The Implication of Patchwork and Compactness in Qualitative Spatio-Temporal Reasoning

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Abstract

We study the qualitative spatiotemporal logic that results by combining the propositional temporal logic (PTL) with a qualitative spatial constraint language, namely, the $\mathcal{L}_1$ logic, and investigate the implication of the constraint properties of *compactness* and *patchwork* in qualitative spatiotemporal reasoning. We use these properties to strengthen results regarding the complexity of the satisfiability problem in $\mathcal{L}_1$, by replacing the stricter global consistency property used in literature and, consequently, generalizing to more qualitative spatial constraint languages. Further, we identify fragments of the $\mathcal{L}_1$ logic that capture significant aspects of spatiotemporal change. In particular, we address the issue of periodical, and smoothness and continuity constraints between spatial configurations, and obtain results on their computational properties. Regarding periodicity, we use again the properties of compactness and patchwork to strengthen results that exist in literature, by re-establishing conditions that allow for tractability and, again, generalizing to a larger class of qualitative spatial constraint languages.

1 Introduction

Time and *space* are fundamental cognitive concepts that have been the focus of study in many scientific disciplines, including Artificial Intelligence and, in particular, Qualitative Reasoning [Wolter and Zakharyaschev, 2003; Hazarika, 2012]. Towards constraint-based qualitative spatiotemporal reasoning, most of the work has relied on formalisms based on the propositional temporal logic (PTL), also known as linear temporal logic, and the qualitative spatial constraint language RCC-8 [Wolter and Zakharyaschev, 2003; 2000]. PTL [Huth and Ryan, 2004] is the well known temporal logic comprising operators $\mathcal{U}$ (until), $\bigcirc$ (next point in time), $\Box$ (always), and $\Diamond$ (eventually) over various flows in time, such as $\langle \mathbb{N}, < \rangle$. RCC-8 is a fragment of the Region Connection Calculus (RCC) [Randell *et al.*, 1992] and is used to describe regions that are non-empty regular subsets of some topological space by stating their topological relations to each other. The topological relations comprise relations DC (disconnected), EC (externally connected), EQ (equal), PO (partially overlapping), TPP (tangential proper part), $TPPi$ (tangential proper part inverse), $NTPP$ (non-tangential proper part), $NTPPi$ (non-tangential proper part inverse). These 8 relations are depicted in [Randell *et al.*, 1992, Fig. 4]. One of the most important of such formalisms is the $\mathcal{ST}_1^-$ logic [Gabelaia *et al.*, 2003]. For example, one can have the following statement using that formalism: $\Diamond TPP(X, Y)$, which translates to "eventually region X will be a tangential proper part of region Y".

In this paper, we consider a generalization of the $\mathcal{ST}_1^-$ logic, denoted by $\mathcal{L}_1$, which is the product of the combination of PTL [Huth and Ryan, 2004] with any qualitative spatial constraint language, such as RCC-8 [Randell *et al.*, 1992], Cardinal Direction Algebra (CDA) [Frank, 1991; Ligozat, 1998], and Block Algebra (BA) [Balbiani *et al.*, 2002], and make the following contributions: (i) we show that satisfiability checking of a $\mathcal{L}_1$ formula is PSPACE-complete if the qualitative spatial constraint language considered has the constraint properties of *compactness* and *patchwork* [Lutz and Milicic, 2007] for atomic networks, thus, strengthening previous related results that required atomic networks to be *globally consistent* [Balbiani and Condotta, 2002; Demri and D'Souza, 2007], and (ii) we capture properties that deal with spatial behaviour in a temporal universe, such as periodicity, and continuity and smoothness, with particular fragments of the $\mathcal{L}_1$ logic, and investigate their computational properties; regarding periodicity, we use again the properties of compactness and patchwork to obtain a stronger result than the one existing in literature.

As opposed to the $\mathcal{ST}_1^-$ logic [Gabelaia *et al.*, 2003], $\mathcal{L}_1$ does not rely on the semantics or a particular interpretation of the qualitative spatial constraint language used, but rather on constraint properties, namely, *compactness* and *patchwork* [Lutz and Milicic, 2007]. These properties have been found to hold for RCC-8, Cardinal Direction Algebra (CDA), Block Algebra (BA), and their derivatives [Huang, 2012].

