Basic Equivalence in Logic-Based Argumentation

Leila Amgoud
IRIT - CNRS
118, route de Narbonne,
31062 Toulouse Cedex 9 France
amgoud@irit.fr

Srdjan Vesic
IRIT - Université de Toulouse
118, route de Narbonne,
31062 Toulouse Cedex 9 France
vesic@irit.fr

***Abstract*—An argumentation system consists mainly of a set of interacting arguments and a semantics for evaluating them. In this paper, we study when two argumentation systems are *equivalent*. We focus on argumentation systems defined around a Tarskian logic. We propose different equivalence criteria, study their links and finally show under which conditions two systems are equivalent wrt each of the proposed criteria.**

***Keywords*-Argumentation systems; Equivalence criteria;**

I. INTRODUCTION

Argumentation is a reasoning process in which interacting arguments are built and evaluated using a semantics. It is widely studied in Artificial Intelligence, namely for reasoning about defeasible information and modeling agents interactions ([1]).

The most abstract argumentation system was proposed by Dung ([4]). It takes as inputs a set of arguments and a binary relation encoding attacks among them, and returns acceptable subsets of arguments, called extensions. Knowing when two such systems are *equivalent* is important. For instance, when building an argumentation system over a given knowledge base, it is very common that several attack relations may be used. Thus, knowing which relations induce equivalent systems may be useful. Under some logics, like propositional logic, an infinite number of arguments is built from (a finite) knowledge base. It would be convenient to know whether such a system can be exchanged with an equivalent finite sub-system.

Recently, a study on when two Dung's abstract systems are *equivalent* has been carried out and three particular *criteria* have been proposed ([5]). According to those criteria, two systems are equivalent if they return exactly the same extensions (resp. skeptical or credulous arguments) even after being extended by the same arbitrary set of arguments. While these criteria are meaningful, they are too rigid that generally two systems are equivalent only if they are equal. Let us consider two argumentation systems built under propositional logic: the first system has the set $\{(\{x\}, x)\}$ as its unique extension while the second system has $\{(\{x\}, x \wedge x)\}$ as extension. These two systems will never be considered as equivalent by the two above criteria since the two arguments $(\{x\}, x)$ and $(\{x\}, x \wedge x)$ are considered different. However, they are almost the same and they return logically equivalent results. Thus, in order to define more accurately the notion of equivalence between two systems, the structure of arguments should be taken into account. In a recent paper ([3]), we focused on logic-based instantiations of Dung's argumentation system, particularly those defined under a Tarskian logic ([6]). We proposed three criteria of equivalence which are themselves based on a particular relation of equivalence between arguments.

In this paper, we extend that work and propose new criteria of equivalence. Some of them are based on two new relations of equivalence between arguments while others are defined on other intuitive outputs of an argumentation system. We study the interdependencies between the criteria and identify conditions under which two argumentation systems are equivalent wrt a given equivalence criterion. We focus on two particular cases: the case of infinite systems to which we identify equivalent finite sub-systems, and the case of two argumentation systems which may be built from distinct knowledge bases but use the same definition of attack relation (like rebut, undercut, ...).

The paper is structured as follows: Section II introduces the logic-based systems that will be studied. Section III defines different equivalence criteria whose links are investigated in Section IV. Section V shows under which conditions two systems are equivalent wrt each of the proposed criteria. The last section concludes. Due to space limitation, only some proofs are included in the paper.

II. LOGIC-BASED SYSTEMS

This section describes the logical argumentation systems we are interested in. They use Tarski's monotonic logic in order to build the two components of a system: the arguments and the attacks among them.

According to Tarski ([6]), a *monotonic logic* is a pair $(\mathcal{L}, \mathsf{CN})$ where $\mathcal{L}$ is a set of *well-formed formulae* and CN is a *consequence operator*, i.e. a function from $2^{\mathcal{L}}$ to $2^{\mathcal{L}}$ that satisfies the following axioms:

- $X \subseteq \mathsf{CN}(X)$ **(Expansion)**
- $\mathsf{CN}(\mathsf{CN}(X)) = \mathsf{CN}(X)$ **(Idempotence)**

- $\mathsf{CN}(X) = \bigcup_{Y \subseteq_f X} \mathsf{CN}(Y)$[1] **(Finiteness)**
- $\mathsf{CN}(\{x\}) = \mathcal{L}$ for some $x \in \mathcal{L}$ **(Absurdity)**
- $\mathsf{CN}(\emptyset) \neq \mathcal{L}$ **(Coherence)**

Intuitively, $\mathsf{CN}(X)$ returns the set of formulae that are logical consequences of X according to the logic at hand. Almost all well-known logics (classical logic, intuitionistic logic, modal logics etc.) are special cases of Tarski's notion of monotonic logic. In such a logic, a set X of formulae is *consistent* iff its set of consequences is not the whole set $\mathcal{L}$.

Arguments are built from a *knowledge base* Σ which is a finite subset of $\mathcal{L}$.

Definition 1 (Argument): An *argument* is a pair (X, x) s.t. $X \subseteq \Sigma$, X is consistent, $x \in \mathsf{CN}(X)$, and $\nexists X' \subset X$ s.t. $x \in \mathsf{CN}(X')$. X is the *support* of the argument and x its *conclusion*.

Notations: For an argument $a = (X, x)$, $\texttt{Conc}(a) = x$ and $\texttt{Supp}(a) = X$. For a set $\mathcal{S} \subseteq \mathcal{L}$, $\texttt{Arg}(\mathcal{S}) = \{a \mid a$ is an argument (in the sense of Definition 1) and $\texttt{Supp}(a) \subseteq \mathcal{S}\}$. For any $\mathcal{E} \subseteq \texttt{Arg}(\mathcal{L})$, $\texttt{Base}(\mathcal{E}) = \bigcup_{a \in \mathcal{E}} \texttt{Supp}(a)$.

An argumentation system defined over a given knowledge base is defined as follows.

Definition 2 (Argumentation system): An *argumentation system* (AS) defined over a knowledge base Σ is a pair $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ where $\mathcal{A} \subseteq \texttt{Arg}(\Sigma)$ and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is an attack relation. The writing $a\mathcal{R}b$ means that argument a *attacks* argument b.

In the previous definition, we did not consider the whole set $\texttt{Arg}(\Sigma)$ of arguments that may be built from Σ. The reason is that we are looking for equivalent systems. Thus, we may be interested by a sub-system which is equivalent to the global system (i.e. the one using $\texttt{Arg}(\Sigma)$). We may also need to compare two sub-systems of the global one.

Recently, a study on how to choose an appropriate attack relation has been carried out ([2]). Some basic properties of an attack relation have also been discussed. Examples of such properties are recalled below.

C1	$\forall a, b, c \in \mathcal{A}$, if $\texttt{Conc}(a) = \texttt{Conc}(b)$ then $(a\mathcal{R}c$ iff $b\mathcal{R}c)$
C1'	$\forall a, b, c \in \mathcal{A}$, if $\texttt{Conc}(a) \equiv \texttt{Conc}(b)$ then $(a\mathcal{R}c$ iff $b\mathcal{R}c)$
C2	$\forall a, b, c \in \mathcal{A}$, if $\texttt{Supp}(a) = \texttt{Supp}(b)$ then $(c\mathcal{R}a$ iff $c\mathcal{R}b)$
C2'	$\forall a, b, c \in \mathcal{A}$, if $\texttt{Supp}(a) \equiv \texttt{Supp}(b)$ then $(c\mathcal{R}a$ iff $c\mathcal{R}b)$

The two first properties say that two arguments having the same (resp. equivalent) conclusions attack exactly the same set of arguments. The two remaining properties say that arguments having the same (resp. equivalent) supports are attacked by the same set of arguments.

