On the equivalence of logic-based argumentation systems

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Abstract. Equivalence between two argumentation systems means mainly that the two systems return the same outputs. It can be used for different purposes, namely in order to show whether two systems that are built over the same knowledge base but with distinct attack relations return the same outputs, and more importantly to check whether an infinite system can be reduced into a finite one.
Recently, the equivalence between abstract argumentation systems was investigated. Two categories of equivalence criteria were particularly proposed. The first category compares directly the outputs of the two systems (e.g. their extensions) while the second compares the outputs of their extended versions (i.e. the systems augmented by the same set of arguments). It was shown that only identical systems are equivalent w.r.t. those criteria.
In this paper, we study when two logic-based argumentation systems are equivalent. We refine existing criteria by considering the internal structure of arguments and propose new ones. Then, we identify cases where two systems are equivalent. In particular, we show that under some reasonable conditions on the logic underlying an argumentation system, the latter has an equivalent finite subsystem. This subsystem constitutes a threshold under which arguments of the system have not yet attained their final status and consequently adding a new argument may result in status change. From that threshold, the statuses of all arguments become stable.

1 Introduction

One of the most abstract argumentation systems was proposed by Dung [6]. It consists of a *set of arguments* and a binary relation representing *conflicts* among arguments. Those conflicts are then solved using a *semantics* which amounts to define acceptable sets of arguments, called *extensions*. From the extensions, a *status* is assigned to each argument. An argument is *skeptically* accepted if it appears in each extension, it is *credulously* accepted if it belongs to at least one extension, and finally it is *rejected* if it is not in any extension.

Several works were done on this system. Some of them extended it with new features like preferences between arguments (e.g. [2, 4]) or weights on attacks (e.g. [7]), others defined new semantics that solve some problems encountered with Dung's ones (e.g. [3, 5]) and another category of works instantiated the system for application purposes. More recently, the question of *equivalence* between two abstract argumentation systems was tackled by Oikarinen and Woltran [9]. To the best of our knowledge this is the only work on this issue. The authors proposed two kinds of equivalence: *basic equivalence* and *strong equivalence*. According to basic equivalence, two systems are equivalent if they have the same extensions (resp. the same sets of skeptically/credulously accepted arguments). However, these criteria were not studied by Oikarinen and Woltran. Instead, they concentrated on strong equivalence. Two systems are strongly equivalent if they have the same extensions (resp. the same sets of skeptically/credulously accepted arguments) even after extending both systems by any set of arguments. The authors investigated under which conditions two systems are strongly equivalent. They have shown that when there are no self-attacking arguments, which is the case in most argumentation systems, and particularly in most logic-based argumentation systems as shown by Amgoud and Besnard [1], then two systems are strongly equivalent if and only if they coincide, i.e. they are the same. This makes the notion of strong equivalence a nice theoretical property, but without any practical applications.

In this paper, we study when two logic-based argumentation systems are equivalent. We refine existing criteria by considering the internal structure of arguments and propose new ones. We identify interesting cases where two systems are equivalent. In particular, we show that under some reasonable conditions on the logic underlying an argumentation system, the latter has an equivalent finite subsystem, which constitutes a threshold under which arguments of the system have not yet attained their final status and consequently any new argument may result in status change. From that threshold, the statuses of all arguments become stable.

The paper is structured as follows: in Section 2, we recall the logic-based argumentation systems we are interested in. In Section 3, we propose three equivalence criteria that refine the basic ones and study when two systems are equivalent w.r.t. each criterion. In Section 4, we refine the three criteria of strong equivalence and give the conditions under which they hold. Section 5 studies when the status of an argument may change when a new argument is received or removed from a system. The last section is devoted to some concluding remarks and perspectives. All the proofs are put in an appendix.

2 Logic-Based Argumentation Systems

This section describes the logical instantiations of Dung's argumentation system we are interested in. They are built around *any* monotonic logic whose consequence operator satisfies the five postulates proposed by Tarski [10]. Indeed, according to those postulates, a *monotonic logic* is a pair $(\mathcal{L}, \mathtt{CN})$ where $\mathcal{L}$ is any set of *well-formed formulae* and $\mathtt{CN}$ is a *consequence operator*, i.e. a function from $2^{\mathcal{L}}$ to $2^{\mathcal{L}}$ that satisfies the following five postulates:

- $X \subseteq \mathtt{CN}(X)$ **(Expansion)**
- $\mathtt{CN}(\mathtt{CN}(X)) = \mathtt{CN}(X)$ **(Idempotence)**
- $\mathtt{CN}(X) = \bigcup_{Y \subseteq_f X} \mathtt{CN}(Y)$[1] **(Finiteness)**

[1] The notation $Y \subseteq_f X$ means that Y is a finite subset of X.

– $\mathtt{CN}(\{x\}) = \mathcal{L}$ for some $x \in \mathcal{L}$ **(Absurdity)**
– $\mathtt{CN}(\emptyset) \neq \mathcal{L}$ **(Coherence)**

Intuitively, $\mathtt{CN}(X)$ returns the set of formulae that are logical consequences of X according to the logic at hand. Almost all well-known logics (classical logic, intuitionistic logic, modal logics, ...) are special cases of Tarski's notion of monotonic logic. In such a logic, a set X of formulae is *consistent* iff its set of consequences is not the set $\mathcal{L}$. For two formulae $x, y \in \mathcal{L}$, we say that x and y are equivalent, denoted by $x \equiv y$, iff $\mathtt{CN}(\{x\}) = \mathtt{CN}(\{y\})$. Arguments are built from a *knowledge base* Σ which is a finite subset of $\mathcal{L}$.

Definition 1 (Argument). *Let $(\mathcal{L}, \mathtt{CN})$ be a Tarskian logic and $\Sigma \subseteq \mathcal{L}$. An* argument *built from Σ is a pair (X, x) s.t.*

– $X \subseteq \Sigma$*,*
– X *is consistent,*
– $x \in \mathtt{CN}(X)$*,*
– $\nexists X' \subset X$ *s.t.* $x \in \mathtt{CN}(X')$.

X *is the* support *of the argument and x its* conclusion*.*

Notations: For an argument $a = (X, x)$, $\mathtt{Conc}(a) = x$ and $\mathtt{Supp}(a) = X$. For a set $\mathcal{S} \subseteq \mathcal{L}$, $\mathtt{Arg}(\mathcal{S}) = \{a \mid a$ is an argument (in the sense of Definition 1) and $\mathtt{Supp}(a) \subseteq \mathcal{S}\}$. The set of all arguments that can be built from the language $\mathcal{L}$ will be denoted by $\mathtt{Arg}(\mathcal{L})$. For any $\mathcal{E} \subseteq \mathtt{Arg}(\mathcal{L})$, $\mathtt{Base}(\mathcal{E}) = \bigcup_{a \in \mathcal{E}} \mathtt{Supp}(a)$.

The previous definition specified what we accept as an argument. An attack relation $\mathcal{R}$ is defined on a given set $\mathcal{A}$ of arguments, i.e. $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$. The writing $a\mathcal{R}b$ or $(a, b) \in \mathcal{R}$ means that argument a *attacks* argument b. A study on how to choose an appropriate attack relation was recently carried out by Amgoud and Besnard [1]. Some basic properties of an attack relation were also discussed by Gorogiannis and Hunter [8]. Examples of such properties are recalled below.

