respect to Inc if and only if Inc satisfies strong dominance.

Proposition 6. *Let* Inc *be an inconsistency measure.* Inc *satisfies strong dominance if and only if for every two knowledge bases Σ and Σ', $\Sigma \succeq \Sigma'$ implies $\mathtt{Inc}(\Sigma) \geq \mathtt{Inc}(\Sigma')$.*

Proof. Let us first show that if Inc satisfies strong dominance then $\Sigma \succeq \Sigma'$ implies $\mathtt{Inc}(\Sigma) \geq \mathtt{Inc}(\Sigma')$. Let Inc satisfy strong dominance and let $\Sigma \succeq \Sigma'$. Denote $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ and $\Sigma' = \{\varphi'_1, \ldots, \varphi'_n\}$, where $\vdash \varphi_i \to \varphi'_i$ for every i. From strong dominance, we have that $\Sigma \succeq \mathtt{Inc}(\{\varphi'_1, \varphi_2, \ldots, \varphi_n\}) \geq \mathtt{Inc}(\{\varphi'_1, \varphi'_2, \varphi_3, \ldots, \varphi_n\}) \geq \ldots \geq \mathtt{Inc}(\Sigma')$.

We now show that if Inc does not satisfy strong dominance then there exist Σ, Σ' such that $\Sigma \succeq \Sigma'$ and $\mathtt{Inc}(\Sigma) < \mathtt{Inc}(\Sigma')$. Since Inc does not satisfy strong dominance then there exist Σ^0, α, β such that $\alpha \vdash \beta$ and $\mathtt{Inc}(\Sigma \cup \{\alpha\}) < \mathtt{Inc}(\Sigma \cup \{\beta\})$. Letting $\Sigma = \Sigma^0 \cup \{\alpha\}$ and $\Sigma' = \Sigma^0 \cup \{\beta\}$ ends the proof. $\square$

5 STEPWISE VS DIRECT INCONSISTENCY RESOLUTION

Every inconsistency resolution approach based on weakening changes the given set of formulae, and consequently reduces the quality of information. We consider two interrelated notions that every resolution procedure should take into account:

1. Minimizing the change of the given knowledge base.
2. Saving as much information as possible.

We show that stepwise resolution proposed in the previous section correctly addresses the second item. On the other hand, it can make some unnecessary changes to the given knowledge base.

In order to measure the information of a knowledge base, we use the definition by Hunter and Konieczny (2005), which is the propositional version of the definition proposed earlier in the literature (Lozinskii, 1997).

Definition 7. *A measure of quantity of information is a function* Inf *from the set of knowledge bases to $[0, +\infty)$, such that the following conditions hold:*

1. $\mathtt{Inf}(\emptyset) = 0$
2. $\mathtt{Inf}(\mathtt{Var} \cup \{\neg x \mid x \in \mathtt{Var}\}) = 0$
3. *If* $\Sigma \nvdash \bot$ *and* $\Sigma \vdash \varphi$, *then* $\mathtt{Inf}(\Sigma \cup \{\varphi\}) = \mathtt{Inf}(\Sigma)$

4. *If* $\Sigma \cup \{\varphi\} \nvdash \bot$ *and* $\Sigma \nvdash \varphi$*, then* $\mathtt{Inf}(\Sigma \cup \{\varphi\}) > \mathtt{Inf}(\Sigma)$
5. *If* $\Sigma \nvdash \bot$ *and* $\Sigma \vdash \varphi$*, then* $\mathtt{Inf}(\Sigma \cup \{\neg\varphi\}) < \mathtt{Inf}(\Sigma)$
6. *If* $\Sigma \nvdash \bot$ *and for all* $x \in \mathtt{Var}$ *we have* $\Sigma \vdash x$ *or* $\Sigma \vdash \neg x$*, then for all* Σ' $\mathtt{Inf}(\Sigma) \geq \mathtt{Inf}(\Sigma')$

The item 3 of the previous definition, called *Closed* (Grant and Hunter, 2011b), states that adding a logical consequence of a knowledge base does not change the quantity of information. The item 6 states that a complete knowledge base, i.e. the set of formulae that has a unique model, has the highest possible information content.

Grant and Hunter introduce a weaker definition of inconsistency measure, but additional useful conditions are also considered (Grant and Hunter, 2011b). One of them, named *Equivalence*, states that

$$\mathtt{Inf}(\Sigma) = \mathtt{Inf}(\Sigma'), \text{ whenever } \Sigma \nvdash \bot \text{ and } \Sigma \equiv \Sigma'.$$

They show that Equivalence implies the item 3 from Definition 7. Actually, those two properties are equivalent.