The organization of the paper is as follows. In Section 2 we recall the definition of a qualitative spatial constraint language, along with the properties of compactness, patchwork, and global consistency. Section 3 introduces the $\mathcal{L}_1$ logic, and in Section 4 we explain its implication with compactness and patchwork. In Section 5 we present the fragments that capture spatiotemporal behaviour and analyse their computational properties. In Section 6 we conclude and give directions for future work.

2 Preliminaries

A (binary) qualitative temporal or spatial constraint language [Renz and Ligozat, 2005] is based on a finite set B of *jointly exhaustive and pairwise disjoint* (JEPD) relations defined on a domain D, called the set of base relations. The base relations of set B of a particular qualitative constraint language can be used to represent the definite knowledge between any two entities with respect to the given level of granularity. B contains the identity relation Id, and is closed under the inverse operation ($^{-1}$). Indefinite knowledge can be specified by disjunctions of possible base relations, and is represented by the set containing them. Hence, 2^{B} represents the total set of relations. 2^{B} is equipped with the usual set-theoretic operations (union and intersection), the inverse operation, and the weak composition operation denoted by $\diamond$ [Renz and Ligozat, 2005]. A network from any qualitative spatial constraint language, such as RCC-8 [Randell *et al.*, 1992], Cardinal Direction Algebra (CDA) [Frank, 1991; Ligozat, 1998], Block Algebra (BA) [Balbiani *et al.*, 2002], or Interval Algebra (IA) [Allen, 1983], can be formulated as a qualitative constraint network (QCN) as follows (a RCC-8 example of which is shown in Figure 1).

Definition 1 *A* QCN *is a tuple* (V, C) *where* V *is a non-empty finite set of variables and* C *is a mapping that associates a relation* $C(v, v') \in 2^{\mathsf{B}}$ *to each pair* (v, v') *of* $V \times V$*. Mapping* C *is such that* $C(v, v) = \{\mathsf{Id}\}$ *and* $C(v, v') = (C(v', v))^{-1}$ *for every* $v, v' \in V$*.*

If b is a base relation, $\{b\}$ is a singleton relation. An *atomic* QCN is a QCN where each constraint is a singleton relation. Given two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$, $\mathcal{N} \cup \mathcal{N}'$ denotes the QCN $\mathcal{N}'' = (V'', C'')$, where $V'' = V \cup V'$, $C''(u, v) = C''(v, u) = \mathsf{B}$ for all $(u, v) \in (V \setminus V') \times (V' \setminus V)$, $C''(u, v) = C(u, v) \cap C'(u, v)$ for every $u, v \in V \cap V'$, $C''(u, v) = C(u, v)$ for every $(u, v) \in (V \times V) \setminus (V' \times V')$, and $C''(u, v) = C'(u, v)$ for every $(u, v) \in (V' \times V') \setminus (V \times V)$. A QCN $\mathcal{N} = (V, C)$ is said to be *trivially inconsistent* iff $\exists v, v' \in V$ with $C(v, v') = \emptyset$.

We can interpret any QCN $\mathcal{N} = (V, C)$ using a structure of the form $\mathcal{M}_{\mathsf{S}} = (\mathsf{D}, \alpha)$, where α is a mapping that associates elements of D to elements of V. For the case of RCC-8 for example, if $\mathcal{T}$ is some topological space [Munkres, 2000], let $\mathcal{R}(\mathcal{T})$ denote the set of all non-empty regular closed subsets in $\mathcal{T}$. Then, the domain D of RCC-8 is the set $\mathcal{R}(\mathcal{T})$, which can be infinite. A structure $\mathcal{M}_{\mathsf{S}} = (\mathsf{D}, \alpha)$ is a model for a QCN $\mathcal{N} = (V, C)$, also called a *solution*, if mapping α can yield a spatial configuration where the relations between the spatial variables can be described by C. It follows that a QCN is satisfiable if there exists a model for it. A *partial solution* for $\mathcal{N}$ on $V' \subseteq V$ is the mapping α restricted to V'.