Property 1: Let $\mathcal{R}$ be an attack relation.
- If $\mathcal{R}$ satisfies $C1'$ then it satisfies $C1$.
- If $\mathcal{R}$ satisfies $C2'$ then it satisfies $C2$.

If not mentioned otherwise, we do not assume that the attack relation enjoys the above properties. Arguments are evaluated using stable semantics. Note that this is not a substantial limitation since the main purpose of this paper is to explore general ways to define equivalence in logical argumentation and not to study the subtleties of different semantics. For all the main results of this paper, similar ones can be proved for all well-known semantics.

Definition 3 (Acceptability semantics): Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS and $\mathcal{E} \subseteq \mathcal{A}$.
- $\mathcal{E}$ is *conflict-free* iff $\nexists a, b \in \mathcal{E}$ s.t. $a\mathcal{R}b$.
- $\mathcal{E}$ is a *stable extension* iff it is a conflict-free set that attacks wrt $\mathcal{R}$ all arguments in $\mathcal{A} \setminus \mathcal{E}$.

Let $\texttt{Ext}(\mathcal{F})$ denote the set of all extensions of $\mathcal{F}$.

In general, an argumentation system may have an infinite number of extensions even if the knowledge base Σ is finite. Let us consider the following example.

Example 1: Let $(\mathcal{L}, \mathsf{CN})$ be a Tarski's logic s.t. $\mathcal{L} = \{x_0, x_1, x_2, \ldots\}$, $\mathsf{CN}(\emptyset) = \emptyset$ and $\forall X \subseteq \mathcal{L}$, if $X \neq \emptyset$ then $\mathsf{CN}(X) = \{x_i, x_{i+1}, x_{i+2}, \ldots\}$, where i is the minimal number s.t. $x_i \in X$. Let $\Sigma = \{x_1\}$ and $\mathcal{A} = \texttt{Arg}(\Sigma)$. Assume that $a\mathcal{R}b$ iff $\texttt{Conc}(a) \neq \texttt{Conc}(b)$. It is clear that there are infinitely many stable extensions: $\{(\{x_1\}, x_1)\}$, $\{(\{x_1\}, x_2)\}$, $\{(\{x_1\}, x_3)\}, \ldots$.

The following result shows that under some reasonable conditions, an argumentation system built from a knowledge base Σ has a finite number of extensions.

Proposition 1: Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system built from Σ. If Σ is finite and $\mathcal{R}$ satisfies $C2$, then $(\mathcal{A}, \mathcal{R})$ has a finite number of extensions.

A status is assigned to each argument as follows.

Definition 4 (Status of arguments): Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS and $a \in \mathcal{A}$.
- a is *skeptically accepted* iff $\texttt{Ext}(\mathcal{F}) \neq \emptyset$ and $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{F})$, $a \in \mathcal{E}$
- a is *credulously accepted* iff $\exists\, \mathcal{E} \in \texttt{Ext}(\mathcal{F})$ s.t. $a \in \mathcal{E}$
- a is *rejected* iff $\nexists \mathcal{E} \in \texttt{Ext}(\mathcal{F})$ s.t. $a \in \mathcal{E}$

Any sceptical argument is also credulous. However, there are exactly three disjunct cases, since an argument can be: i) sceptical (and credulous), ii) credulous and not sceptical iii) rejected. Let $\texttt{Status}(a, \mathcal{F})$ be a function which returns the status of an argument a in argumentation system $\mathcal{F}$. This function simply returns three different values in those three disjunct cases. In addition to extensions and the status of arguments, other outputs are returned by an AS. These are summarized in the next definition.

Definition 5 (Outputs of an AS): Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ be an AS defined over a knowledge base Σ.

[1]The notation $Y \subseteq_f X$ means that Y is a finite subset of X.

- $\texttt{Sc}(\mathcal{F}) = \{a \in \mathcal{A} \mid$ a is skeptically accepted $\}$
- $\texttt{Cr}(\mathcal{F}) = \{a \in \mathcal{A} \mid$ a is credulously accepted $\}$
- $\texttt{Output}_{sc}(\mathcal{F}) = \{\texttt{Conc}(a) \mid$ a is skeptically accepted $\}$
- $\texttt{Output}_{cr}(\mathcal{F}) = \{\texttt{Conc}(a) \mid$ a is credulously accepted $\}$
- $\texttt{Bases}(\mathcal{F}) = \{\texttt{Base}(\mathcal{E}) \mid \mathcal{E} \in \texttt{Ext}(\mathcal{F})\}$

The first four sets contain the skeptically and credulously accepted arguments (resp. conclusions). $\texttt{Bases}(\mathcal{F})$ contains the subbases of Σ which are returned by the extensions of $\mathcal{F}$.

III. EQUIVALENCE CRITERIA

Throughout this section, we assume a fixed Tarskian logic $(\mathcal{L}, \mathsf{CN})$ and two arbitrary argumentation systems $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ that are defined using this logic. Note that the two systems may be built from different knowledge bases. We study when $\mathcal{F}$ and $\mathcal{F}'$ are equivalent. We propose two families of *equivalence criteria*. The first family compares directly the outputs of the two systems while the second family takes advantage of similarities between arguments and logical equivalence between formulae. The following definition introduces the criteria of the first family.

Definition 6 (Equivalence criteria): Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built using the same Tarski's logic $(\mathcal{L}, \mathsf{CN})$. The two systems $\mathcal{F}$ and $\mathcal{F}'$ are *EQi-equivalent* iff criterion EQi below holds:

EQ1: $\texttt{Ext}(\mathcal{F}) = \texttt{Ext}(\mathcal{F}')$
EQ2: $\texttt{Sc}(\mathcal{F}) = \texttt{Sc}(\mathcal{F}')$
EQ3: $\texttt{Cr}(\mathcal{F}) = \texttt{Cr}(\mathcal{F}')$
EQ4: $\texttt{Output}_{sc}(\mathcal{F}) = \texttt{Output}_{sc}(\mathcal{F}')$
EQ5: $\texttt{Output}_{cr}(\mathcal{F}) = \texttt{Output}_{cr}(\mathcal{F}')$
EQ6: $\texttt{Bases}(\mathcal{F}) = \texttt{Bases}(\mathcal{F}')$

Note that the first three criteria were mentioned but not studied by Oikarinen and Woltran ([5]). Let us consider again the example from the introduction.

Example 2: Assume two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$ having respectively $\{(\{x\}, x)\}$ and $\{(\{x\}, x \wedge x)\}$ as extensions. These two systems are equivalent wrt criterion EQ6 since $\texttt{Bases}(\mathcal{F}) = \texttt{Bases}(\mathcal{F}') = \{\{x\}\}$. However, they are not equivalent wrt the remaining criteria.

Let us now consider another tricky example.