C1 $\forall a, b, c \in \mathcal{A}$, if $\mathtt{Conc}(a) \equiv \mathtt{Conc}(b)$ then $a\mathcal{R}c$ iff $b\mathcal{R}c$
C2 $\forall a, b, c \in \mathcal{A}$, if $\mathtt{Supp}(a) = \mathtt{Supp}(b)$ then $c\mathcal{R}a$ iff $c\mathcal{R}b$

The first property says that two arguments having equivalent conclusions attack exactly the same arguments. The second property says that arguments having the same supports are attacked by the same arguments. In this paper, we study attack relations verifying these two properties. That is, from now on, we suppose that an attack relation verifies $C1$ and $C2$.

An argumentation system is defined as follows.

Definition 2 (Argumentation system). *An* argumentation system *(AS) built from a knowledge base Σ is a pair $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ where $\mathcal{A} \subseteq \mathtt{Arg}(\Sigma)$ and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is an attack relation which verifies $C1$ and $C2$.*

In the rest of the paper, we do *not* implicitly suppose that two arbitrary AS are built from the *same* knowledge base. We also assume that arguments are evaluated using stable semantics. Note that this is not a substantial limitation since the main purpose of this paper is to explore equivalence and strong equivalence in logic-based argumentation and not to study the subtleties of different semantics. For all the main results of this paper, similar ones can be proved for all well-known semantics.

Definition 3 (Stable semantics). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS and $\mathcal{E} \subseteq \mathcal{A}$.*

- $\mathcal{E}$ *is* conflict-free *iff* $\nexists a, b \in \mathcal{E}$ *s.t.* $a\mathcal{R}b$.
- $\mathcal{E}$ *is a* stable extension *iff* $\mathcal{E}$ *is conflict-free and attacks any argument in* $\mathcal{A} \setminus \mathcal{E}$.

Let $\mathtt{Ext}(\mathcal{F})$ *denote the set of all the stable extensions of* $\mathcal{F}$.

A status is assigned to each argument as follows.

Definition 4 (Status of arguments). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS and $a \in \mathcal{A}$.*

- a *is* skeptically accepted *iff* $\mathtt{Ext}(\mathcal{F}) \neq \emptyset$ *and* $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{F})$, $a \in \mathcal{E}$
- a *is* credulously accepted *iff* $\exists\, \mathcal{E} \in \mathtt{Ext}(\mathcal{F})$ *s.t.* $a \in \mathcal{E}$
- a *is* rejected *iff* $\nexists \mathcal{E} \in \mathtt{Ext}(\mathcal{F})$ *s.t.* $a \in \mathcal{E}$

Note that there are three possible statuses of an argument. An argument is either 1) skeptically and credulously accepted, or 2) only credulously accepted, or 3) rejected. Let $\mathtt{Status}(a, \mathcal{F})$ be a function which returns the status of an argument a in an AS $\mathcal{F}$. We assume that this function returns three different values corresponding to the three possible situations.

Property 1. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system and $a, a' \in \mathcal{A}$. If $\mathtt{Supp}(a) = \mathtt{Supp}(a')$, then $\mathtt{Status}(a, \mathcal{F}) = \mathtt{Status}(a', \mathcal{F})$.

In addition to extensions and the status of arguments, other outputs are returned by an AS. These are summarized in the next definition.

Definition 5 (Outputs of an AS). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS built over a knowledge base Σ.*

- $\mathtt{Sc}(\mathcal{F}) = \{a \in \mathcal{A} \mid$ *a is skeptically accepted* $\}$
- $\mathtt{Cr}(\mathcal{F}) = \{a \in \mathcal{A} \mid$ *a is credulously accepted* $\}$
- $\mathtt{Output}_{sc}(\mathcal{F}) = \{\mathtt{Conc}(a) \mid$ *a is skeptically accepted*$\}$
- $\mathtt{Output}_{cr}(\mathcal{F}) = \{\mathtt{Conc}(a) \mid$ *a is credulously accepted*$\}$
- $\mathtt{Bases}(\mathcal{F}) = \{\mathtt{Base}(\mathcal{E}) \mid \mathcal{E} \in \mathtt{Ext}(\mathcal{F})\}$

3 Basic Equivalence of Argumentation Systems

Three criteria for the notion of basic equivalence were proposed [9]. They compare the outputs of systems as follows. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems. The following three criteria are used:

- $\mathtt{Ext}(\mathcal{F}) = \mathtt{Ext}(\mathcal{F}')$

– $\text{Sc}(\mathcal{F}) = \text{Sc}(\mathcal{F}')$
– $\text{Cr}(\mathcal{F}) = \text{Cr}(\mathcal{F}')$

While these criteria are meaningful, they are too rigid. Let us consider two argumentation systems grounded on propositional logic. Assume that the first system has one stable extension which is $\{(\{x\}, x)\}$ while the second system has $\{(\{x\}, x \wedge x)\}$ as its unique stable extension. According to the three previous criteria, the two systems are not equivalent. In what follows, we refine the three criteria by taking into account the internal structure of arguments via a notion of equivalent arguments.

Definition 6 (Equivalent arguments). *For two arguments $a, a' \in \text{Arg}(\mathcal{L})$, a is* equivalent *to a', denoted by $a \approx a'$, iff $\text{Supp}(a) = \text{Supp}(a')$ and $\text{Conc}(a) \equiv \text{Conc}(a')$.*

Note that this relation of equivalence was also used by Gorogiannis and Hunter [8].

The following property shows that equivalent arguments w.r.t. relation $\approx$ behave in the same way w.r.t. attacks.

Property 2. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system. For all $a, a', b, b' \in \mathcal{A}$, if $a \approx a'$ and $b \approx b'$, then $a\mathcal{R}b$ iff $a'\mathcal{R}b'$.

Note that relation $\approx$ is an *equivalence relation* (i.e. reflexive, symmetric and transitive). The equivalence between two arguments is extended to equivalence between sets of arguments as follows.

Definition 7 (Equivalent sets of arguments). *Let $\mathcal{E}, \mathcal{E}' \subseteq \text{Arg}(\mathcal{L})$. $\mathcal{E}$ is* equivalent *to $\mathcal{E}'$, denoted by $\mathcal{E} \sim \mathcal{E}'$, iff $\forall a \in \mathcal{E}, \exists a' \in \mathcal{E}'$ s.t. $a \approx a'$ and $\forall a' \in \mathcal{E}', \exists a \in \mathcal{E}$ s.t. $a \approx a'$.*

We can now define a flexible notion of equivalence between argumentation systems.

Definition 8 (Equivalence between two AS). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems grounded on the same logic $(\mathcal{L}, \text{CN})$. The two systems $\mathcal{F}$ and $\mathcal{F}'$ are* EQi-equivalent *iff criterion EQi below holds:*

EQ1 $\exists f : \text{Ext}(\mathcal{F}) \to \text{Ext}(\mathcal{F}')$ *s.t. f is a bijection and $\forall \mathcal{E} \in \text{Ext}(\mathcal{F})$, $\mathcal{E} \sim f(\mathcal{E})$*
EQ2 $\text{Sc}(\mathcal{F}) \sim \text{Sc}(\mathcal{F}')$
EQ3 $\text{Cr}(\mathcal{F}) \sim \text{Cr}(\mathcal{F}')$

For two equivalent argumentation systems $\mathcal{F}$ and $\mathcal{F}'$, we will write $\mathcal{F} \equiv_{EQX} \mathcal{F}'$, with $X \in \{1, 2, 3\}$.