Checking the satisfiability of a RCC-8, CDA, or BA network is NP-complete in the general case [Renz, 1999; Ligozat, 1998; Balbiani *et al.*, 2002]. However, there exist large maximal tractable subclasses of RCC-8, CDA, and BA, which allow for practical and efficient reasoning. In particular, checking the satisfiability of a QCN (V, C) of RCC-8, CDA, or BA comprising only relations from one of its maximal tractable subclasses containing all singleton relations and the universal relation B, can be done in $O(|V|^3)$ time using the $\diamond$-consistency algorithm (also called *algebraic closure*), that iteratively performs the following operation until a fixed point $\overline{C}$ is reached: $\forall v, v', v'' \in V$, $C(v, v') \leftarrow C(v, v') \cap (C(v, v'') \diamond C(v'', v'))$ [Renz and Ligozat, 2005]. Let us recall the definition of global consistency.

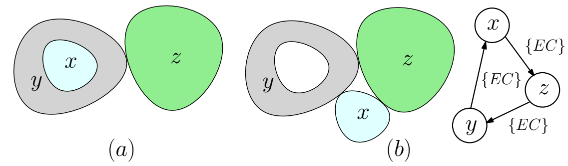

Figure 1: RCC-8 configurations

Definition 2 *A* QCN $\mathcal{N} = (V, C)$ *is* globally consistent *if and only if, for any* $V' \subset V$*, every partial solution on* V' *can be extended to a partial solution on* $V' \cup \{v\} \subseteq V$*, for any* $v \in V \setminus V'$*.*

We now recall the definitions of the constraint properties of *patchwork* and *compactness* in the context of qualitative reasoning, and give an example of how the former properties combined are less strict than global consistency alone. (To be precise, [Lutz and Milicic, 2007] introduced patchwork for atomic QCNs, and [Huang, 2012] generalized it also for non-atomic ones, a property that we will use later in this paper).

Definition 3 ([Huang, 2012; Lutz and Milicic, 2007]) *A qualitative temporal or spatial constraint language has* patchwork*, if for any finite satisfiable constraint networks* $\mathcal{N} = (V, C)$ *and* $\mathcal{N}' = (V', C')$ *defined in this language where for any* $u, v \in V \cap V'$ *we have that* $C(u, v) = C'(u, v)$*, the constraint network* $\mathcal{N} \cup \mathcal{N}'$ *is satisfiable.*

In light of patchwork, which concerns finite networks, compactness ensures satisfiability of an infinite sequence of finite satisfiable extensions of a network.

Definition 4 ([Huang, 2012]) *A qualitative temporal or spatial constraint language has* compactness*, if any infinite set of constraints defined in this language is satisfiable whenever all its finite subsets are satisfiable.*

Intuitively, patchwork ensures that the combination of two satisfiable constraint networks that agree on their common part, i.e., on the constraints between their common variables, continues to be satisfiable, while compactness allows for defining satisfiable networks of infinite size. Global consistency implies patchwork, but the opposite is not true. Even though RCC-8 has patchwork [Huang, 2012], it does not have global consistency [Renz and Ligozat, 2005].

Example. Let us consider the spatial configuration shown in Figure 1(a). Region y is a doughnut, and region x is externally connected to it, by occupying its hole. Further, region z is externally connected to region y. For RCC-8 we know that the constraint network $\{EC(x, y), EC(y, z), EC(x, z)\}$ is satisfiable as it is $\diamond$-consistent. However, the valuation of region variables x and y is such that it is impossible to extend it with a valuation of region variable z so that $EC(x, z)$

may hold. Patchwork allows us to disregard any partial valuations and focus on the satisfiability of the network. Then, we can consider a valuation that respects the constraint network. Such a valuation is, for example, the one presented in Figure 1(b) along with its atomic QCN on the right.

As briefly mentioned earlier, due to Huang we have the following result:

Proposition 1 ([Huang, 2012]) RCC-*8,* CDA*,* BA*, and* IA *have compactness, and patchwork for not trivially inconsistent and* $\diamond$*-consistent* QCN*s defined on one of the maximal tractable subclasses* $\hat{\mathcal{H}}_8$*,* $\mathcal{C}_8$*, or* $\mathcal{Q}_8$*,* $\mathcal{B}_{\mathsf{CDA}}$*,* $\mathcal{H}^n_{\mathsf{IA}}$*, and* $\mathcal{H}_{\mathsf{IA}}$ *respectively.*

3 The $\mathcal{L}_1$ spatiotemporal logic

In general, a spatial QCN, as described in Section 2, constitutes a static spatial configuration in some domain, over a set of spatial variables V. To be able to describe a spatial configuration that changes over time, we can combine PTL [Huth and Ryan, 2004] with a qualitative spatial constraint language in a unique formalism. The domain D of a QCN will always remain the same, but the spatial variables in it may spatially change with the passing time (e.g., in shape, size, or orientation). We can interpret formulas of such a spatiotemporal formalism using a spatiotemporal structure defined as follows.