Example 3: Assume two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$ built on propositional logic, having respectively $\{(\{x\}, x)\}$ and $\{(\{x \wedge y\}, x \wedge x)\}$ as extensions. These two systems are not equivalent wrt any of the above criteria. However, we would like to conclude that the systems are equivalent since their conclusions are logically equivalent.

The following example shows two systems which return equivalent subbases of Σ.

Example 4: Assume two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$ having respectively $\{(\{x, \neg\neg y\}, x \wedge y)\}$ and $\{(\{x, y\}, x \wedge y)\}$ as extensions. The two systems are equivalent wrt EQ4 and EQ5 but are not equivalent wrt the remaining criteria, for example EQ6. However, $\texttt{Bases}(\mathcal{F}) = \{\{x, \neg\neg y\}\}$ and $\texttt{Bases}(\mathcal{F}) = \{\{x, y\}\}$ contain equivalent formulae.

In order to have more refined notions of equivalence of argumentation systems, we take advantage of logical equivalence between formulae and between sets of formulae.

Definition 7 (Equivalence between sets and formulae): Let $x, y \in \mathcal{L}$ and $X, Y \subseteq \mathcal{L}$.

- x and y are equivalent, denoted by $x \equiv y$, iff $\mathsf{CN}(\{x\}) = \mathsf{CN}(\{y\})$.
- X and Y are equivalent, denoted by $X \cong Y$, iff $\forall x \in X$, $\exists y \in Y$ s.t. $x \equiv y$ and $\forall y \in Y, \exists x \in X$ s.t. $x \equiv y$. We write $X \ncong Y$ iff X and Y are not equivalent.

In case of propositional logic, this allows to say that the two sets $\{x, \neg\neg y\}$ and $\{x, y\}$ are equivalent. Note that if $X \cong Y$, then $\mathsf{CN}(X) = \mathsf{CN}(Y)$. However, the converse is not true. For instance, $\mathsf{CN}(\{x \wedge y\}) = \mathsf{CN}(\{x, y\})$ while $\{x \wedge y\} \ncong \{x, y\}$. One may ask why not to use the equality of $\mathsf{CN}(X)$ and $\mathsf{CN}(Y)$ in order to say that X and Y are equivalent? The answer is given by the following counter-example of two AS whose credulous conclusions are respectively $\{x, \neg x\}$ and $\{y, \neg y\}$. It is clear that $\mathsf{CN}(\{x, \neg x\}) = \mathsf{CN}(\{y, \neg y\})$ while the two sets are in no way similar.

In order to define an accurate notion of equivalence between argumentation systems, we also take advantage of equivalence of arguments. Two arguments are equivalent if they have the same or equivalent supports and conclusions.

Definition 8 (Equivalence between arguments): For two arguments $a, a' \in \texttt{Arg}(\mathcal{L})$.

- $a \approx_1 a'$ iff $\texttt{Supp}(a) = \texttt{Supp}(a')$ and $\texttt{Conc}(a) \equiv \texttt{Conc}(a')$
- $a \approx_2 a'$ iff $\texttt{Supp}(a) \equiv \texttt{Supp}(a')$ and $\texttt{Conc}(a) = \texttt{Conc}(a')$
- $a \approx_3 a'$ iff $\texttt{Supp}(a) \equiv \texttt{Supp}(a')$ and $\texttt{Conc}(a) \equiv \texttt{Conc}(a')$

Note that each relation $\approx_i$ is an *equivalence relation* (i.e. reflexive, symmetric and transitive). The equivalence between two arguments is extended to equivalence between sets of arguments as follows.

Definition 9 (Equivalence between sets of arguments): Let $\mathcal{E}, \mathcal{E}' \subseteq \texttt{Arg}(\mathcal{L})$ and $\approx_i$ be an equivalence relation between arguments with $i \in \{1, 2, 3\}$. $\mathcal{E} \sim_i \mathcal{E}'$ iff

$\forall a \in \mathcal{E}, \exists a' \in \mathcal{E}'$ s.t. $a \approx_i a'$ and $\forall a' \in \mathcal{E}', \exists a \in \mathcal{E}$ s.t. $a \approx_i a'$.

We are now ready to introduce the second family of equivalence criteria.

Definition 10 (Equivalence criteria cont.): Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built using the same Tarski's logic $(\mathcal{L}, \mathsf{CN})$. Let $\sim_i$ with $i \in \{1,2,3\}$. The two systems $\mathcal{F}$ and $\mathcal{F}'$ are *EQi-equivalent* iff criterion EQi below holds:

- EQ1i: $\exists f : \texttt{Ext}(\mathcal{F}) \to \texttt{Ext}(\mathcal{F}')$ s.t. f is a bijection and $\forall \mathcal{E} \in \texttt{Ext}(\mathcal{F})$, $\mathcal{E} \sim_i f(\mathcal{E})$
- EQ2i: $\texttt{Sc}(\mathcal{F}) \sim_i \texttt{Sc}(\mathcal{F}')$
- EQ3i: $\texttt{Cr}(\mathcal{F}) \sim_i \texttt{Cr}(\mathcal{F}')$
- EQ4b $\texttt{Output}_{sc}(\mathcal{F}) \cong \texttt{Output}_{sc}(\mathcal{F}')$
- EQ5b $\texttt{Output}_{cr}(\mathcal{F}) \cong \texttt{Output}_{cr}(\mathcal{F}')$
- EQ6b $\forall S \in \texttt{Bases}(\mathcal{F})$, $\exists S' \in \texttt{Bases}(\mathcal{F}')$ s.t. $S \cong S'$ and $\forall S' \in \texttt{Bases}(\mathcal{F}')$, $\exists S \in \texttt{Bases}(\mathcal{F})$ s.t. $S \cong S'$

Each of the above criteria refines a criterion in Definition 6 by considering equivalences either between sets of arguments or sets of formulae. The three first criteria use an index i since they are built upon an equivalence relation $\sim_i$ between sets of arguments (with $i \in \{1,2,3\}$). For instance, EQ11 stands for a criterion which use relation $\sim_1$.

Example 2 (Cont): The two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$ of Example 2 are equivalent wrt criteria EQ11 and EQ13 since the two arguments $(\{x\}, x)$ and $(\{x\}, x \wedge x)$ are equivalent wrt relations $\approx_1$ and $\approx_3$. The two systems are also equivalent wrt criteria EQ21 and EQ23 for the same reasons. Finally, they are equivalent wrt EQ4b since the two conclusions x and $x \wedge x$ are logically equivalent.

Example 3 (Cont): The two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$ of Example 3 are equivalent wrt criterion EQ4b.

Example 4 (Cont): The two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$ of Example 4 are equivalent wrt criterion EQ6b since the two bases $\{x, \neg\neg y\}$ and $\{x, y\}$ are equivalent wrt relation $\cong$.

Notation: If two argumentation systems $\mathcal{F}$ and $\mathcal{F}'$ are equivalent wrt criterion c, then we write $\mathcal{F} \equiv_c \mathcal{F}'$.
It is easy to check that each criterion is an *equivalence relation*, that is reflexive, symmetric and transitive.

Property 2: Each criterion is an *equivalence relation*.

Note that rejected arguments are not considered when comparing two argumentation systems. The reason is that rejected arguments are not an important output of a system compared to skeptical arguments which support the conclusions to infer from a knowledge base. Let us consider the following example.