It is easy to show that each criterion EQi refines one criterion among those proposed by Oikarinen and Woltran.

Property 3. Let $\mathcal{F}$ and $\mathcal{F}'$ be two argumentation systems grounded on the same logic $(\mathcal{L}, \text{CN})$.

– If $\text{Ext}(\mathcal{F}) = \text{Ext}(\mathcal{F}')$, then $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$.
– If $\text{Sc}(\mathcal{F}) = \text{Sc}(\mathcal{F}')$, then $\mathcal{F} \equiv_{EQ2} \mathcal{F}'$.
– If $\text{Cr}(\mathcal{F}) = \text{Cr}(\mathcal{F}')$, then $\mathcal{F} \equiv_{EQ3} \mathcal{F}'$.

Note that the converses are not always true. We show also that when two systems are equivalent w.r.t. EQ1, then they are also equivalent w.r.t. EQ2 and EQ3. This means that criterion EQ1 is more general than the others.

Theorem 1. *Let $\mathcal{F}$ and $\mathcal{F}$ be two argumentation systems. If $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, then $\mathcal{F} \equiv_{EQ2} \mathcal{F}'$ and $\mathcal{F} \equiv_{EQ3} \mathcal{F}'$.*

It can also be checked that equivalent arguments from equivalent systems have the same status.

Theorem 2. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems. If $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, then for all $a \in \mathcal{A}$ and for all $a' \in \mathcal{A}'$, if $a \approx a'$ then $\texttt{Status}(a, \mathcal{F}) = \texttt{Status}(a', \mathcal{F}')$.*

In order to show that outputs of equivalent systems are equivalent as well, we need the following notion.

Definition 9 (Equivalent sets of formulae). *Let $X, Y \subseteq \mathcal{L}$. We say that X and Y are equivalent, denoted by $X \cong Y$, iff $\forall x \in X, \exists y \in Y$ s.t. $x \equiv y$ and $\forall y \in Y, \exists x \in X$ s.t. $x \equiv y$.*

For example, in case of the propositional logic, this allows to say that the two sets $\{x, \neg\neg y\}$ and $\{x, y\}$ are equivalent. Note that if $X \cong Y$, then $\texttt{CN}(X) = \texttt{CN}(Y)$. However, the converse is not true. For instance, $\texttt{CN}(\{x \wedge y\}) = \texttt{CN}(\{x, y\})$ while $\{x \wedge y\} \ncong \{x, y\}$. One may ask why not to use the equality of $\texttt{CN}(X)$ and $\texttt{CN}(Y)$ in order to say that X and Y are equivalent? The answer is given by the following example of two AS whose credulous conclusions are respectively $\{x, \neg x\}$ and $\{y, \neg y\}$. It is clear that $\texttt{CN}(\{x, \neg x\})$ = $\texttt{CN}(\{y, \neg y\})$ while the two sets are different.

The next result shows that if two argumentation systems are equivalent w.r.t. $EQ1$, then their sets of skeptical (credulous) conclusions are equivalent, and the bases of their extensions coincide (i.e. are the same).

Theorem 3. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS. If $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, then:*

- $\texttt{Output}_{sc}(\mathcal{F}) \cong \texttt{Output}_{sc}(\mathcal{F}')$
- $\texttt{Output}_{cr}(\mathcal{F}) \cong \texttt{Output}_{cr}(\mathcal{F}')$
- $\texttt{Bases}(\mathcal{F}) = \texttt{Bases}(\mathcal{F}')$

Since equivalent systems preserve all their important outputs, then we can exchange a given system with an equivalent one. In what follows, we show how we can take advantage of this notion of equivalence in order to reduce the number of arguments in an AS. The idea is to take exactly one argument from each equivalence class of $\mathcal{A}/\approx$. A resulting system is called *core*. Let X be a given set and $\sim$ an equivalence relation on it. For all $x \in X$, we write $[x] = \{x' \in X \mid x' \sim x\}$ and $X/\sim = \{[x] \mid x \in X\}$.

Definition 10 (Core). *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system. An argumentation system $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ is a* core *of $\mathcal{F}$ iff:*

- $\mathcal{A}' \subseteq \mathcal{A}$

– $\forall C \in \mathcal{A}/\approx,\ \ |C \cap \mathcal{A}'| = 1$
– $\mathcal{R}' = \mathcal{R}_{|\mathcal{A}'}$, *where* $\mathcal{R}_{|\mathcal{A}'} = \{(a,b) \mid (a,b) \in \mathcal{R}$ *and* $a,b \in \mathcal{A}'\}$, *i.e. the* restriction *of* $\mathcal{R}$ *on* $\mathcal{A}'$.

The fact that at least one representative of each equivalence class is included in a core allows us to show that any core of an AS is equivalent with the latter.

Theorem 4. *If* $\mathcal{F}'$ *is a core of an argumentation system* $\mathcal{F}$*, then* $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$.

We now provide a condition which guarantees that any core of any argumentation system built from a finite knowledge base is finite. This is the case for logics in which any consistent finite set of formulae has finitely many logically non-equivalent consequences. To formalize this, we use the following notation for a set of logical consequences made from consistent subsets of a given set: For any $X \subseteq \mathcal{L}$, $\mathtt{Cncs}(X) = \{x \in \mathcal{L} \mid \exists Y \subseteq X$ s.t. $\mathtt{CN}(Y) \neq \mathcal{L}$ and $x \in \mathtt{CN}(Y)\}$. We show that if $\mathtt{Cncs}(\Sigma)$ has a finite number of equivalence classes, then any core of $\mathcal{F}$ is finite (i.e. with a finite set of arguments).

Theorem 5. *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *be an argumentation system built from a finite knowledge base* Σ*. If* $\mathtt{Cncs}(\Sigma)/\equiv$ *is finite, then any core of* $\mathcal{F}$ *is finite.*

This result is of great importance since it shows that instead of working with an infinite argumentation system which is costly, one can focus only on its core which is finite. Recall that generally, logic-based argumentation systems are infinite. This is for instance the case of systems that are grounded on propositional logic.

4 Strong Equivalence of Argumentation Systems

In this section, we study strong equivalence between logic-based argumentation systems. As mentioned before, two argumentation systems are strongly equivalent iff after adding the same set of arguments to both systems, the new systems are equivalent w.r.t. any of the basic criteria given in Definition 8.

Recall that $\mathtt{Arg}(\mathcal{L})$ is the set of all arguments that can be built from a logical language $(\mathcal{L}, \mathtt{CN})$. Let $\mathcal{R}(\mathcal{L})$ be an attack relation on the set $\mathtt{Arg}(\mathcal{L})$, i.e. $\mathcal{R}(\mathcal{L}) \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. As in the first part of the paper, we assume that $\mathcal{R}(\mathcal{L})$ verifies properties $C1$ and $C2$.

Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system where $\mathcal{A} \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}(\mathcal{L})_{|\mathcal{A}}$. Augmenting $\mathcal{F}$ by an arbitrary set $\mathcal{B}$ of arguments ($\mathcal{B} \subseteq \mathtt{Arg}(\mathcal{L})$) results in a new system denoted by $\mathcal{F} \oplus \mathcal{B}$, where $\mathcal{F} \oplus \mathcal{B} = (\mathcal{A}_b, \mathcal{R}_b)$ with $\mathcal{A}_b = \mathcal{A} \cup \mathcal{B}$ and $\mathcal{R}_b = \mathcal{R}(\mathcal{L})_{|\mathcal{A}_b}$.