Definition 5 *A* ST-structure *is a tuple* $\mathcal{M}_{\mathsf{ST}} = (\mathsf{D}, \mathbb{N}, \alpha)$*, where* α *is a mapping that associates elements of* D *to the spatial variables of a set* V *at a point of time* $i \in \mathbb{N}$*. Thus,* $\alpha(i)$ *denotes the set of elements of* D *that are associated with the spatial variables of* V *at point of time* i*. By extending notation,* $\alpha(v, i)$*, where* $v \in V$*, denotes the element of* D *that is associated with spatial variable* v *at point of time* i*.*

For example, in the case of RCC-8, α would be a mapping associating elements of $\mathcal{R}(\mathcal{T})$ to spatial region variables at a point of time $i \in \mathbb{N}$. The set of atomic propositions AP in the case of standalone PTL [Huth and Ryan, 2004] is replaced by the set of base relations B of the qualitative spatial constraint language considered. We will call such a spatiotemporal formula over B a $\mathcal{L}_0$ formula. Thus, the set of $\mathcal{L}_0$ formulas over B is inductively defined as follows: if $P \in \mathsf{B}$ then P is a $\mathcal{L}_0$ formula, and if ψ and ϕ are $\mathcal{L}_0$ formulas then $\neg\phi$, $\phi \vee \psi$, $\bigcirc\phi$, $\Box\phi$, $\Diamond\phi$, and $\phi \mathcal{U} \psi$ are $\mathcal{L}_0$ formulas.

A simple example of a $\mathcal{L}_0$ formula is $\Box NTPP(Athens, Greece)$, stating that Athens will always be located in Greece. To increase the expressiveness of the $\mathcal{L}_0$ logic we can allow the application of operator $\bigcirc$ to spatial variables, i.e., we can have the following statement: $\Box EQ(Greece, \bigcirc Greece)$, which translates to "Greece will never change its borders". We call the enriched logic the $\mathcal{L}_1$ logic.

Definition 6 *Given a* $\mathcal{L}_1$ *formula* ϕ *over* B*, we write* $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \phi$ *for the fact that* $\mathcal{M}_{\mathsf{ST}}$ *satisfies* ϕ *at point of time* i*, with* $i \in \mathbb{N}$ *(or formula* ϕ *is true in* $\mathcal{M}_{\mathsf{ST}}$ *at point of time* i*). The semantics is then defined as follows:*

- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models P(\bigcirc^n v, \bigcirc^m v')$ *iff the relation that holds between* $\alpha(v, i+n)$ *and* $\alpha(v', i+m)$ *is the relation* P*, with* $P \in \mathsf{B}$
- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \neg\phi$ *iff* $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \not\models \phi$
- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \phi \vee \psi$ *iff* $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \phi$ *or* $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \psi$
- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \phi \mathcal{U} \psi$ *if there exists a natural number* k *such that* $i \leq k$*,* $\langle \mathcal{M}_{\mathsf{ST}}, k\rangle \models \psi$*, and for all natural numbers* j*, if* $i \leq j$ *and* $j < k$ *then* $\langle \mathcal{M}_{\mathsf{ST}}, j\rangle \models \phi$

Formulas of the form $\Diamond\phi$ and $\Box\phi$ are abbreviations for $\top \mathcal{U} \phi$ and $\neg(\top \mathcal{U} \neg\phi)$ respectively. A structure $\mathcal{M}_{\mathsf{ST}} = (\mathsf{D}, \mathbb{N}, \alpha)$, for which $\langle \mathcal{M}_{\mathsf{ST}}, 0\rangle \models \phi$, is a model for ϕ. It follows that a $\mathcal{L}_1$ formula ϕ is satisfiable if there exists a model for it. Note that a formula of the form $\bigcirc^k P(\bigcirc^l v, \bigcirc^m v')$ is equivalent to formula $P(\bigcirc^{l+k} v, \bigcirc^{m+k} v')$. The size of $P(\bigcirc^{l+k} v, \bigcirc^{m+k} v')$ is then defined to be equal to $\max\{l+k, m+k\}$. Like in [Balbiani and Condotta, 2002], we define the size of any $\mathcal{L}_1$ formula ϕ, denoted by $|\phi|$, inductively as follows: $P(\bigcirc^l v, \bigcirc^m v') = \max\{l, m\}$; $|\neg\phi| = |\phi|$; $|\phi \vee \psi| = |\phi \mathcal{U} \psi| = \max\{|\phi|, |\psi|\}$.