Example 5: Let $(\mathcal{L}, \mathsf{CN})$ be propositional logic, let $a_1 = (\{t \wedge \neg x\}, \neg x)$, $a_2 = (\{x, y\}, x \wedge y)$, $a_3 = (\{w \wedge \neg y\}, \neg y)$, $\mathcal{A} = \{a_1, a_2\}$, $\mathcal{A}' = \{a_2, a_3\}$, $\mathcal{R} = \{(a_1, a_2)\}$, $\mathcal{R}' = \{(a_3, a_2)\}$. It is easy to see that $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ would be equivalent if we compare rejected arguments, since their sets of rejected arguments coincide, i.e. for both systems that is the set $\{a_2\}$. However, those two systems have almost nothing in common since neither their conclusions nor their arguments coincide. Note also that arguments of those systems are not equivalent wrt any reasonable equivalence relation.

IV. LINKS BETWEEN CRITERIA

It is clear that not all criteria are equally demanding and that they are not completely independent. For example, it is easy to see that when two argumentation systems are equivalent wrt EQ1, then they are also equivalent wrt EQ11, EQ12 and EQ13. In this section, we investigate all dependencies between the criteria proposed so far.

Theorem 1: Let $\mathcal{F}$ and $\mathcal{F}'$ be two AS built on the same logic $(\mathcal{L}, \mathsf{CN})$. Table I summarizes the dependencies in the following form: $(\mathcal{F} \equiv_c \mathcal{F}') \Rightarrow (\mathcal{F} \equiv_{c'} \mathcal{F}')$.

Note that if two argumentation systems are equivalent wrt $EQ1$, then they are equivalent wrt any of the other criteria. This is not the case for its refined versions, i.e. for $EQ11, EQ12$ and $EQ13$. For instance, if two systems are equivalent wrt to $EQ11$, they are not necessarily equivalent wrt $EQ21$, $EQ23$ and $EQ4b$. The following result shows that under some reasonable constraints, these implications exist. Indeed, if two argumentation systems are equivalent wrt to $EQ11$, then they are also equivalent wrt the three criteria $EQ21$, $EQ23$ and $EQ4b$ provided that the two systems use attack relations which verify properties $C1'$ and $C2$. Before presenting formally this result, let us show how the two properties $C1'$ and $C2$ of an attack relation are related to the equivalence relation $\approx_1$ between arguments which is used in criterion $EQ11$.

The following property shows that equivalent arguments wrt relation $\approx_1$ behave in the same way wrt attacks in case the attack relation enjoys the two properties $C1'$ and $C2$.

Property 3: Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1'$ and $C2$. For all $a, a', b, b' \in \mathcal{A}$, $(a \approx_1 a'$ and $b \approx_1 b') \Rightarrow (a\mathcal{R}b$ iff $a'\mathcal{R}b')$.

	EQ1	EQ11	EQ12	EQ13	EQ2	EQ21	EQ22	EQ23	EQ3	EQ31	EQ32	EQ33	EQ4	EQ4b	EQ5	EQ5b	EQ6	EQ6b
EQ1	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
EQ11		+		+						+		+				+	+	+
EQ12			+	+							+	+			+	+		+
EQ13				+								+				+		+
EQ2					+	+	+	+					+	+				
EQ21						+		+						+				
EQ22							+	+					+	+				
EQ23								+						+				
EQ3									+	+	+	+			+	+		
EQ31										+		+				+		
EQ32											+	+			+	+		
EQ33												+				+		
EQ4													+	+				
EQ4b														+				
EQ5															+	+		
EQ5b																+		
EQ6																	+	+
EQ6b																		+

Table I

IF TWO SYSTEMS ARE EQUIVALENT WRT CRITERION c IN ROW i THEN THEY ARE EQUIVALENT WRT CRITERION c' IN COLUMN j.

The next result shows that equivalent arguments wrt relation $\approx_1$ belong to the same extensions.

Property 4: Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1'$ and $C2$. For all $a, a' \in \mathcal{A}$, if $a \approx_1 a'$, then $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{F})$, $a \in \mathcal{E}$ iff $a' \in \mathcal{E}$.

It can also be checked that when two argumentation systems are equivalent wrt $EQ11$, then if we consider two equivalent arguments (one from each system), then the two arguments have the same status.

Property 5: Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built using the same logic $(\mathcal{L}, \mathsf{CN})$, and let $\mathcal{R}$ and $\mathcal{R}'$ verify $C1'$ and $C2$, and $\mathcal{F} \equiv_{EQ11} \mathcal{F}'$. For all $a \in \mathcal{A}$ and for all $a' \in \mathcal{A}'$, if $a \approx_1 a'$ then $\mathtt{Status}(a, \mathcal{F}) = \mathtt{Status}(a', \mathcal{F}')$.

In general, when two argumentation systems are equivalent wrt $EQ11$, they are not necessarily equivalent wrt $EQ21, EQ23$ and $EQ4b$. The following result shows that when the attack relations of both systems verify $C1'$ and $C2$, the previous implications hold.

Theorem 2: Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built using the same logic $(\mathcal{L}, \mathsf{CN})$, $\mathcal{R}$ and $\mathcal{R}'$ verify $C1'$ and $C2$. If $\mathcal{F} \equiv_{EQ11} \mathcal{F}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ21, EQ23, EQ4b\}$.

When two argumentation systems are equivalent wrt $EQ12$, they are also equivalent wrt $EQ22, EQ23, EQ4$ and $EQ4b$ in case the attack relations of the two systems enjoy properties $C1$ and $C2'$. The reason is that there is a correlation between an attack relation which satisfies these two properties and the equivalence relation $\approx_2$ between arguments. Indeed, equivalent arguments wrt $\approx_2$ behave in the same way wrt an attack relation satisfying $C1$ and $C2'$.

Property 6: Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1$ and $C2'$. For all $a, a', b, b' \in \mathcal{A}$, $(a \approx_2 a'$ and $b \approx_2 b') \Rightarrow (a\mathcal{R}b$ iff $a'\mathcal{R}b')$.

Equivalent arguments wrt $\approx_2$ belong to the same extensions of an argumentation system.

Property 7: Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1$ and $C2'$. For all $a, a' \in \mathcal{A}$, if $a \approx_2 a'$ then $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{F})$, $a \in \mathcal{E}$ iff $a' \in \mathcal{E}$.

Finally, two equivalent arguments pertaining to two systems whose attack relations satisfy $C1$ and $C2'$ have the same status.

Property 8: Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built from the same logic $(\mathcal{L}, \mathsf{CN})$, $\mathcal{R}$ and $\mathcal{R}'$ verify $C1$ and $C2'$, and $\mathcal{F} \equiv_{EQ12} \mathcal{F}'$. For all $a \in \mathcal{A}$ and for all $a' \in \mathcal{A}'$, if $a \approx_2 a'$ then $\mathtt{Status}(a, \mathcal{F}) = \mathtt{Status}(a', \mathcal{F}')$.

From the above properties, it follows that two argumentation systems which are equivalent wrt $EQ12$ are also equivalent wrt $EQ22, EQ23, EQ4$ and $EQ4b$.

Theorem 3: Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built using the same logic $(\mathcal{L}, \mathsf{CN})$, $\mathcal{R}$ and $\mathcal{R}'$ verify $C1$ and $C2'$. If $\mathcal{F} \equiv_{EQ12} \mathcal{F}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ22, EQ23, EQ4, EQ4b\}$.