Definition 11 (Strong equivalence between two AS). *Let* $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ *and* $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ *be two argumentation systems built using the same logic* $(\mathcal{L}, \mathtt{CN})$*. The two systems* $\mathcal{F}$ *and* $\mathcal{F}'$ *are* EQi-strongly equivalent *iff criterion EQiS below holds:*

EQ1S $\forall \mathcal{B} \subseteq \mathtt{Arg}(\mathcal{L})$, $\mathcal{F} \oplus \mathcal{B} \equiv_{EQ1} \mathcal{F}' \oplus \mathcal{B}$
EQ2S $\forall \mathcal{B} \subseteq \mathtt{Arg}(\mathcal{L})$, $\mathcal{F} \oplus \mathcal{B} \equiv_{EQ2} \mathcal{F}' \oplus \mathcal{B}$

EQ3S $\forall \mathcal{B} \subseteq \mathtt{Arg}(\mathcal{L})$, $\mathcal{F} \oplus \mathcal{B} \equiv_{EQ3} \mathcal{F}' \oplus \mathcal{B}$.

In the remainder of the paper, we will use the terms 'strongly equivalent w.r.t. EQi' and 'equivalent w.r.t. EQiS' to denote the same thing (where $i \in \{1, 2, 3\}$).

Property 4. If two argumentation systems are strongly equivalent w.r.t. EQ1S (resp. EQ2S, EQ3S), then they are equivalent w.r.t. EQ1 (resp. EQ2, EQ3).

The following property establishes the links between the three criteria of strong equivalence.

Property 5. Let $\mathcal{F}$ and $\mathcal{F}'$ be two argumentation systems. If $\mathcal{F} \equiv_{EQ1S} \mathcal{F}'$, then $\mathcal{F} \equiv_{EQ2S} \mathcal{F}'$ and $\mathcal{F} \equiv_{EQ3S} \mathcal{F}'$.

We have already pointed out that in logic-based argumentation, there are no self-attacking arguments [1]. Formally, $\nexists a \in \mathtt{Arg}(\mathcal{L})$ such that $(a, a) \in \mathcal{R}(\mathcal{L})$. Furthermore, it was proved that if there are no self-attacking arguments, then any two argumentation systems are strongly equivalent (w.r.t. any of the three criteria used by Oikarinen and Woltran) if and only if they coincide [9]. In what follows, we show that if the structure of arguments is taken into account and if criteria are relaxed as we proposed in Definition 11, then there are cases where different systems are strongly equivalent. More precisely, we show that if $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ where $\mathcal{R}$ and $\mathcal{R}'$ are restrictions of $\mathcal{R}(\mathcal{L})$ on $\mathcal{A}$ and $\mathcal{A}'$, and if $\mathcal{A} \sim \mathcal{A}'$ then $\mathcal{F} \equiv_{EQ1S} \mathcal{F}'$.

Theorem 6. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems. If $\mathcal{A} \sim \mathcal{A}'$ then $\mathcal{F} \equiv_{EQ1S} \mathcal{F}'$.*

From the previous theorem, we conclude that if the sets of arguments of two systems are equivalent w.r.t. $\sim$, then they are also strongly equivalent w.r.t. EQ2 and EQ3.

Corollary 1. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems. If $\mathcal{A} \sim \mathcal{A}'$ then $\mathcal{F} \equiv_{EQ2S} \mathcal{F}'$ and $\mathcal{F} \equiv_{EQ3S} \mathcal{F}'$.*

As in the basic case, strong equivalence can be used in order to reduce the computational cost of an argumentation system by removing unnecessary arguments. We provide a condition under which a given argumentation system has a *finite strongly equivalent system*.

Theorem 7. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system built over a knowledge base Σ. If $\mathtt{Cncs}(\Sigma)/\equiv$ is finite, then there exists an argumentation system $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ such that $\mathcal{F} \equiv_{EQ1S} \mathcal{F}'$ and $\mathcal{A}'$ is finite.*

The following corollary follows directly.

Corollary 2. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system built over a knowledge base Σ. If $\mathtt{Cncs}(\Sigma)/\equiv$ is finite, then there exists an argumentation system $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ s.t. $\mathcal{F} \equiv_{EQ2S} \mathcal{F}'$ and $\mathcal{F} \equiv_{EQ3S} \mathcal{F}'$ and $\mathcal{A}'$ is finite.*

This result is of great importance. It shows that our criteria are useful since on the one hand, there are situations when *different* systems are equivalent, and on the other hand, our criteria allow to reduce an infinite system to a finite one.

5 Dynamics of Argument Status

Let us now show when the previous results may be used when studying dynamics of argumentation systems. The problem we are interested in is defined as follows: Given an argumentation system $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ where $\mathcal{A} \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}(\mathcal{L})_{|\mathcal{A}}$, when the status of any argument $a \in \mathcal{A}$ may evolve if a new argument $e \in \mathtt{Arg}(\mathcal{L})$ is received or if an argument $e \in \mathcal{A}$ is removed. When $\mathcal{F}$ is extended by e, the resulting system is denoted by $\mathcal{F} \oplus \{e\}$. When an argument e is removed from $\mathcal{F}$, the new system is denoted by $\mathcal{F} \ominus \{e\} = (\mathcal{A}', \mathcal{R}')$ is defined as $\mathcal{A}' = \mathcal{A} \setminus \{e\}$ and $\mathcal{R}' = \mathcal{R}(\mathcal{L})_{|\mathcal{A}'}$.

5.1 Extending an AS by New Argument(s)

Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system and $\mathtt{Base}(\mathcal{A}) = \Sigma$. By definition of $\mathcal{F}$, the set $\mathcal{A}$ is a subset of $\mathtt{Arg}(\Sigma)$ (the set of all arguments that may be built from Σ). Let $\mathcal{F}_c = (\mathtt{Arg}(\Sigma), \mathcal{R}(\mathcal{L})_{|\mathtt{Arg}(\Sigma)})$ denote the *complete* version of $\mathcal{F}$. We also say that $\mathcal{F}$ is *incomplete* iff $\mathcal{A} \subset \mathtt{Arg}(\Sigma)$. Note that, generally for reasoning over a knowledge base, a complete system is considered. However, in dialogues the exchanged arguments do not necessarily constitute a complete system. To say it differently, it may be the case that other arguments may be built using the exchanged information (the formulas of the supports of exchanged arguments).

In what follows, we show that the statuses of arguments in an incomplete system are floating in case that system does not contain a core of the complete system. However, as soon as an incomplete system is a core or contains a core of the complete system, then the status of each argument becomes fixed and will never change when a new argument from $\mathtt{Arg}(\Sigma)$ is received.