Lemma 1. *Let f be a function from the set of knowledge bases to* $[0,+\infty)$*. Function f satisfies* Equivalence *if and only if it satisfies* Closed*.*

Proof. As it is shown by Grant and Hunter (2011b), if $\mathtt{Inf}$ satisfies Equivalence, then for any Σ such that $\Sigma \nvdash \bot$ and $\Sigma \vdash \varphi$ we have that $\Sigma \equiv \Sigma \cup \{\varphi\}$. By the assumption, $\mathtt{Inf}(\Sigma \cup \{\varphi\}) = \mathtt{Inf}(\Sigma)$.

Now suppose that $\mathtt{Inf}$ satisfies Closed. Let $\Sigma = \{\varphi_1, \dots, \varphi_n\}$, $\Sigma \nvdash \bot$ and $\Sigma \equiv \Sigma'$. We define $\Sigma_0 = \Sigma'$ and $\Sigma_i = \Sigma_{i-1} \cup \{\varphi_i\}$, for $i \in \{1, \dots, n\}$. Then we have $\Sigma' \vdash \varphi_i$, so $\Sigma_i \vdash \varphi_i$, for $i \in \{1, \dots, n\}$. By the assumption, $\mathtt{Inf}(\Sigma') = \mathtt{Inf}(\Sigma_0) = \mathtt{Inf}(\Sigma_1) = \cdots = \mathtt{Inf}(\Sigma_n) = \mathtt{Inf}(\Sigma' \cup \Sigma)$. Similarly, $\mathtt{Inf}(\Sigma) = \mathtt{Inf}(\Sigma' \cup \Sigma)$, so $\mathtt{Inf}(\Sigma) = \mathtt{Inf}(\Sigma')$. □

As expected, our partial order on knowledge bases agrees with information measures.

Proposition 7. *Let Σ and Σ' be two consistent knowledge bases and let $\mathtt{Inf}$ be an information measure. If* $\Sigma \succeq \Sigma'$ *then* $\mathtt{Inf}(\Sigma) \geq \mathtt{Inf}(\Sigma')$*.*

Proof. Suppose that Σ consistent. If $\Sigma \succeq \Sigma'$, then for every $\varphi' \in \Sigma'$ there is $\varphi \in \Sigma$ such that $\vdash \varphi \rightarrow \varphi'$, so $\Sigma \cup \Sigma' \equiv \Sigma$. Let φ_Σ be a formula such that $\{\varphi_\Sigma\} \equiv \Sigma$. Hence, $\{\varphi_\Sigma\} \cup \Sigma' \equiv \Sigma$. Thus, $\{\varphi_\Sigma\} \cup \Sigma'$ is also consistent. By Lemma 1, $\mathtt{Inf}(\Sigma) = \mathtt{Inf}(\{\varphi_\Sigma\} \cup \Sigma')$. On the other hand, $\mathtt{Inf}(\{\varphi_\Sigma\} \cup \Sigma') \geq \mathtt{Inf}(\Sigma')$, by items 3 and 4 of Definition 7. Thus, we obtained $\mathtt{Inf}(\Sigma) \geq \mathtt{Inf}(\Sigma')$. □

By item 6 of Definition 7 and the following proposition, *stepwise inconsistency resolution based on minimal weakening* (i.e. the process of obtaining a consistent knowledge base by consecutively applying minimal changes) leads to a maximally informative consistent knowledge base.

Proposition 8. *Let Σ be an inconsistent knowledge base, and let Σ' be the knowledge base obtained from Σ by stepwise inconsistency resolution based on minimal weakening. Then Σ' is complete.*

Proof. If Σ' is a consistent knowledge base, obtained from Σ by the stepwise inconsistency resolution based on minimal weakening, i.e. $\Sigma = \Sigma_0 \succ \Sigma_1 \succ \cdots \succ \Sigma_{n-1} \succ \Sigma_n = \Sigma'$, then, by Proposition 2, there is a co-atom that does not appear in any formula from Σ'. Also, since Σ_{n-1} is inconsistent, each co-atom appears in at least one formula from Σ_{n-1}. If $\Sigma' = \Sigma_{n-1}(\varphi, del(\alpha_i))$, then obviously α_i is the only co-atom that does not appear in any formula from Σ'. The valuation v_{α_i} defined in the proof of Proposition 2 is a model of Σ'. Suppose that there is a valuation $v \neq v_{\alpha_i}$ that is also a model of Σ'. We define the co-atom

$$\alpha^v = \bigvee_{x \in \mathtt{Var}, v(x) = false} x \vee \bigvee_{x \in \mathtt{Var}, v(x) = true} \neg x.$$

If $\psi \in \Sigma'$ is a formula such that α^v appears in ψ, then ψ is false under the valuation v, since

$$\neg\alpha^v = \bigwedge_{x \in \mathtt{Var}, v(x) = false} \neg x \wedge \bigwedge_{x \in \mathtt{Var}, v(x) = true} x$$

is obviously true under v. Thus, the valuation v_{α_i} is the only model of Σ'. □

Nevertheless, some of the changes made to the knowledge bases might be unnecessary. Consider the following example.