Finally, similar results can be shown when considering an attack relation satisfying the two properties $C1'$ and $C2'$ and the equivalence relation $\approx_3$ between arguments.

Property 9: Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1'$ and $C2'$. For all $a, a', b, b' \in \mathcal{A}$, $(a \approx_3 a'$ and $b \approx_3 b') \Rightarrow (a\mathcal{R}b$ iff $a'\mathcal{R}b')$.

The following property shows that equivalent arguments wrt $\approx_3$ belong to the same extensions in an argumentation system whose attack relation satisfies $C1'$ and $C2'$.

Property 10: Let $(\mathcal{A}, \mathcal{R})$ be an argumentation system s.t. $\mathcal{R}$ enjoys $C1'$ and $C2'$. For all $a, a' \in \mathcal{A}$, if $a \approx_3 a'$ then $\forall \mathcal{E} \in \mathtt{Ext}(\mathcal{F})$, $a \in \mathcal{E}$ iff $a' \in \mathcal{E}$.

A similar result as Property 8 is found in case of argumentation systems with attack relations satisfying $C1'$ and $C2'$ and using the equivalence relation $\approx_3$.

Property 11: Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built from the same logic $(\mathcal{L}, \mathsf{CN})$, $\mathcal{R}$ and $\mathcal{R}'$ verify $C1'$ and $C2'$, and $\mathcal{F} \equiv_{EQ13} \mathcal{F}'$. For all $a \in \mathcal{A}$ and for all $a' \in \mathcal{A}'$, if $a \approx_3 a'$ then $\mathtt{Status}(a, \mathcal{F}) = \mathtt{Status}(a', \mathcal{F}')$.

Finally, we show that if two argumentation systems whose attack relations enjoy $C1'$ and $C2'$ are equivalent wrt $EQ13$, then they are also equivalent wrt $EQ23$ and $EQ4b$.

Theorem 4: Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$, $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two argumentation systems built using the same logic $(\mathcal{L}, \mathsf{CN})$, $\mathcal{R}$ and $\mathcal{R}'$ verify $C1'$ and $C2'$. If $\mathcal{F} \equiv_{EQ13} \mathcal{F}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ23, EQ4b\}$.

In sum, the comparative study revealed that the most general equivalence criteria are $EQ1, EQ11, EQ12$ and $EQ13$. Indeed, if two systems are equivalent wrt to one of these criteria, then they are so with most of the remaining criteria.

V. CONDITIONS FOR EQUIVALENCE

In section III, we have proposed different intuitive criteria for the equivalence of two argumentation systems built from the same logic. An important question now is "are there conditions under which two distinct argumentation systems may be equivalent wrt to those criteria?" Recall that the answer is no in case of the criteria used by Oikarinen and Woltran ([5]). In that paper, it has been shown that two systems are equivalent if they coincide. This means that the notion of equivalence defined in that paper is useless. In this section, we show that our refined criteria, in particular $EQ11, EQ12$ and $EQ13$, make it possible to compare different systems.

We study two particular situations. In the first one, we focus on the global argumentation system that may be built from a knowledge base Σ, that is the system which uses the whole set $\mathtt{Arg}(\Sigma)$ of arguments. Since generally this set is infinite, we identify a finite subset $\mathcal{A}$ of $\mathtt{Arg}(\Sigma)$ such that the corresponding system is equivalent to the global one. Such a result is of great importance from a computational point of view. Instead of working with an infinite set of arguments, we only handle its finite subset. Before presenting the formal result, let us first introduce some useful notations.

Notation: For an arbitrary set X, an arbitrary equivalence relation $\sim$ on X, and $x \in X$, $[x] = \{x' \in X \mid x' \sim x\}$ and $X/\sim = \{[x] \mid x \in X\}$. For any $X \subseteq \mathcal{L}$, $\mathtt{Cncs}(X) = \{x \in \mathcal{L} \mid \exists Y \subseteq X \text{ s.t.} \mathsf{CN}(Y) \neq \mathcal{L} \text{ and } x \in \mathsf{CN}(Y)\}$.

We show that if the attack relation used in $\mathcal{F} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ verifies properties $C1'$ and $C2$ and $\mathtt{Cncs}(\Sigma)$ has a finite number of equivalence classes, then there exists a finite sub-system (i.e. with a finite set of arguments) of $\mathcal{F}$ which is equivalent to $\mathcal{F}$ wrt most of the proposed criteria.

Theorem 5: Let $\mathcal{F} = (\mathtt{Arg}(\Sigma), \mathcal{R})$ be an argumentation system built from a knowledge base Σ. If $\mathcal{R}$ satisfies $C1'$ and $C2$ and $\mathtt{Cncs}(\Sigma)/\equiv$ is finite, then there exists $\mathcal{A} \subseteq \mathtt{Arg}(\Sigma)$ s.t. $\mathcal{A}$ is finite and $\mathcal{F}' = (\mathcal{A}, \mathcal{R}|_{\mathcal{A}})^2 \equiv_x \mathcal{F}$ with $x \in \{EQ11, EQ13, EQ21, EQ23, EQ31, EQ33, EQ4b, EQ5b, EQ6, EQ6b\}$.

The second situation we are interested in is the case of two argumentation systems that may be built from two distinct knowledge bases but use the same attack relation. For instance, both systems use 'rebut' relation or both systems use 'assumption attack', etc. Recall that $\mathtt{Arg}(\mathcal{L})$ is the set of all arguments that can be built from a fixed logical language $\mathcal{L}$ using a fixed consequence operator CN. We denote by $\mathcal{R}_{\mathcal{L}}$ the attack relation which is used in the two systems with $\mathcal{R}_{\mathcal{L}} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. The following result shows when two systems $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$, where $\mathcal{R}, \mathcal{R}' \subseteq \mathcal{R}_{\mathcal{L}}$, are equivalent wrt $EQ11$.

Theorem 6: Let $(\mathcal{L}, \mathsf{CN})$ be a fixed logic, $\mathtt{Arg}(\mathcal{L})$ a set of arguments and $\mathcal{R}_{\mathcal{L}} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS s.t. $\mathcal{A}, \mathcal{A}' \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}}, \mathcal{R}' = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}'}$. If $\mathcal{R}_{\mathcal{L}}$ satisfies $C1'$ and $C2$ and $\mathcal{A} \sim_1 \mathcal{A}'$, then $\mathcal{F} \equiv_{EQ11} \mathcal{F}'$.

The following result follows from Theorem 2.

Corollary 1: Let $(\mathcal{L}, \mathsf{CN})$ be a fixed logic, $\mathtt{Arg}(\mathcal{L})$ a set of arguments and $\mathcal{R}_{\mathcal{L}} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS s.t. $\mathcal{A}, \mathcal{A}' \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}}, \mathcal{R}' = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}'}$. If $\mathcal{R}_{\mathcal{L}}$ satisfies $C1'$ and $C2$ and $\mathcal{A} \sim_1 \mathcal{A}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ13, EQ21, EQ23, EQ31, EQ33, EQ4b, EQ5b, EQ6, EQ6b\}$.

A similar result is shown for argumentation systems which use the same attack relation provided that this latter satisfies properties $C1$ and $C2'$.