Theorem 8. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}_c = (\mathtt{Arg}(\mathtt{Base}(\mathcal{A})), \mathcal{R}(\mathcal{L})_{|\mathtt{Arg}(\mathtt{Base}(\mathcal{A}))})$ be two argumentation systems. If there exists a core $(\mathcal{A}', \mathcal{R}')$ of $\mathcal{F}_c$ s.t. $\mathcal{A}' \subseteq \mathcal{A}$, then $\forall e \in \mathtt{Arg}(\mathtt{Base}(\mathcal{A}))$ the following hold:*

- $\mathcal{F} \equiv_{EQ1} \mathcal{F} \oplus \{e\}$
- $\forall a \in \mathcal{A}$, $\mathtt{Status}(a, \mathcal{F}) = \mathtt{Status}(a, \mathcal{F} \oplus \{e\})$
- $\mathtt{Status}(e, \mathcal{F} \oplus \{e\}) = \mathtt{Status}(a, \mathcal{F})$, *where $a \in \mathcal{A}$ is any argument s.t.* $\mathtt{Supp}(a) = \mathtt{Supp}(e)$.

We now show that when a system does not contain a core of the system built over its base, new arguments may change the status of the existing ones.

Example 1. Let $(\mathcal{L}, \mathtt{CN})$ be propositional logic and let us consider the attack relation defined as follows: $\forall a, b \in \mathtt{Arg}(\mathcal{L})$, $a\mathcal{R}b$ iff $\exists h \in \mathtt{Supp}(b)$ s.t. $\mathtt{Conc}(a) \equiv \neg h$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ with $\mathcal{A} = \{a_1, a_2\}$ s.t. $a_1 = (\{x, x \to y\}, y)$ and $a_2 = (\{\neg x\}, \neg x)$. It can be checked that $a_2\mathcal{R}a_1$. Thus, a_2 is skeptically accepted while a_1 is rejected. Note that $\mathtt{Base}(\mathcal{A}) = \{x, \neg x, x \to y\}$, thus $e = (\{x\}, x) \in \mathtt{Arg}(\mathtt{Base}(\mathcal{A}))$. In the new system $\mathcal{F} \oplus \{e\}$, the two arguments both change their statuses.

The previous example illustrates a situation where an argumentation system does not contain a core of the system constructed from its base. This means that not all crucial arguments are considered in $\mathcal{F}$; thus, it is not surprising that it is possible to revise arguments' statuses.

5.2 Removing Argument(s) from an AS

We have already seen that extracting a core of an argumentation system is a compact way to represent the original system. In that process, redundant arguments are deleted from the original system. In this subsection, we show under which conditions deleting argument(s) does not influence the status of other arguments.

As one may expect, if an argument e is deleted from an argumentation system $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and if the resulting system $\mathcal{F} \ominus \{e\}$ is a core or contains a core of the complete version of $\mathcal{F}$, then all arguments in $\mathcal{A}$ keep their original status.

Theorem 9. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an argumentation system, $\mathcal{F}_c = (\mathtt{Arg}(\mathtt{Base}(\mathcal{A})), \mathcal{R}(\mathcal{L})_{|\mathtt{Arg}(\mathtt{Base}(\mathcal{A}))})$ its complete version. Let $e \in \mathcal{A}$. If $\mathcal{F} \ominus \{e\}$ contains a core of $\mathcal{F}_c$, then the following hold:*

– $\mathcal{F} \equiv_{EQ1} \mathcal{F} \ominus \{e\}$
– $\forall a \in \mathcal{A} \setminus \{e\}$, $\mathtt{Status}(a, \mathcal{F}) = \mathtt{Status}(a, \mathcal{F} \ominus \{e\})$.

It can be shown that Theorem 8 (resp. Theorem 9) is true even if a (finite or infinite) set of arguments is added (resp. deleted) to $\mathcal{F}$. In order to simplify the presentation, only results when one argument is added (deleted) were presented. The general result (when an arbitrary set of arguments is added/removed) is proved in Lemma 2 in the Appendix.

6 Conclusion

In this paper, we studied the problem of equivalence and strong equivalence between logic-based argumentation systems. While there are no works on equivalence in argumentation, previous works on strong equivalence are disappointing, since according to the proposed criteria [9] no different systems may be equivalent, the only exception being a case when systems contain self-attacking arguments, which is never a case in logical based argumentation [1]. Thus, this notion has no practical application since two different systems are never strongly equivalent.

In this paper, we have refined existing criteria and defined new ones by taking into account the structure of arguments. Since almost all applications of Dung's abstract argumentation system are obtained by constructing arguments from a given knowledge base, using a given logic, we studied the most general case in logic-based argumentation: we conducted our study for *any* logic which satisfies five basic properties proposed by Tarski [10]. We proposed flexible equivalence criteria and we showed when two systems are equivalent and strongly equivalent w.r.t. those criteria. The results show that for almost all well-known logics, even for an infinite argumentation system, it can be possible to find a finite system which is strongly equivalent to it.

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Appendix

Proof. of Property 1. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS and $a, a' \in \mathcal{A}$ such that $\texttt{Supp}(a) = \texttt{Supp}(a')$. We prove that for every stable extension $\mathcal{E}$, we have $a \in \mathcal{E}$ iff $a' \in \mathcal{E}$. Let us assume that $a \in \mathcal{E}$ and $a' \notin \mathcal{E}$. Since $\mathcal{E}$ is a stable extension, then $\exists b \in \mathcal{E}$ s.t. $b\mathcal{R}a'$. Since $\mathcal{R}$ satisfies property $C2$, then $b\mathcal{R}a$ which contradicts the fact that $\mathcal{E}$ is a stable extension. The case $a \notin \mathcal{E}$ and $a' \in \mathcal{E}$ is symmetric. This means that each extension of $\mathcal{F}$ either contains both a and a' or does not contain any of those two arguments. Consequently, the statuses of those arguments must coincide.

Proof. of Property 2. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS and $a, a', b, b' \in \mathcal{A}$ such that $a \approx a'$ and $b \approx b'$. Assume that $a\mathcal{R}b$. Since $\texttt{Supp}(b) = \texttt{Supp}(b')$ then from $C2$, it follows that $a\mathcal{R}b'$. From $C1$ and the fact that $\texttt{Conc}(a) \equiv \texttt{Conc}(a')$, we get $a'\mathcal{R}b'$. To show that $a'\mathcal{R}b'$ implies $a\mathcal{R}b$ is similar.

Proof. of Theorem 1. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS such that $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$.

– Let us prove that $\texttt{Sc}(\mathcal{F}) \sim \texttt{Sc}(\mathcal{F}')$. If $\texttt{Ext}(\mathcal{F}) = \emptyset$, then from $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, $\texttt{Ext}(\mathcal{F}') = \emptyset$. In this case, $\texttt{Sc}(\mathcal{F}) \sim \texttt{Sc}(\mathcal{F}')$ holds trivially, since $\texttt{Sc}(\mathcal{F}) = \texttt{Sc}(\mathcal{F}') = \emptyset$. Assume now that $\texttt{Ext}(\mathcal{F}) \neq \emptyset$.
  Let $\texttt{Sc}(\mathcal{F}) = \emptyset$. We will prove that $\texttt{Sc}(\mathcal{F}') = \emptyset$. Suppose the contrary and let $a' \in \texttt{Sc}(\mathcal{F}')$. Let $\mathcal{E}' \in \texttt{Ext}(\mathcal{F}')$. Argument a' is skeptically accepted, thus $a' \in \mathcal{E}'$. Let f be a bijection from $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, and let us denote $\mathcal{E} = f^{-1}(\mathcal{E}')$. From $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, we obtain $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$. Furthermore, $\mathcal{E} \sim \mathcal{E}'$, and, consequently, $\exists a \in \mathcal{E}$ s.t. $a \approx a'$. Theorem 2 implies that a is skeptically accepted in $\mathcal{F}$, contradiction.
  Let $\texttt{Sc}(\mathcal{F}) \neq \emptyset$ and let $a \in \texttt{Sc}(\mathcal{F})$. Since $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, and a is in at least one extension, then $\exists a' \in \mathcal{A}'$ s.t. $a' \approx a$. From $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$ and from Theorem 2, a' is skeptically accepted in $\mathcal{F}'$. Thus $\forall a \in \texttt{Sc}(\mathcal{F})$, $\exists a' \in \texttt{Sc}(\mathcal{F}')$ s.t. $a' \approx a$. To prove

that $\forall a' \in \texttt{Sc}(\mathcal{F}')$, $\exists a \in \texttt{Sc}(\mathcal{F})$ s.t. $a \approx a'$ is similar.