Example 3. *Let* $Var = \{x, y\}$*, and let* $\Sigma = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$*, where* $\varphi_1 = (x \vee y) \wedge (x \vee \neg y)$*,* $\varphi_2 = (\neg x \vee y) \wedge (\neg x \vee \neg y)$*,* $\varphi_3 = (x \vee \neg y) \wedge (\neg x \vee y)$ *and* $\varphi_4 = (x \vee y) \wedge (\neg x \vee \neg y)$*. Let Σ' be obtained by replacing φ_1 with* $\varphi_1' = (x \vee y)$*, i.e.* $\varphi_1' = \varphi_{1\,del(x \vee \neg y)}$*. By Proposition 2, Σ' is inconsistent. If Σ'' is obtained from Σ' by replacing φ_2 with* $\varphi_2' = \varphi_{2\,del(\neg x \vee y)}$*, then Σ'' is also inconsistent. Let Σ''' be obtained from Σ'' by replacing φ_3 with* $\varphi_3' = \varphi_{3\,del(x \vee \neg y)}$*. By Proposition 2,* $\Sigma''' = \{\varphi_1', \varphi_2', \varphi_3', \varphi_4\}$ *is consistent, since $x \vee \neg y$ does not appear in the formulae of Σ'''. However, note that the second step of the resolution was not necessary, since the knowledge base* $\{\varphi_1', \varphi_2, \varphi_3', \varphi_4\}$ *is also consistent.*

Now we propose $\Sigma_{del(\alpha_i)}$ as candidates for the resolving inconsistent knowledge base Σ. Note that, in the case when we want to end up with a consistent knowledge base, it can be argued that deleting a co-atom α_i in some but not in all formulae does not make sense. Namely, whether a co-atom appears once or several times in Σ does not make any difference concerning consistency of Σ. Thus, in a scenario where we continue applying changes until obtaining a consistent knowledge base, we should either remove a co-atom from all formulae or not remove it at all (in order to avoid information loss without consistency gain).

The next result shows that if we delete all occurrences of an arbitrary co-atom, Σ becomes not only consistent, but also maximally informative, in the sense that adding information (in terms of semantics) would result in its inconsistency. Recall that we call a knowledge base complete if and only if it has exactly one model.

Proposition 9. *If Σ is an inconsistent knowledge base in CCNF then for every $\alpha_i \in \texttt{Coatoms}(\Sigma)$, $\Sigma_{del(\alpha_i)}$ is complete.*

Proof. If Σ is inconsistent, by Proposition 2, each co-atom appears in at least one formula from Σ. Consequently, α_i is the only co-atom that does not appear in any formula from $\Sigma_{del(\alpha_i)}$. Similarly as in the proof of Proposition 8, one can show that the valuation v_{α_i} is the only model of $\Sigma_{del(\alpha_i)}$, so it is complete. □

By the following proposition, however the resolution of inconsistency of Σ is conducted, if the obtained set is not of the form $\Sigma_{del(\alpha_i)}$, the inconsistency can be resolved by applying less changes to Σ.

Proposition 10. *Let Σ, Σ' be two knowledge bases in CCNF. If Σ is inconsistent then:*

1. *If $\Sigma \succeq \Sigma'$ and Σ' is consistent, then there exists $\alpha_i \in \texttt{Coatoms}(\Sigma)$ such that $\Sigma \succeq \Sigma_{del(\alpha_i)} \succeq \Sigma'$.*
2. *For every $\alpha_i \in \texttt{Coatoms}(\Sigma)$, we have that if $\Sigma \succeq \Sigma' \succ \Sigma_{del(\alpha_i)}$, then Σ' is inconsistent.*

Proof.