We then have the following result from [Balbiani and Condotta, 2002], which is proven by the authors using a nondeterministic decision-based procedure:

Theorem 1 ([Balbiani and Condotta, 2002]) *Checking the satisfiability of a* $\mathcal{L}_1$ *formula* ϕ *in a* ST*-structure is* PSPACE*-complete in the length of* ϕ *if atomic* QCN*s defined in the considered qualitative language are globally consistent.*

4 Revisiting the satisfiability problem in $\mathcal{L}_1$

In this section, we revisit a result regarding the satisfiability of $\mathcal{L}_1$ formulas in a ST-structure, using patchwork and compactness. These properties strengthen previous results, in that we do not longer need to restrict atomic QCNs to being globally consistent as in [Balbiani and Condotta, 2002; Demri and D'Souza, 2007], but we can consider atomic QCNs that have compactness and patchwork. As explained in Section 2, compactness and patchwork combined are less strict than global consistency alone.

Given a $\mathcal{L}_1$ formula ϕ, Balbiani and Condotta in [Balbiani and Condotta, 2002] show that the satisfiability of formula ϕ can be checked by characterizing a particular infinite sequence of finite satisfiable atomic QCNs representing an infinite consistent valuation of ϕ. Each of the QCNs of such a sequence represents a set of spatial constraints in a fixed-width window of time. The set of spatial constraints at point of time i, is given by the i-th QCN in the infinite sequence, and shares spatial constraints with the next QCN. Moreover, in such a sequence, there exists a point of time after which the corresponding QCNs replicate the same set of spatial constraints. The global consistency property is then used for the following two tasks: (i) to prove that by considering all the QCNs of the aforementioned sequence we obtain a consistent set of constraints, and (ii) to prove that in such an infinite sequence, a sub-sequence which begins and ends with two QCNs representing the same set of spatial constraints can be reduced to just considering the first QCN.

In the sequel, we formally show that tasks (i) and (ii) can be performed using the properties of patchwork and compactness instead. As a consequence, we can generalize a result regarding the satisfiability of a $\mathcal{L}_1$ formula ϕ to a larger class of calculi than the previously considered in literature. We now introduce the two aforementioned tasks in the form of two propositions.

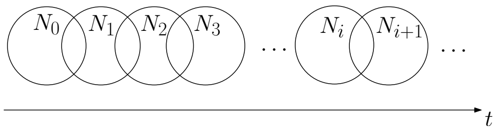

Figure 2: A countably infinite sequence of satisfiable atomic QCNs that agree on their common part

Proposition 2 *Let $V = \{v_0, \ldots, v_n\}$ be a set of variables, $w \geq 0$ an integer, and $\mathcal{S} = (\mathcal{N}_0 = (V_0, C_0), \mathcal{N}_1 = (V_1, C_1), \ldots)$ a countably infinite sequence of satisfiable atomic* QCN*s, as shown in Figure 2, such that:*

- *for each $i \geq 0$, V_i is defined by the set of variables $\{v_0^0,\ldots,v_n^0,\ldots,v_0^w,\ldots,v_n^w\}$,*
- *for each $i \geq 0$, for all $m, m' \in \{0, \ldots, n\}$, and for all $k, k' \in \{1, \ldots, w\}$, $C_i(v_m^k, v_{m'}^{k'}) = C_{i+1}(v_m^{k-1}, v_{m'}^{k'-1})$.*

We have that if the constraint language considered has compactness and patchwork for atomic QCN*s, then $\mathcal{S}$ defines a consistent set of qualitative constraints.*