Theorem 7: Let $(\mathcal{L}, \mathsf{CN})$ be a fixed logic, $\mathtt{Arg}(\mathcal{L})$ a set of arguments and $\mathcal{R}_{\mathcal{L}} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS s.t. $\mathcal{A}, \mathcal{A}' \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}}, \mathcal{R}' = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}'}$. If $\mathcal{R}_{\mathcal{L}}$ satisfies $C1$ and $C2'$ and

[2] $\mathcal{R}|_{\mathcal{A}} = \{(a, b) \in \mathcal{R} \text{ s.t. } a, b \in \mathcal{A}\}$.

$\mathcal{A} \sim_2 \mathcal{A}'$, then $\mathcal{F} \equiv_{EQ12} \mathcal{F}'$.

Due to the dependencies between criterion $EQ12$ and some other equivalence criteria, the next result holds.

Corollary 2: Let $(\mathcal{L}, \mathsf{CN})$ be a fixed logic, $\mathtt{Arg}(\mathcal{L})$ a set of arguments and $\mathcal{R}_\mathcal{L} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS s.t. $\mathcal{A}, \mathcal{A}' \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_\mathcal{L}|_\mathcal{A}, \mathcal{R}' = \mathcal{R}_\mathcal{L}|_{\mathcal{A}'}$. If $\mathcal{R}_\mathcal{L}$ satisfies $C1$ and $C2'$ and $\mathcal{A} \sim_2 \mathcal{A}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ13, EQ22, EQ23, EQ32, EQ33, EQ4, EQ4b, EQ5, EQ5b, EQ6b\}$.

The following result shows under which conditions two systems are equivalent wrt $EQ13$.

Theorem 8: Let $(\mathcal{L}, \mathsf{CN})$ be a fixed logic, $\mathtt{Arg}(\mathcal{L})$ a set of arguments, $\mathcal{R}_\mathcal{L} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS s.t. $\mathcal{A}, \mathcal{A}' \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_\mathcal{L}|_\mathcal{A}, \mathcal{R}' = \mathcal{R}_\mathcal{L}|_{\mathcal{A}'}$. If $\mathcal{R}_\mathcal{L}$ satisfies $C1'$ and $C2'$ and $\mathcal{A} \sim_3 \mathcal{A}'$, then $\mathcal{F} \equiv_{EQ13} \mathcal{F}'$.

The following corollary follows from Theorem 4.

Corollary 3: Let $(\mathcal{L}, \mathsf{CN})$ be a fixed logic, $\mathtt{Arg}(\mathcal{L})$ a set of arguments and $\mathcal{R}_\mathcal{L} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$. Let $\mathcal{F} = (\mathcal{A}, \mathcal{R})$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$ be two AS s.t. $\mathcal{A}, \mathcal{A}' \subseteq \mathtt{Arg}(\mathcal{L})$ and $\mathcal{R} = \mathcal{R}_\mathcal{L}|_\mathcal{A}, \mathcal{R}' = \mathcal{R}_\mathcal{L}|_{\mathcal{A}'}$. If $\mathcal{R}_\mathcal{L}$ satisfies $C1'$ and $C2'$ and $\mathcal{A} \sim_3 \mathcal{A}'$, then $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ23, EQ33, EQ4b, EQ5b, EQ6b\}$.

VI. CONCLUSION

In this paper, we have tackled the problem of equivalence between two argumentation systems. We have shown the benefits that can be obtained when the internal structure of arguments is taken into account. We have proposed different equivalence criteria for argumentation systems that use the same underlying logic. We have provided conditions under which two systems are equivalent wrt a given criterion. We have also shown how an infinite system may be exchanged by an equivalent, finite system.

An extension of this work would be to study the conditions under which two argumentation systems that use different attack relations are equivalent wrt the proposed criteria. This is important in order to compare the results returned by different attack relations. Another natural extension of this work would be to consider the strong equivalence as discussed by Oikarinen and Woltran ([5]), that is, two argumentation systems are strongly equivalent if they are equivalent wrt any of proposed criteria even after adding the same set of arguments in both systems. Finally, we will study the notion of equivalence between two systems based on different logics.

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APPENDIX

Proof of Proposition 1. Let $S_1, \ldots, S_n \subseteq \Sigma$ be all the consistent subsets of Σ. We will use the notation $\mathcal{A}_i = \{a \in \mathcal{A} \mid \mathtt{Supp}(a) = S_i\}$. (Note that some of the sets in $\mathcal{A}_1, \ldots, \mathcal{A}_n$ may be empty, but that is not important for the proof.) We will now prove that for every stable extension $\mathcal{E}$, for any i, for any $a, a' \in \mathcal{A}_i$ we have $a \in \mathcal{E}$ iff $a' \in \mathcal{E}$. Let us suppose that $a \in \mathcal{E}$ and $a' \notin \mathcal{E}$. Since $\mathcal{E}$ is a stable extension, then $\exists b \in \mathcal{E}$ s.t. $b\mathcal{R}a'$. This means that $b\mathcal{R}a$ which contradicts the fact that $\mathcal{E}$ is a stable extension. The case $a \notin \mathcal{E}$ and $a' \in \mathcal{E}$ is symmetric. This means that for any i, any extension either contains all elements of A_i or neither of them. Formally, for any extension $\mathcal{E}$, $\forall i \in \{1, \ldots, n\}$, we have $\mathcal{E} \cap \mathcal{A}_i = \mathcal{A}_i$ or $\mathcal{E} \cap \mathcal{A}_i = \emptyset$. Consequently, there is at most 2^n different extensions.

Proof of Theorem 2. Let us prove that EQ21 is verified. If $\mathtt{Ext}(\mathcal{F}) = \emptyset$, then from EQ11, $\mathtt{Ext}(\mathcal{F}') = \emptyset$. In this case, EQ21 is trivial, since $\mathtt{Sc}(\mathcal{F}) = \mathtt{Sc}(\mathcal{F}') = \emptyset$. Else, let $\mathtt{Ext}(\mathcal{F}) \neq \emptyset$.

Let $\mathtt{Sc}(\mathcal{F}) = \emptyset$. We will prove that $\mathtt{Sc}(\mathcal{F}') = \emptyset$. Suppose the contrary and let $a' \in \mathtt{Sc}(\mathcal{F}')$. Let $\mathcal{E}' \in \mathtt{Ext}(\mathcal{F}')$. Argument a' is skeptically accepted, thus $a' \in \mathcal{E}'$. Let f be a bijection from EQ11, and let us denote $\mathcal{E} = f^{-1}(\mathcal{E}')$. From $\mathcal{F} \equiv_{EQ11} \mathcal{F}'$, we obtain $\mathcal{E} \in \mathtt{Ext}(\mathcal{F})$. Furthermore, $\mathcal{E} \sim_1 \mathcal{E}'$, and, consequently, $\exists a \in \mathcal{E}$ s.t. $a \approx_1 a'$. Property 5 implies that a is skeptically accepted in $\mathcal{F}$, contradiction.