1. Let $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ and $\Sigma' = \{\varphi'_1, \ldots, \varphi'_n\}$ such that for every j, it holds that $\vdash \varphi_j \rightarrow \varphi'_j$. Since Σ' is consistent, there exists a co-atom α_i that is absent from all φ'_j. Thus, for every j, $\vdash \varphi_j \rightarrow \varphi_{j\,del(\alpha_i)}$ and $\vdash \varphi_{j\,del(\alpha_i)} \rightarrow \varphi'_j$ hold. Consequently, $\Sigma \succeq \Sigma_{del(\alpha_i)} = \{\varphi_{1\,del(\alpha_i)}, \ldots, \varphi_{n\,del(\alpha_i)}\} \succeq \Sigma'$.
2. First, recall that v_{α_i} is the only model of $\Sigma_{del(\alpha_i)}$. From $\Sigma' \succ \Sigma_{del(\alpha_i)}$ we obtain that v is not a model of Σ', for every $v \neq v_{\alpha_i}$. If $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ and $\Sigma' = \{\varphi'_1, \ldots, \varphi'_n\}$, then for every j, $\varphi'_j = \varphi_j$ or $\varphi'_j = \varphi_{j\,del(\alpha_i)}$. From $\Sigma' \succ \Sigma_{del(\alpha_i)}$ we obtain that there is k such that $\varphi'_k \neq \varphi_{k\,del(\alpha_i)}$. Thus, $\varphi'_k = \varphi_{k\,del(\alpha_i)} \wedge \alpha_i$. Then the valuation v_{α_i} is not a model of φ'_k, so Σ' is inconsistent.

□

The results from this section allow us to draw several conclusions. First, we see that minimal ways to weaken Σ are to apply deletion of exactly one co-atom. If one weakens it more, then we lose more than necessary in terms of information. Does stepwise inconsistency resolution make sense if we want to end up with a *consistent* knowledge base (given that it will never yield a better result in terms of information loss than one-step approach based on deleting one co-atom from all the formulae)?

Indeed, one might just want to go directly to the closest consistent knowledge base. But how to know which knowledge base is the closest? We answer this question in the next section.

6 DEFINING METRICS ON THE SET OF KNOWLEDGE BASES

Relation $\succeq$ allows to compare knowledge bases and enjoys some desirable properties related to inconsistency and information measures. However, since this relation is not a total order, some knowledge bases are not comparable. Consider the following example.

Example 4. *Suppose that $\texttt{Var} = \{x, y, z\}$. Then there are 8 co-atoms, $\alpha_1, \alpha_2, \ldots, \alpha_8$. Let*

$$\varphi_1 = \alpha_1 \wedge \alpha_2$$
$$\varphi_2 = \alpha_3 \wedge \alpha_4 \wedge \alpha_5 \wedge \alpha_6 \wedge \alpha_7$$
$$\varphi_3 = \alpha_3 \wedge \alpha_8$$
$$\varphi_4 = \alpha_2 \wedge \alpha_3$$

By Proposition 2, the knowledge base $\Sigma = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ is inconsistent. By the previous section, minimal ways to weaken Σ are to apply deletion of exactly one co-atom. Note that $\Sigma_{del(\alpha_1)}$ can be obtained from Σ by only one step in the stepwise inconsistency resolution (replacing φ_1 by $\varphi_{1\,del(\alpha_1)}$). On the other hand, we need three steps to obtain $\Sigma_{del(\alpha_3)}$ from Σ, so $\Sigma_{del(\alpha_1)}$ is intuitively closer to Σ. Nevertheless, the partial order $\succeq$ can not capture this intuition, since $\Sigma_{del(\alpha_1)}$ and $\Sigma_{del(\alpha_3)}$ are incomparable, i.e. neither $\Sigma_{del(\alpha_1)} \succeq \Sigma_{del(\alpha_3)}$ nor $\Sigma_{del(\alpha_3)} \succeq \Sigma_{del(\alpha_1)}$.

Motivated by the previous example, we propose to define a family of metrics to measure distance from a given set to the closest consistent set. Let us recall the definition of a metric.

Definition 8. *A metric on a set S is any function $d : S^2 \longrightarrow \mathbb{R}$, such that for all $a, b, c \in S$ the following conditions hold:*

1. $d(a,b) \geq 0$
2. $d(a,b) = 0$ *iff* $a = b$
3. $d(a,b) = d(b,a)$
4. $d(a,b) + d(b,c) \geq d(a,c)$.

The pair $\langle S,d \rangle$ is called metric space. For $A \subseteq S$, distance from element a to A is defined as $d(a,A) = \inf_{b \in A} d(a,b)$.

We are primarily interested in applying the metrics in order to analyze how the distance between a knowledge base and the closest consistent knowledge base changes when we apply the approach proposed in Sections 4 and 5. Since our weakenings do not change the cardinality of a knowledge base, as in the case of partial order $\succeq$, it suffices to measure the distances between knowledge bases of the same cardinality.

In the following definition, $\triangle$ denotes the symmetric difference of two sets[3] and $\Pi(n)$ is the set of permutations of the set $\{1,\ldots,n\}$. Furthermore, since we work with models, we identify the formulae having the same models. In other words, we do not distinguish between φ and ψ if $\{\varphi\} \equiv \{\psi\}$.