- We can easily see that $\texttt{Ext}(\mathcal{F}) = \emptyset$ iff $\texttt{Ext}(\mathcal{F}') = \emptyset$ and that $\texttt{Ext}(\mathcal{F}) = \{\emptyset\}$ iff $\texttt{Ext}(\mathcal{F}') = \{\emptyset\}$. Let $a \in \texttt{Cr}(\mathcal{F})$. We prove that $\exists a' \in \texttt{Cr}(\mathcal{F}')$ s.t. $a \approx a'$. Since $a \in \texttt{Cr}(\mathcal{F})$ then $\exists \mathcal{E} \in \texttt{Ext}(\mathcal{F})$ s.t. $a \in \mathcal{E}$. Let f be a bijection between from $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$ and let $\mathcal{E}' = f(\mathcal{E})$. From $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, we obtain that $\mathcal{E} \sim \mathcal{E}'$, thus $\exists a' \in \mathcal{E}'$ s.t. $a \approx a'$. This means that $\forall x \in \texttt{Cr}(\mathcal{F})$, $\exists x' \in \texttt{Cr}(\mathcal{F}')$ such that $x \approx x'$. To prove that $\forall a' \in \texttt{Cr}(\mathcal{F}')$, $\exists a \in \texttt{Cr}(\mathcal{F})$ such that $a \approx a'$ is similar. Thus, $\texttt{Cr}(\mathcal{F}) \sim \texttt{Cr}(\mathcal{F}')$.

Proof. of Theorem 2. If $\mathcal{F}$ has no extensions, then all arguments in $\mathcal{F}$ and $\mathcal{F}'$ are rejected. Thus, in the rest of the proof, we study the case when $\texttt{Ext}(\mathcal{F}) \neq \emptyset$. We will prove that for any extension $\mathcal{E}$ of $\mathcal{F}$, $a \in \mathcal{E}$ iff $a' \in f(\mathcal{E})$, where $f : \texttt{Ext}(\mathcal{F}) \to \texttt{Ext}(\mathcal{F}')$ is a bijection s.t. $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{F})$, $\mathcal{E} \sim f(\mathcal{E})$. Let $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$, let $a \in \mathcal{E}$ and let $a' \in \mathcal{A}'$ with $a \approx a'$. Let $\mathcal{E}' = f(\mathcal{E})$; we will prove that $a' \in \mathcal{E}'$. From $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, one obtains $\exists a'' \in \mathcal{E}'$ s.t. $a \approx a''$. (Note that we do not know whether $a' = a''$ or not.) We will prove that $\{a'\} \cup \mathcal{E}'$ is conflict-free. Let us suppose the contrary. This means that $\exists x \in \mathcal{E}'$ s.t. $x\mathcal{R}'a'$ or $a'\mathcal{R}'x$. From $(x\mathcal{R}'a' \vee a'\mathcal{R}'a')$, we have $(x\mathcal{R}'a'' \vee a''\mathcal{R}'x)$, which contradicts the fact that $\mathcal{E}'$ is a stable extension. We conclude that $\{a'\} \cup \mathcal{E}'$ is conflict-free. Since $\mathcal{E}'$ is a stable extension, it attacks any argument $y \notin \mathcal{E}'$. Since $\mathcal{E}'$ does not attack a', then $a' \in \mathcal{E}'$.

This means that we showed that for any $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$, if $a \in \mathcal{E}$ then $a' \in f(\mathcal{E})$. Let $a \notin \mathcal{E}$ and let us prove that $a' \notin f(\mathcal{E})$. Suppose the contrary, i.e. suppose that $a' \in f(\mathcal{E})$. Since we made exactly the same hypothesis on $\mathcal{F}$ and $\mathcal{F}'$, by using the same reasoning as in the first part of the proof, we can prove that $a \in \mathcal{E}$, contradiction. This means that $a' \notin f(\mathcal{E})$. So, we proved that for any extension $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$, we have $a \in \mathcal{E}$ iff $a' \in f(\mathcal{E})$.

If a is skeptically accepted, then for any $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$, $a \in \mathcal{E}$. Let $\mathcal{E}' \in \texttt{Ext}(\mathcal{F}')$. Then, from $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$, there exists $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$ s.t. $\mathcal{E}' = f(\mathcal{E})$. Since $a \in \mathcal{E}$, then $a' \in \mathcal{E}'$. If a is not skeptically accepted, then $\exists \mathcal{E} \in \texttt{Ext}(\mathcal{F})$ s.t. $a \notin \mathcal{E}$. It is clear that $\mathcal{E}' = f(\mathcal{E})$ is an extension of $\mathcal{F}'$ and that $a' \notin \mathcal{E}'$. Thus, in this case a' is not skeptically accepted in $\mathcal{F}'$.

Let a be credulously accepted in $\mathcal{F}$ and let $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$ be an extension s.t. $a \in \mathcal{E}$. Then, $a' \in f(\mathcal{E})$, thus a' is credulously accepted in $\mathcal{F}'$. It is easy to see that the case when a is not credulously accepted in $\mathcal{F}$ and a' is credulously accepted in $\mathcal{F}'$ is not possible.

If a is rejected in $\mathcal{F}$, then a is not credulously accepted, thus a' is not credulously accepted which means that it is rejected.

Proof. of Theorem 3. Let $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$. From Theorem 1, we obtain $\mathcal{F} \equiv_{EQ2} \mathcal{F}'$ and $\mathcal{F} \equiv_{EQ3} \mathcal{F}'$. It is easy to see that this implies $\texttt{Output}_{sc}(\mathcal{F}) \cong \texttt{Output}_{sc}(\mathcal{F}')$ and $\texttt{Output}_{cr}(\mathcal{F}) \cong \texttt{Output}_{cr}(\mathcal{F}')$. Considering the third part of the theorem, let f be a bijection from $\mathcal{F} \equiv \mathcal{F}'$, let $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$ and $\mathcal{E}' = f(\mathcal{E})$. One can easily check that $\texttt{Base}(\mathcal{E}) = \texttt{Base}(\mathcal{E}')$. This means that $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{F})$, $\exists \mathcal{E}' \in \texttt{Ext}(\mathcal{F}')$ s.t. $\texttt{Base}(\mathcal{E}) = \texttt{Base}(\mathcal{E}')$. To see that $\forall \mathcal{E}' \in \texttt{Ext}(\mathcal{F}')$, $\exists \mathcal{E} \in \texttt{Ext}(\mathcal{F})$ s.t. $\texttt{Base}(\mathcal{E}) = \texttt{Base}(\mathcal{E}')$ is similar. Consequently, $\texttt{Bases}(\mathcal{F}) = \texttt{Bases}(\mathcal{F}')$.