Proof. Given $\mathcal{N}_i$, we rewrite its set of variables to $\{v_0^i,\ldots,v_n^i,\ldots,v_0^{w+i},\ldots,v_n^{w+i}\}$. Then, by patchwork we can assert that for each integer $k \geq 0$, $\bigcup_{k \geq i \geq 0} \mathcal{N}_i$ is a consistent set of qualitative constraints. Suppose though, that $\bigcup_{i \geq 0} \mathcal{N}_i$ is an inconsistent set. By compactness we know that there exists an integer $k' \geq 0$ for which $\bigcup_{k' \geq i \geq 0} \mathcal{N}_i$ is inconsistent. This is a contradiction. Thus, $\mathcal{S}$ defines a consistent set of qualitative constraints. ⊣

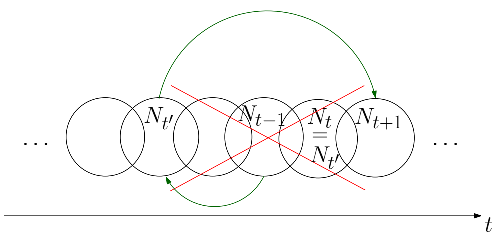

Figure 3: A countably infinite sequence of satisfiable atomic QCNs that contains a sub-sequence which begins and ends with two QCNs representing the same set of spatial constraints; we can reduce the sub-sequence to just considering the first QCN and patch it with the QCN following the sub-sequence

Proposition 3 *Let $V = \{v_0, \ldots, v_n\}$ be a set of variables, $w \geq 0, t > t' \geq 0$ three integers, and $\mathcal{S} = (\mathcal{N}_0 = (V_0, C_0), \mathcal{N}_1 = (V_1, C_1), \ldots)$ a countably infinite sequence of satisfiable atomic* QCN*s, as shown in Figure 3, such that:*

- *for each $i \geq 0$, V_i is defined by the set of variables $\{v_0^0,\ldots,v_n^0,\ldots,v_0^w,\ldots,v_n^w\}$,*
- *for each $i \geq 0$, for all $m, m' \in \{0, \ldots, n\}$, and for all $k, k' \in \{1, \ldots, w\}$, $C_i(v_m^k, v_{m'}^{k'}) = C_{i+1}(v_m^{k-1}, v_{m'}^{k'-1})$,*
- *for all $m, m' \in \{0, \ldots, n\}$ and all $k, k' \in \{0, \ldots, w\}$, $C_{t'}(v_m^k, v_{m'}^{k'}) = C_t(v_m^k, v_{m'}^{k'})$.*

Let $\mathcal{S}' = (\mathcal{N}'_0 = (V'_0, C'_0), \mathcal{N}'_1 = (V'_1, C'_1), \ldots)$ be the infinite sequence defined by:

- *for all $i \in \{0, \ldots, t'\}$, $\mathcal{N}'_i = \mathcal{N}_i$,*
- *for all $i > t'$, $V'_i = V_i$, and for all $m, m' \in \{0, \ldots, n\}$ and all $k, k' \in \{0, \ldots, w\}$, $C'_i(v_m^k, v_{m'}^{k'}) = C_{i+(t-t')}(v_m^k, v_{m'}^{k'})$.*

We have that if the constraint language considered has compactness and patchwork for atomic QCN*s, then $\mathcal{S}'$ defines a consistent set of qualitative constraints.*

Proof. We have $\mathcal{N}_i$ which is a satisfiable QCN for all $i \geq 0$. From this, we can deduce that $\mathcal{N}'_i$ is a satisfiable QCN for all $i \geq 0$. By Proposition 2 we can deduce that $\mathcal{S}'$ defines a consistent set of qualitative constraints. ⊣

We now can obtain the following result:

Theorem 2 *Checking the satisfiability of a $\mathcal{L}_1$ formula ϕ in a* ST*-structure is* PSPACE*-complete in the length of ϕ if the qualitative spatial constraint language considered has compactness and patchwork for atomic* QCN*s.*

Proof. (Sketch) Consider the approach in [Balbiani and Condotta, 2002] where a proof of PSPACE-completeness is given for a logic that considers qualitative constraint languages for which satisfiable atomic QCNs are globally consistent (see Theorem 1). To be able to replace the use of global consistency with the use of patchwork and compactness, we need to use Propositions 2 and 3 in the proofs of Lemmas 3 and 4 in [Balbiani and Condotta, 2002]. The interested reader can verify that the aforementioned proofs make use of global consistency to perform exactly the tasks described by Propositions 2 and 3. Since these propositions build on compactness and patchwork, we can prove PSPACE-completeness using these properties instead. ⊣