Definition 9. *For a fixed $n \in \mathbb{N}$ and $p \in [1,+\infty)$, we define the binary function d_p on the set of knowledge bases of cardinality n as follows. For $\Sigma = \{\varphi_1, \ldots \varphi_n\}$ and $\Sigma' = \{\psi_1, \ldots \psi_n\}$ let*

$$d_p(\Sigma,\Sigma') = \min_{\pi \in \Pi(n)} (\sum_{i=1}^{n} |Mod(\varphi_i) \triangle Mod(\psi_{\pi(i)})|^p)^{\frac{1}{p}}.$$

We also define

$$d_\infty(\Sigma,\Sigma') = \min_{\pi \in \Pi(n)} \max_{i \in \{1,\ldots,n\}} |Mod(\varphi_i) \triangle Mod(\psi_{\pi(i)})|.$$

The previous definition uses a variant of p-norms. Intuitively, it takes the minimum over all the permutations of Σ' in order to "try to find" the alignment of Σ' that is "the most similar" to Σ. The bigger the difference in number of models between φ and φ', the bigger the distance between the bases.

By $[1,+\infty]$ we denote the set $[1,+\infty) \cup \{+\infty\}$.

Proposition 11. *For each $p \in [1,+\infty]$, d_p is a metric.*

Proof. The proofs of $d_p(\Sigma,\Sigma') \geq 0$, $d_p(\Sigma,\Sigma') = 0$ iff $\Sigma = \Sigma'$, and $d_p(\Sigma,\Sigma') = d_p(\Sigma',\Sigma)$ are trivial, so we only prove $d_p(\Sigma,\Sigma') + d_p(\Sigma',\Sigma'') \geq d_p(\Sigma,\Sigma'')$. We only consider the case when $p \in [1,+\infty)$; the case when $p = +\infty$ can be proved in the similar way. Let $\Sigma = \{\varphi_1,\ldots,\varphi_n\}$, $\Sigma' = \{\varphi'_1,\ldots,\varphi'_n\}$ and $\Sigma'' = \{\varphi''_1,\ldots,\varphi''_n\}$. Let $\sigma \in \Pi(n)$ be the permutation such that $d_p(\Sigma,\Sigma') = (\sum_{i=1}^n |Mod(\varphi_i) \triangle Mod(\varphi'_{\sigma(i)})|^p)^{\frac{1}{p}}$. Then $d_p(\Sigma,\Sigma') + d_p(\Sigma',\Sigma'') = (\sum_{i=1}^n |Mod(\varphi_i) \triangle Mod(\varphi'_{\sigma(i)})|^p)^{\frac{1}{p}} + \min_{\pi \in \Pi(n)} (\sum_{i=1}^n |Mod(\varphi'_i) \triangle Mod(\varphi''_{\pi(i)})|^p)^{\frac{1}{p}} = (\sum_{i=1}^n |Mod(\varphi_i) \triangle Mod(\varphi'_{\sigma(i)})|^p)^{\frac{1}{p}} + \min_{\pi \in \Pi(n)} (\sum_{i=1}^n |Mod(\varphi'_{\sigma(i)}) \triangle Mod(\varphi''_{\pi(\sigma(i))})|^p)^{\frac{1}{p}} = \min_{\pi \in \Pi(n)} ((\sum_{i=1}^n |Mod(\varphi_i) \triangle Mod(\varphi'_{\sigma(i)})|^p)^{\frac{1}{p}} + (\sum_{i=1}^n |Mod(\varphi'_{\sigma(i)}) \triangle Mod(\varphi''_{\pi(\sigma(i))})|^p)^{\frac{1}{p}})$. From the inequality

$$(\sum_{i=1}^n a_i^p)^{\frac{1}{p}} + (\sum_{i=1}^n b_i^p)^{\frac{1}{p}} \geq (\sum_{i=1}^n (a_i+b_i)^p)^{\frac{1}{p}}, \quad a_i, b_i \geq 0$$

we obtain $d_p(\Sigma,\Sigma') + d_p(\Sigma',\Sigma'') \geq \min_{\pi \in \Pi(n)} (\sum_{i=1}^n (|Mod(\varphi_i) \triangle Mod(\varphi'_{\sigma(i)})| + |Mod(\varphi'_{\sigma(i)}) \triangle Mod(\varphi''_{\pi(\sigma(i))})|)^p)^{\frac{1}{p}}$.