Lemma 1. *Let $\mathcal{R}(\mathcal{L}) \subseteq \texttt{Arg}(\mathcal{L}) \times \texttt{Arg}(\mathcal{L})$ be an attack relation on the set of all arguments built from $\mathcal{L}$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS such that $\mathcal{A}, \mathcal{A}' \subseteq \texttt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}(\mathcal{L})_{|\mathcal{A}}$, $\mathcal{R}' = \mathcal{R}(\mathcal{L})_{|\mathcal{A}'}$. If $\mathcal{A} \sim \mathcal{A}'$, then $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$.*

Proof. Let us first suppose that $\texttt{Ext}(\mathcal{F}) \neq \emptyset$ and let us define the function $f' : 2^{\mathcal{A}} \to 2^{\mathcal{A}'}$ as follows: $f'(B) = \{a' \in \mathcal{A}' \mid \exists a \in B \text{ s.t. } a' \approx a\}$.

Let f be the restriction of f' to $\texttt{Ext}(\mathcal{F})$. We will prove that the image of this function is $\texttt{Ext}(\mathcal{F}')$ and that f is a bijection between $\texttt{Ext}(\mathcal{F})$ and $\texttt{Ext}(\mathcal{F}')$ which verifies EQ1.

- First, we will prove that for any $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$, $f(\mathcal{E}) \in \texttt{Ext}(\mathcal{F}')$. Let $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$ and let $\mathcal{E}' = f(\mathcal{E})$. We will prove that $\mathcal{E}'$ is conflict-free. Let $a', b' \in \mathcal{E}'$. There must exist $a, b \in \mathcal{E}$ s.t. $a \approx a'$ and $b \approx b'$. Since $\mathcal{E}$ is an extension, $\neg(a\mathcal{R}b)$ and $\neg(b\mathcal{R}a)$. By applying Property 2 on $(\texttt{Arg}(\mathcal{L}), \mathcal{R}(\mathcal{L}))$, we have that $\neg(a'\mathcal{R}'b')$ and $\neg(b'\mathcal{R}'a')$. Let $x' \in \mathcal{A}' \setminus \mathcal{E}'$. Then $\exists x \in \mathcal{A}$ s.t. $x \approx x'$. Note also that it must be that $x \notin \mathcal{E}$. Since $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$, then $\exists y \in \mathcal{E}$ s.t. $y\mathcal{R}x$. Note that $\exists y' \in \mathcal{E}'$ s.t. $y' \approx y$. From Property 2, $y'\mathcal{R}'x'$.
- We have shown that the image of f is the set $\texttt{Ext}(\mathcal{F}')$. We will now prove that $f : \texttt{Ext}(\mathcal{F}) \to \texttt{Ext}(\mathcal{F}')$ is injective. Let $\mathcal{E}_1, \mathcal{E}_2 \in \texttt{Ext}(\mathcal{F})$ with $\mathcal{E}_1 \neq \mathcal{E}_2$ and $\mathcal{E}' = f(\mathcal{E}_1) = f(\mathcal{E}_2)$. We will show that if $\mathcal{E}_1 \sim \mathcal{E}_2$ then $\mathcal{E}_1 = \mathcal{E}_2$. Without loss of generality, let $\exists x \in \mathcal{E}_1 \setminus \mathcal{E}_2$. Then, from $\mathcal{E}_1 \sim \mathcal{E}_2$, $\exists x' \in \mathcal{E}_2$, s.t. $x' \approx x$. Then, since $x \in \mathcal{E}_1$ and $x \notin \mathcal{E}_2$, from the proof of Property 1 we obtain that $x' \in \mathcal{E}_1$ and $x' \notin \mathcal{E}_2$. Contradiction with $x' \in \mathcal{E}_2$. This means that $\neg(\mathcal{E}_1 \sim \mathcal{E}_2)$. Without loss of generality, $\exists a_1 \in \mathcal{E}_1 \setminus \mathcal{E}_2$ s.t. $\nexists a_2 \in \mathcal{E}_2$ s.t. $a_1 \approx a_2$. Let $a' \in \mathcal{A}'$ s.t. $a' \approx a_1$. Recall that $\mathcal{E}' = f(\mathcal{E}_2)$. Thus, $\exists a_2 \in \mathcal{E}_2$ s.t. $a_2 \approx a'$. Contradiction.
- We show that $f : \texttt{Ext}(\mathcal{F}) \to \texttt{Ext}(\mathcal{F}')$ is surjective. Let $\mathcal{E}' \in \texttt{Ext}(\mathcal{F}')$, and let us show that $\exists \mathcal{E} \in \texttt{Ext}(\mathcal{F})$ s.t. $\mathcal{E}' = f(\mathcal{E})$. Let $\mathcal{E} = \{a \in \mathcal{A} \mid \exists a' \in \mathcal{E}' \text{ s.t. } a \approx a'\}$. From Property 2 we see that $\mathcal{E}$ is conflict-free. For any $b \in \mathcal{A} \setminus \mathcal{E}$, $\exists b' \in \mathcal{A}' \setminus \mathcal{E}'$ s.t. $b \approx b'$. Since $\mathcal{E}' \in \texttt{Ext}(\mathcal{F}')$, then $\exists a' \in \mathcal{E}'$ s.t. $a'\mathcal{R}'b'$. Now, $\exists a \in \mathcal{E}$ s.t. $a \approx a'$; from Property 2, $a\mathcal{R}b$. Thus, $\mathcal{E}$ is a stable extension in $\mathcal{F}$.
- We will now show that $f : \texttt{Ext}(\mathcal{F}) \to \texttt{Ext}(\mathcal{F}')$ verifies the condition of EQ1. Let $\mathcal{E} \in \texttt{Ext}(\mathcal{F})$ and $\mathcal{E}' = f(\mathcal{E})$. Let $a \in \mathcal{E}$. Then, $\exists a' \in \mathcal{A}'$ s.t. $a' \approx a$. From the definition of f, it must be that $a' \in \mathcal{E}'$. Similarly, if $a' \in \mathcal{E}'$, then must be an argument $a \in \mathcal{A}$ s.t. $a \approx a'$, and again from the definition of the function f, we conclude that $a \in \mathcal{E}$.

From all above, we conclude that $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$. Let us take a look at the case when $\texttt{Ext}(\mathcal{F}) = \emptyset$. We will show that $\texttt{Ext}(\mathcal{F}') = \emptyset$. Suppose not and let $\mathcal{E}' \in \texttt{Ext}(\mathcal{F}')$. Let us define $\mathcal{E} = \{a \in \mathcal{A} \mid \exists a' \in \mathcal{E}' \text{ s.t. } a \approx a'\}$. From Property 2, $\mathcal{E}$ must be conflict-free. The same property shows that for any $b \in \mathcal{A} \setminus \mathcal{E}$, $\exists a \in \mathcal{E}$ s.t. $a\mathcal{R}b$. Thus, $\mathcal{E}$ is a stable extension in $\mathcal{F}$. Contradiction with the hypothesis that $\texttt{Ext}(\mathcal{F}) = \emptyset$.