Let $\mathtt{Sc}(\mathcal{F}) \neq \emptyset$ and let $a \in \mathtt{Sc}(\mathcal{F})$. Since EQ11 is verified, and a is in at least one extension, then $\exists a' \in \mathcal{A}'$ s.t. $a' \approx_1 a$. Since EQ11 is verified then, from Property 5, a' is skeptically accepted in $\mathcal{F}'$. Thus $\forall a \in \mathtt{Sc}(\mathcal{F})$, $\exists a' \in \mathtt{Sc}(\mathcal{F}')$ s.t. $a' \approx_1 a$. To prove that $\forall a' \in \mathtt{Sc}(\mathcal{F}')$, $\exists a \in \mathtt{Sc}(\mathcal{F})$ s.t. $a \approx_1 a'$ is similar.
Since EQ21 implies EQ23 and EQ4b in the general case, as shown in Theorem 1, then we conclude that $\mathcal{F}$ and $\mathcal{F}'$ must be equivalent wrt. EQ21, EQ23 and EQ4b.

Proof of of Theorem 5. Let $\mathcal{A}' \subseteq \mathcal{A}$ be a set such that $\forall a \in \mathcal{A}$, $\exists! a' \in \mathcal{A}'$ s.t. $a \approx_1 a'$. Let $\mathcal{R}' = \mathcal{R}|_{\mathcal{A}'}$ and $\mathcal{F}' = (\mathcal{A}', \mathcal{R}')$. From Theorem 6 and Corollary 1 applied on $\mathcal{F}$ and $\mathcal{F}'$, we obtain that $\mathcal{F} \equiv_x \mathcal{F}'$

with $x \in \{EQ11, EQ13, EQ21,\ EQ23,\ EQ31, EQ33, EQ4b, EQ5b, EQ6, EQ6b\}$. There are two remarks that should be made here. The first one is purely a technical issue: namely, we use Theorem 6 and Corollary 1 in the proof of Theorem 5. Note, however, that this is not a problem, since Theorem 6 and Corollary 1 are proved without a reference to the present result. We decided to present Theorem 5 before Theorem 6 and Corollary 1 since we estimated that it is easier to follow Section V that way. The second remark is more complex. Recall that, at one hand, we supposed that set $\mathcal{A}$ and attack relation $\mathcal{R}$ on that set are given. On the other hand, Theorem 6 and Corollary 1 require an attack relation $\mathcal{R}_{\mathcal{L}} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$ s.t. $\mathcal{R} = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}}$ where $\mathcal{R}_{\mathcal{L}}$ verifies C1' and C2. We will now show that, given a relation $\mathcal{R}$ on set $\mathcal{A}$ which enjoys C1' and C2, we can always define (at least one) relation $\mathcal{R}_{\mathcal{L}} \subseteq \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$ s.t. $\mathcal{R} = \mathcal{R}_{\mathcal{L}}|_{\mathcal{A}}$ and $\mathcal{R}_{\mathcal{L}}$ verifies C1' and C2. It is clear that in our case, the behavior of $\mathcal{R}_{\mathcal{L}}$ outside of $\mathcal{A}$ is not important, but formally we must prove that such a relation exists. The most obvious option is to define $\mathcal{R}_{\mathcal{L}}$ as follows: for any $x, y \in \mathtt{Arg}(\mathcal{L})$, $x\mathcal{R}_{\mathcal{L}}y$ iff $\exists x' \in \mathcal{A}$, $\exists y' \in \mathcal{A}$ s.t. $\mathtt{Conc}(x) \equiv \mathtt{Conc}(x')$ and $\mathtt{Supp}(y) = \mathtt{Supp}(y')$ and $x\mathcal{R}y$. Let us verify that $\mathcal{R}_{\mathcal{L}}$ defined in this way verifies C1' and C2. Let $a, b, c \in \mathtt{Arg}(\mathcal{L})$ and let $\mathtt{Conc}(a) \equiv \mathtt{Conc}(b)$ and $a\mathcal{R}_{\mathcal{L}}c$. Since $a\mathcal{R}_{\mathcal{L}}c$ then $\exists a' \in \mathcal{A}$ $\exists c' \in \mathcal{A}$ s.t. $\mathtt{Conc}(a) \equiv \mathtt{Conc}(a')$ and $\mathtt{Supp}(c) = \mathtt{Supp}(c')$ and $a'\mathcal{R}c'$. Since $\mathtt{Conc}(b) \equiv \mathtt{Conc}(a)$, then $\mathtt{Conc}(b) \equiv \mathtt{Conc}(a')$. Thus, $\exists a' \in \mathcal{A}$, $\exists c' \in \mathcal{A}$ s.t. $\mathtt{Conc}(a') \equiv \mathtt{Conc}(b)$ and $\mathtt{Supp}(c) = \mathtt{Supp}(c')$ and $a'\mathcal{R}c'$. Thus, $b\mathcal{R}_{\mathcal{L}}c$. Let us suppose that $\neg(a\mathcal{R}_{\mathcal{L}}c)$. In that case, $\neg(\exists a' \in \mathcal{A}, \exists c' \in \mathcal{A}$ s.t. $\mathtt{Conc}(a') \equiv \mathtt{Conc}(c)$ and $\mathtt{Supp}(c) = \mathtt{Supp}(c')$ and $a'\mathcal{R}c')$. Since $\mathtt{Conc}(b) \equiv \mathtt{Conc}(a)$ then $\neg(b\mathcal{R}_{\mathcal{L}}c)$. Thus, C1' is verified by $\mathcal{R}_{\mathcal{L}}$. Let us see why $\mathcal{R}_{\mathcal{L}}$ enjoys C2. Let $a, b, c \in \mathtt{Arg}(\mathcal{L})$ and let $\mathtt{Supp}(a) = \mathtt{Supp}(b)$. Let $c\mathcal{R}_{\mathcal{L}}a$. This means that $\exists c' \in \mathcal{A}$, $\exists a' \in \mathcal{A}$ s.t. $\mathtt{Conc}(c) \equiv \mathtt{Conc}(c')$ and $\mathtt{Supp}(a') = \mathtt{Supp}(a)$ and $c\mathcal{R}a$. Since $\mathtt{Supp}(a) = \mathtt{Supp}(b)$ then $c\mathcal{R}_{\mathcal{L}}b$. It is easy to see that if $\neg(c\mathcal{R}_{\mathcal{L}}a)$ then $\neg(c\mathcal{R}_{\mathcal{L}}b)$. All this guarantee that we can apply Theorem 6 and Corollary 1. We obtain that $\mathcal{F} \equiv_x \mathcal{F}'$ with $x \in \{EQ11, EQ13, EQ21,\ EQ23,\ EQ31, EQ33, EQ4b, EQ5b, EQ6, EQ6b\}$.

In addition to that, we will show that $\mathcal{A}'$ is finite. Since Σ is finite, then $\{\mathtt{Supp}(a) \mid a \in \mathcal{A}'\}$ must be finite. If for any $H \in \{\mathtt{Supp}(a) \mid a \in \mathcal{A}'\}$, the set $\{a \in \mathcal{A}' \mid \mathtt{Supp}(a) = H\}$, is finite, then the set $\mathcal{A}'$ is clearly finite. Else, there exists $H_0 \in \{\mathtt{Supp}(a) \mid a \in \mathcal{A}'\}$, s.t. the set $\mathcal{A}_{H_0} = \{a \in \mathcal{A}' \mid \mathtt{Supp}(a) = H_0\}$ is infinite. By the definition of $\mathcal{A}'$, one obtains that $\forall a, b \in \mathcal{A}_{H_0}$, $\mathtt{Conc}(a) \not\equiv \mathtt{Conc}(b)$. It is clear that $\forall a \in \mathcal{A}_{H_0}$, $\mathtt{Conc}(a) \in \mathtt{Cncs}(\Sigma)$. This implies that there are infinitely many different formulae having logically non-equivalent conclusions in $\mathtt{Cncs}(\Sigma)$, formally the set $\mathtt{Cncs}(\Sigma)/\equiv$ is infinite, contradiction.