Note that, for any sets A, B and C, $(A \triangle B) \cup (B \triangle C) \supseteq A \triangle C$, so $|A \triangle B| + |B \triangle C| \geq |(A \triangle B) \cup (B \triangle C)| \geq |A \triangle C|$. Thus, $d_p(\Sigma,\Sigma') + d_p(\Sigma',\Sigma'') \geq \min_{\pi \in \Pi(n)} ((\sum_{i=1}^n |Mod(\varphi_i) \triangle Mod(\varphi''_{\pi(\sigma(i))})|^p)^{\frac{1}{p}} = \min_{\pi' \in \Pi(n)} ((\sum_{i=1}^n |Mod(\varphi_i) \triangle Mod(\varphi''_{\pi'(i)})|^p)^{\frac{1}{p}} = d_p(\Sigma,\Sigma'')$. □

The following example illustrates that the metrics d_1 and d_2 are not compatible. Namely, it shows that it can be the case that $d_1(\Sigma,\Sigma') < d_1(\Sigma,\Sigma'')$, but $d_2(\Sigma,\Sigma') > d_2(\Sigma,\Sigma'')$.

Example 5. *Suppose that* `Var` $= \{x_1,\ldots,x_6\}$. *Then there are 64 co-atoms. Let*

$\varphi_1 = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{10}$
$\varphi_2 = \alpha_{21} \wedge \alpha_{22} \wedge \cdots \wedge \alpha_{30}$
$\varphi_3 = \alpha_{21} \wedge \alpha_{22} \wedge \cdots \wedge \alpha_{40}$
$\varphi_4 = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{15}$
$\varphi_5 = \alpha_{21} \wedge \alpha_{22} \wedge \cdots \wedge \alpha_{36}$

Let $\Sigma = \{\varphi_1,\varphi_2\}$, $\Sigma' = \{\varphi_1,\varphi_3\}$ and $\Sigma'' = \{\varphi_4,\varphi_5\}$. Then

$d_1(\Sigma,\Sigma') = |Mod(\varphi_1) \triangle Mod(\varphi_1)| + |Mod(\varphi_2) \triangle Mod(\varphi_3)| = 0 + 10 = 10$
$d_2(\Sigma,\Sigma') = (|Mod(\varphi_1) \triangle Mod(\varphi_1)|^2 + |Mod(\varphi_2) \triangle Mod(\varphi_3)|^2)^{\frac{1}{2}} = \sqrt{0^2+10^2} = 10$
$d_1(\Sigma,\Sigma'') = |Mod(\varphi_1) \triangle Mod(\varphi_4)| + |Mod(\varphi_2) \triangle Mod(\varphi_5)| = 5 + 6 = 11$
$d_2(\Sigma,\Sigma'') = (|Mod(\varphi_1) \triangle Mod(\varphi_4)|^2 + |Mod(\varphi_2) \triangle Mod(\varphi_5)|^2)^{\frac{1}{2}} = \sqrt{5^2+6^2} = \sqrt{61}$

Thus, we obtain that $d_1(\Sigma,\Sigma') < d_1(\Sigma,\Sigma'')$, but $d_2(\Sigma,\Sigma') > d_2(\Sigma,\Sigma'')$.

[3] Recall that symmetric difference of two sets S_1 and S_2 is defined as $S_1 \triangle S_2 = (S_1 \setminus S_2) \cup (S_2 \setminus S_1)$.

Moreover, there are no distinct p_1 and p_2 such that d_{p_1} is compatible with d_{p_2} for every Var.

Proposition 12. *For all $p_1, p_2 \in [1, +\infty]$ with $p_1 \neq p_2$, there exist* Var, *Σ, Σ' and Σ'' with $\mathcal{P}(\Sigma), \mathcal{P}(\Sigma'), \mathcal{P}(\Sigma'') \subseteq$* Var *such that $d_{p_1}(\Sigma, \Sigma') < d_{p_1}(\Sigma, \Sigma'')$ but $d_{p_2}(\Sigma, \Sigma') > d_{p_2}(\Sigma, \Sigma'')$.*

Proof. Let $p_1 < p_2 < +\infty$. Then $2^{\frac{1}{p_2}} < 2^{\frac{1}{p_1}}$, so there is $m \in \mathbb{N}$ such that $m(2^{\frac{1}{p_1}} - 2^{\frac{1}{p_2}}) > 1$. Let us choose $n \in \mathbb{N}$ such that $m2^{\frac{1}{p_2}} < n < m2^{\frac{1}{p_1}}$. We suppose that the set of propositional letters is large enough, and we denote co-atoms by α_i and β_j, assuming that α's and β's are always different. Let