Proof. of Theorem 4. The result is obtained by applying Lemma 1 on $\mathcal{F}$ and $\mathcal{F}'$.

Proof. of Theorem 5. Let $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be a core of $\mathcal{F}$ and let us prove that $\mathcal{F}'$ is finite. Since Σ is finite, then $\{\texttt{Supp}(a) \mid a \in \mathcal{A}'\}$ must be finite. If for all $H \in \{\texttt{Supp}(a) \mid a \in \mathcal{A}'\}$, the set $\{a \in \mathcal{A}' \mid \texttt{Supp}(a) = H\}$, is finite, then the set $\mathcal{A}'$ is clearly finite. Else, there exists $H_0 \in \{\texttt{Supp}(a) \mid a \in \mathcal{A}'\}$, s.t. the set $\mathcal{A}_{H_0} = \{a \in \mathcal{A}' \mid \texttt{Supp}(a) = H_0\}$ is infinite. By the definition of $\mathcal{A}'$, one obtains that $\forall a, b \in \mathcal{A}_{H_0}$, $\texttt{Conc}(a) \not\equiv \texttt{Conc}(b)$. It is clear that $\forall a \in \mathcal{A}_{H_0}$, $\texttt{Conc}(a) \in \texttt{Cncs}(\Sigma)$. This implies that there are infinitely many different formulae having logically non-equivalent conclusions in $\texttt{Cncs}(\Sigma)$, formally, set $\texttt{Cncs}(\Sigma)/\equiv$ is infinite, contradiction.

Proof. of Theorem 6. Let $\mathcal{B} \subseteq \texttt{Arg}(\mathcal{L})$. Since $\mathcal{A} \sim \mathcal{A}'$ then clearly $\mathcal{A} \cup \mathcal{B} \sim \mathcal{A}' \cup \mathcal{B}$. From Lemma 1, we obtain that $\mathcal{F} \oplus \mathcal{B} \equiv_{EQ1} \mathcal{F}' \oplus \mathcal{B}$. Thus, $\mathcal{F} \equiv_{EQ1S} \mathcal{F}'$.

Proof. of Theorem 7. Let $\mathcal{A}' \subseteq \mathcal{A}$ be a set defined as follows: $\forall a \in \mathcal{A}\ \exists! a' \in \mathcal{A}'$ s.t. $a' \approx a$. It is clear that $\mathcal{F}' = (\mathcal{A}', \mathcal{R}' = \mathcal{R}_{|\mathcal{A}'})$ is a core of $\mathcal{F}$. Since $\mathcal{A} \sim \mathcal{A}'$, then from Theorem 6, $\mathcal{F} \equiv_{EQ1S} \mathcal{F}'$. From Theorem 5, $\mathcal{F}'$ is finite.

Lemma 2. *Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS built from Σ which contains a core of $\mathcal{G} = (\mathcal{A}_g = \texttt{Arg}(\Sigma), \mathcal{R}_g = \mathcal{R}(\mathcal{L})|_{\mathcal{A}_g})$ and let $\mathcal{E} \subseteq \texttt{Arg}(\Sigma)$. Then:*

- $\mathcal{F} \equiv_{EQ1} \mathcal{F} \oplus \mathcal{E}$
- $\forall a \in \mathcal{A}, \texttt{Status}(a, \mathcal{F}) = \texttt{Status}(a, \mathcal{F} \oplus \mathcal{E})$
- $\forall e \in \mathcal{E} \setminus \mathcal{A}, \texttt{Status}(e, \mathcal{F} \oplus \mathcal{E}) = \texttt{Status}(a, \mathcal{F})$*, where $a \in \mathcal{A}$ is any argument s.t.* $\texttt{Supp}(a) = \texttt{Supp}(e)$.

Proof. Let $\mathcal{F}' = \mathcal{F} \oplus \mathcal{E}$ with $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ and let $\mathcal{H} = (\mathcal{A}_h, \mathcal{R}_h)$ be a core of $\mathcal{G}$ s.t. $\mathcal{A}_h \subseteq \mathcal{A}$. We will first show that $\mathcal{H}$ is a core of both $\mathcal{F}$ and $\mathcal{F}'$. Let us first show that $\mathcal{H}$ is a core of $\mathcal{F}$. We will show that all conditions of Definition 10 are verified.

- From what we supposed, we have that $\mathcal{A}_h \subseteq \mathcal{A}$.
- We will show that $\forall a \in \mathcal{A}\ \exists! a' \in \mathcal{A}_h$ s.t. $a' \approx a$. Let $a \in \mathcal{A}$. Since $a \in \mathcal{A}_g$ and $\mathcal{H}$ is a core of $\mathcal{G}$, then $\exists! a' \in \mathcal{A}_h$ s.t. $a' \approx a$.
- Since $\mathcal{R} = \mathcal{R}(\mathcal{L})|_{\mathcal{A}}$ and $\mathcal{R}_h = \mathcal{R}(\mathcal{L})|_{\mathcal{A}_h}$ then from $\mathcal{A}_h \subseteq \mathcal{A}$ we obtain that $\mathcal{R}_h = \mathcal{R}|_{\mathcal{A}_h}$.

Thus, $\mathcal{H}$ is a core of $\mathcal{F}$. Let us now show that $\mathcal{H}$ is also a core of $\mathcal{F}'$:

- Since $\mathcal{A}_h \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathcal{A}'$ then $A_h \subseteq \mathcal{A}'$.
- Let $a \in \mathcal{A}'$. Since $a \in \mathcal{A}_g$ and $\mathcal{H}$ is a core of system $\mathcal{G}$, then $\exists! a' \in \mathcal{A}_h$ s.t. $a' \approx a$.
- Since $\mathcal{R}' = \mathcal{R}(\mathcal{L})|_{\mathcal{A}'}$, $\mathcal{R}_h = \mathcal{R}(\mathcal{L})|_{\mathcal{A}_h}$ and $\mathcal{A}_h \subseteq \mathcal{A}'$, then we obtain that $\mathcal{R}_h = \mathcal{R}'|_{\mathcal{A}_h}$.

We have shown that $\mathcal{H}$ is a core of $\mathcal{F}$ and of $\mathcal{F}'$. From Theorem 4, $\mathcal{F} \equiv_{EQ1} \mathcal{H}$ and $\mathcal{F}' \equiv_{EQ1} \mathcal{H}$. Since $\equiv$ is an equivalence relation, then $\mathcal{F} \equiv_{EQ1} \mathcal{F}'$. Let $a \in \mathcal{A}$. From Theorem 2, $\texttt{Status}(a, \mathcal{F}) = \texttt{Status}(a, \mathcal{F}')$.

Let $e \in \mathcal{A}' \setminus \mathcal{A}$ and let $a \in \mathcal{A}$ be an argument such that $\texttt{Supp}(a) = \texttt{Supp}(e)$. From Property 1, we obtain $\texttt{Status}(e, \mathcal{F}') = \texttt{Status}(a, \mathcal{F}')$. Since we have just seen that $\texttt{Status}(a, \mathcal{F}') = \texttt{Status}(a, \mathcal{F})$, then $\texttt{Status}(e, \mathcal{F}') = \texttt{Status}(a, \mathcal{F})$.

Proof. of Theorem 8. This result is a consequence of Lemma 2.

Proof. of Theorem 9. This result is a consequence of Lemma 2.