Proof of Theorem 6. Let us first suppose that $\mathtt{Ext}(\mathcal{F}) \neq \emptyset$ and let us define the function $f' : 2^{\mathcal{A}} \to 2^{\mathcal{A}'}$ as follows: $f'(B) = \{a' \in \mathcal{A}' \mid \exists a \in B \text{ s.t. } a' \approx_1 a\}$.

Let f be the restriction of f' to $\mathtt{Ext}(\mathcal{F})$. We will prove that the image of this function is $\mathtt{Ext}(\mathcal{F}')$ and that f is a bijection between $\mathtt{Ext}(\mathcal{F})$ and $\mathtt{Ext}(\mathcal{F}')$ which verifies EQ11. First, we will prove that for any $\mathcal{E} \in \mathtt{Ext}(\mathcal{F})$, $f(\mathcal{E}) \in \mathtt{Ext}(\mathcal{F}')$. Let $\mathcal{E} \in \mathtt{Ext}(\mathcal{F})$ and let $\mathcal{E}' = f(\mathcal{E})$. We will prove that $\mathcal{E}'$ is conflict-free. Let $a', b' \in \mathcal{E}'$. There must exist $a, b \in \mathcal{E}$ s.t. $a \approx_1 a'$ and $b \approx_1 b'$. Since $\mathcal{E}$ is an extension, $\neg(a\mathcal{R}b)$ and $\neg(b\mathcal{R}a)$. By applying Property 3 on $(\mathtt{Arg}(\mathcal{L}), \mathcal{R}_{\mathcal{L}})$, we have that $\neg(a'\mathcal{R}'b')$ and $\neg(b'\mathcal{R}'a')$. Let $x' \in \mathcal{A}' \setminus \mathcal{E}'$. Then $\exists x \in \mathcal{A}$ s.t. $x \approx_1 x'$. Note also that it must be that $x \notin \mathcal{E}$. Since $\mathcal{E} \in \mathtt{Ext}(\mathcal{F})$, then $\exists y \in \mathcal{E}$ s.t. $y\mathcal{R}x$. Note that $\exists y' \in \mathcal{E}'$ s.t. $y' \approx_1 y$. From Property 3, $y'\mathcal{R}'x'$. We have shown that the image of f is the set $\mathtt{Ext}(\mathcal{F}')$. We will now prove that $f : \mathtt{Ext}(\mathcal{F}) \to \mathtt{Ext}(\mathcal{F}')$ is injective. Let $\mathcal{E}_1, \mathcal{E}_2 \in \mathtt{Ext}(\mathcal{F})$ with $\mathcal{E}_1 \neq \mathcal{E}_2$ and $\mathcal{E}' = f(\mathcal{E}_1) = f(\mathcal{E}_2)$. We will show that if $\mathcal{E}_1 \sim_1 \mathcal{E}_2$ then $\mathcal{E}_1 = \mathcal{E}_2$. Without loss of generality, let $\exists x \in \mathcal{E}_1 \setminus \mathcal{E}_2$. Then, from $\mathcal{E}_1 \sim_1 \mathcal{E}_2$, $\exists x' \in \mathcal{E}_2$, s.t. $x' \approx_1 x$. Then, since $x \in \mathcal{E}_1$ and $x \notin \mathcal{E}_2$, from Property 4 we obtain that $x' \in \mathcal{E}_1$ and $x' \notin \mathcal{E}_2$. Contradiction with $x' \in \mathcal{E}_2$. This means that $\neg(\mathcal{E}_1 \sim_1 \mathcal{E}_2)$. Without loss of generality, $\exists a_1 \in \mathcal{E}_1 \setminus \mathcal{E}_2$ s.t. $\nexists a_2 \in \mathcal{E}_2$ s.t. $a_1 \approx_1 a_2$. Let $a' \in \mathcal{A}'$ s.t. $a' \approx_1 a_1$. Recall that $\mathcal{E}' = f(\mathcal{E}_2)$. Thus, $\exists a_2 \in \mathcal{E}_2$ s.t. $a_2 \approx_1 a'$. Contradiction.

We show that $f : \mathtt{Ext}(\mathcal{F}) \to \mathtt{Ext}(\mathcal{F}')$ is surjective. Let $\mathcal{E}' \in \mathtt{Ext}(\mathcal{F}')$, and let us show that $\exists \mathcal{E} \in \mathtt{Ext}(\mathcal{F})$ s.t. $\mathcal{E}' = f(\mathcal{E})$. Let $\mathcal{E} = \{a \in \mathcal{A} \mid \exists a' \in \mathcal{E}' \text{ s.t. } a \approx_1 a'\}$. From Property 3 we see that $\mathcal{E}$ is conflict-free. For any $b \in \mathcal{A} \setminus \mathcal{E}$, $\exists b' \in \mathcal{A}' \setminus \mathcal{E}'$ s.t. $b \approx_1 b'$. Since $\mathcal{E}' \in \mathtt{Ext}(\mathcal{F}')$, then $\exists a' \in \mathcal{E}'$ s.t. $a'\mathcal{R}'b'$. Now, $\exists a \in \mathcal{E}$ s.t. $a \approx_1 a'$; from Property 3, $a\mathcal{R}b$. Thus, $\mathcal{E}$ is a stable extension in $\mathcal{F}$.

We will now show that $f : \mathtt{Ext}(\mathcal{F}) \to \mathtt{Ext}(\mathcal{F}')$ verifies the condition of EQ11. Let $\mathcal{E} \in \mathtt{Ext}(\mathcal{F})$ and $\mathcal{E}' = f(\mathcal{E})$. Let $a \in \mathcal{E}$. Then, $\exists a' \in \mathcal{A}'$ s.t. $a' \approx_1 a$. From the definition of f, it must be that $a' \in \mathcal{E}'$. Similarly, if $a' \in \mathcal{E}'$, then must be an argument $a \in \mathcal{A}$ s.t. $a \approx_1 a'$, and again from the definition of the function f, we conclude that $a \in \mathcal{E}$.

We conclude that $\mathcal{F} \equiv_{EQ11} \mathcal{F}'$. Let us take a look at the case when $\mathtt{Ext}(\mathcal{F}) = \emptyset$. We show that $\mathtt{Ext}(\mathcal{F}') = \emptyset$. Suppose not and let $\mathcal{E}' \in \mathtt{Ext}(\mathcal{F}')$. Let us define $\mathcal{E} = \{a \in \mathcal{A} \mid \exists a' \in \mathcal{E}' \text{ s.t. } a \approx_1 a'\}$. From Property 3, $\mathcal{E}$ must be conflict-free. The same property shows that for any $b \in \mathcal{A} \setminus \mathcal{E}$, $\exists a \in \mathcal{E}$ s.t. $a\mathcal{R}b$. Thus, $\mathcal{E}$ is a stable extension in $\mathcal{F}$. Contradiction with $\mathtt{Ext}(\mathcal{F}) = \emptyset$.