- $\Sigma = \{\varphi_1, \varphi_2\}$, where $\varphi_1 = \bigwedge_{i=1}^{k} \alpha_i$ and $\varphi_2 = \bigwedge_{i=1}^{k} \beta_i$
- $\Sigma = \{\varphi'_1, \varphi'_2\}$, where $\varphi'_1 = \varphi_1$ and $\varphi'_2 = \bigwedge_{i=1}^{k+n} \beta_i$
- $\Sigma'' = \{\varphi''_1, \varphi''_2\}$, where $\varphi''_1 = \bigwedge_{i=1}^{k+m} \alpha_i$ and $\varphi''_2 = \bigwedge_{i=1}^{k+m} \beta_i$

Then $d_{p_1}(\Sigma, \Sigma') = (|Mod(\varphi_1) \triangle Mod(\varphi'_1)|^{p_1} + |Mod(\varphi_2) \triangle Mod(\varphi'_2)|^{p_1})^{\frac{1}{p_1}} = n$, and $d_{p_2}(\Sigma, \Sigma') = n$. Similarly, $d_{p_2}(\Sigma, \Sigma'') = (|Mod(\varphi_1) \triangle Mod(\varphi''_1)|^{p_2} + |Mod(\varphi_2) \triangle Mod(\varphi''_2)|^{p_2})^{\frac{1}{p_2}} = m2^{\frac{1}{p_1}}$, and $d_{p_2}(\Sigma, \Sigma'') = m2^{\frac{1}{p_2}}$.

Thus, $d_{p_1}(\Sigma, \Sigma') < d_{p_1}(\Sigma, \Sigma'')$ and $d_{p_2}(\Sigma, \Sigma') > d_{p_2}(\Sigma, \Sigma'')$.

In the case where $p_1 = +\infty$ or $p_2 = +\infty$, the proof is similar. □

Nevertheless, the next result shows that for every metric d_p, for each α_i appearing in the least number of formulae of Σ, the distance from Σ to the closest consistent multi-set of the same cardinality is exactly the distance from Σ to $\Sigma_{del(\alpha_i)}$. In other words, independently of parameter p, using $\Sigma_{del(\alpha_i)}$ yields the closest consistent knowledge base.

We first prove the following useful result showing that the distance (with respect to metric d_p) between a knowledge base Σ and the knowledge base Σ' obtained by deleting from Σ a co-atom α_i that appears k times in Σ is $k^{\frac{1}{p}}$.

Lemma 2. *Let Σ be a knowledge base in CCNF and α_i a co-atom such that $|\{\varphi \in \Sigma \mid \alpha_i \in \mathtt{Coatoms}(\varphi)\}| = k$. Then*

$$d_p(\Sigma, \Sigma_{del(\alpha_i)}) = k^{\frac{1}{p}}$$

Proof. Let $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ and, without loss of generality, suppose that α_i appears in formulae $\varphi_1, \ldots, \varphi_k$. On one hand, by denoting $\Sigma' = \{\psi_1, \ldots, \psi_n\}$ with $\psi_i = \varphi_{i,del(\alpha_i)}$ and letting permutation π from Definition 9 be identity, we obtain that $(\sum_{i=1}^{n} |Mod(\varphi_i) \triangle Mod(\psi_{\pi(i)})|^p)^{\frac{1}{p}} = k^{\frac{1}{p}}$. On the other hand, since $\Sigma' = \Sigma_{del(\alpha_i)}$, then there exist k formulae in Σ that do not appear in Σ' (those are formulae $\varphi_1, \ldots, \varphi_k$). Each of those k formulae must be matched with exactly one formula ψ_q from Σ' different from it. Thus, for every permutation, there are at least k matched formulae that differ, so $d_p(\Sigma, \Sigma_{del(\alpha_i)}) \geq k^{\frac{1}{p}}$. From those two facts, we conclude that $d_p(\Sigma, \Sigma_{del(\alpha_i)}) = k^{\frac{1}{p}}$. □

Proposition 13. *Denote by $Cons(n)$ the set of all consistent multi-sets of cardinality n. Then for all Σ in CCNF, for all $p \in [1, +\infty]$, for all $\alpha_i \in \mathtt{Coatoms}(\Sigma)$ such that there exists no $\alpha_j \in \mathtt{Coatoms}(\Sigma)$ such that $|\{\varphi \in \Sigma \mid \alpha_j \in \mathtt{Coatoms}(\varphi)\}| < |\{\varphi \in \Sigma \mid \alpha_i \in \mathtt{Coatoms}(\varphi)\}|$, we have*

$$d_p(\Sigma, Cons(|\Sigma|)) = d_p(\Sigma, \Sigma_{del(\alpha_i)}).$$

Proof. If Σ is consistent, by Proposition 2 there is α_i that is absent from every formula of Σ. Let Σ be inconsistent and let $p \in [1, +\infty)$ (the case when $p = +\infty$ is similar).

First, by Proposition 2, $\Sigma_{del(\alpha_i)}$ is consistent, thus $d_p(\Sigma, Cons(|\Sigma|)) \leq d_p(\Sigma, \Sigma_{del(\alpha_i)})$.

Second, by Lemma 2, we obtain $d_p(\Sigma, \Sigma_{del(\alpha_i)}) = k^{\frac{1}{p}}$.

Third, let us show that for every $\Sigma' \in Cons(|\Sigma|)$ we have $d_p(\Sigma, \Sigma') \geq k^{\frac{1}{p}}$. By Proposition 2, there exists a co-atom α_j such that $\alpha_j \notin \mathtt{Coatoms}(\Sigma')$. Suppose that α_j appears in exactly l formulae of Σ. For each formula $\varphi \in \Sigma$ containing α_j, we have $|Mod(\varphi) \triangle Mod(\psi)| \geq 1$ for every $\varphi' \in \Sigma'$. Consequently, $d_p(\Sigma, \Sigma') \geq l^{\frac{1}{p}} \geq k^{\frac{1}{p}}$.

From the second and the third observation, we have that $d_p(\Sigma, Cons(|\Sigma|)) \geq d_p(\Sigma, \Sigma_{del(\alpha_i)})$. This fact, together with the first observation, ends the proof. □

The previous result confirms that our class of metrics is based on an intuitive distance. Namely, it confirms the natural expectation that deleting a co-atom that appears in the least number of formulae in a knowledge base is the minimal change required to obtain a consistent knowledge base. The following statement is a direct consequence of Proposition 13.

Corollary 1. *For all Σ, Σ' in CCNF such that $\Sigma \succeq \Sigma'$ and for all $p \in [1, +\infty]$, it holds that $d_p(\Sigma, Cons(|\Sigma|)) \geq d_p(\Sigma', Cons(|\Sigma|))$.*

This result proves an important link between the distances defined in Section 6 and the inconsistency resolution method proposed in Sections 4 and 5. Namely, it shows that applying minimal changes to a knowledge base reduces its distance to consistency.

7 CONCLUSION AND RELATED WORK

This paper tackled the question of how to measure inconsistency of a knowledge base, how to compare knowledge bases in terms of inconsistency and information and how to measure distance between knowledge bases. Given an inconsistent knowledge base, our goal is to decrease its inconsistency and/or find the closest consistent knowledge base.

Namely, we proposed to use the operation $\varphi_{del(\alpha_i)}$ as a basic operation for decreasing inconsistency of a knowledge base; its advantage is that it yields a minimal change of a knowledge base. We defined a partial order $\succeq$ on the set of knowledge bases and showed its link with existing inconsistency and information measures. We also introduced operations $\Sigma_{del(\alpha_i)}$ and showed that they define the closest consistent knowledge bases according to $\succeq$. Furthermore, we defined a class of metrics that can be used to measure distances between two knowledge bases as well as the distance from a knowledge base to the closest consistent knowledge base. We formally showed that our class of metrics agrees with an intuitive distance. Namely, whatever the value of parameter p, deleting a co-atom that appears in the least number of formulae in a knowledge base is the minimal change (in the sense of distance measured by d_p) required to obtain a consistent knowledge base. Also, we proved that those metrics are compatible with $\succeq$, which means that the distance between a knowledge base Σ and consistency decreases whenever our minimal weakening is applied to Σ.

The closest related paper is the work by Grant and Hunter discussed in the introduction (Grant and Hunter, 2011b). We are inspired by that paper but we use only one operation instead of three. To be more precise, Grant and Hunter proposed three classes of operations, but given a concrete formula, say $\varphi = x \vee y$ in a concrete knowledge base Σ, they do not say how to practically weaken φ. Our paper offers the answers to this question. Also, we compare stepwise inconsistency resolution and one-step approach where we go directly to the closest consistent knowledge base.

There is another work that might seem very close to the second part of our paper (Grant and Hunter, 2013). But indeed, there is a huge difference, since they introduce metrics to measure distances between *models*, which is not the case in our paper. We measure the distance between *knowledge bases* using a variation of p-norms, applied on symmetric difference of models of formulae of two knowledge bases.

There is also related work on inconsistency measurement for probabilistic logics (Picado-Muiño, 2011; Thimm, 2009), where p-norms are used to calculate measure of inconsistency of a probabilistic knowledge base. But unlike in our paper, in those papers the p-norms are used to determine the distance between all the functions (which are not probability measures) satisfying the (inconsistent) set of probability requirements from the knowledge base, and the closest probability measure.

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