Theorem 2 allows us to consider more calculi than the ones considered in literature for which the combination with PTL yields PSPACE-completeness. Due to the lack of global consistency for RCC-8 [Renz and Ligozat, 2005], in [Gabelaia *et al.*, 2003] the authors restrict themselves to a very particular domain interpretation of RCC-8 to prove that the $\mathcal{ST}_1^-$ logic is PSPACE-complete. As already noted in Section 1, the $\mathcal{ST}_1^-$ logic is the $\mathcal{L}_1$ logic when the considered qualitative constraint language is RCC-8. $\mathcal{L}_1$ does not rely on the semantics of the qualitative constraint language used, but rather on the constraint properties of *compactness* and *patchwork* [Lutz and Milicic, 2007]. Therefore, $\mathcal{L}_1$ is by default able to consider all calculi that have these properties, such as RCC-8 [Randell *et al.*, 1992], Cardinal Direction Algebra (CDA) [Frank, 1991; Ligozat, 1998], Block Algebra (BA) [Balbiani *et al.*, 2002], and even Interval Algebra (IA) [Allen, 1983] when viewed as a spatial calculus. The most notable languages that have patchwork and compactness are listed in [Huang, 2012]. In particular, by Proposition 1 and Theorem 2 we can have the following result:

Corollary 1 *Checking the satisfiability of a $\mathcal{L}_1$ formula in a* ST*-structure is* PSPACE*-complete for* RCC-8*,* CDA, BA, IA.

5 Capturing spatiotemporal behaviour

In this section, we use particular fragments of the $\mathcal{L}_1$ logic to capture properties that deal with spatial behaviour in a temporal universe, such as periodicity, and continuity and smoothness. We first define a sublanguage of $\mathcal{L}_1$ that will be of use in studying our fragments. In particular, let $\mathcal{L}_{\mathsf{QCN}}$ be the sublanguage defined by statements of the form $\phi = \bigwedge(R(\bigcirc^n x_i, \bigcirc^m x_j))$, where $R \in 2^{\mathsf{B}}$. It is easy to see that a $\mathcal{L}_{\mathsf{QCN}}$ formula ϕ can be expressed by a QCN $\mathcal{N} = (V, C)$ as follows. If formula ϕ comprises the set of variables $\{x_0, \ldots, x_k\}$, V will be the set $\{v_0^0, \ldots, v_k^0, \ldots, v_0^{|\phi|}, \ldots, v_k^{|\phi|}\}$. Then, $C(v_i^n, v_j^m) = R$ if $R(\bigcirc^n x_i, \bigcirc^m x_j)$ is a subformula of ϕ, and $C(v_i^n, v_j^m) = (\mathsf{B}$ if $v_i^n \neq v_j^m$ else $\{\mathsf{Id}\})$ otherwise. Expressing a QCN as a $\mathcal{L}_{\mathsf{QCN}}$ formula is also straight-forward. As such, checking the satisfiability of a $\mathcal{L}_{\mathsf{QCN}}$ formula in a ST-structure has the same complexity as the satisfiability problem for the corresponding QCN in the considered qualitative constraint language, and, in particular, is NP-complete for the considered languages of RCC-8, CDA, and BA.

5.1 Spatiotemporal periodicity

In this section, we capture the notion of an ultimately periodic qualitative constraint network (UPQCN) [Condotta *et al.*, 2005]. A UPQCN is a temporalized QCN that evolves over time with a recurrent pattern. It is defined as follows.

Definition 7 ([Condotta *et al.*, 2005]) *A* UPQCN *is a structure* (V, C, t_{min}, t_{max})*, where* $V = \{v_0^0, \ldots, v_n^0, \ldots, v_0^{t_{max}}, \ldots, v_n^{t_{max}}\}$ *is a finite set of variables,* t_{min} *and* t_{max} *are two integers such that* $0 \leq t_{min} \leq t_{max}$*, and* C *is a mapping that associates a relation* $C(v, v') \in 2^{\mathsf{B}}$ *to each pair* (v, v') *of* $V \times V$*. Mapping* C *is such that* $C(v, v) = \{\mathsf{Id}\}$ *and* $C(v, v') = (C(v', v))^{-1}$ *for every* $v, v' \in V$*.*

Intuitively, in a spatial context, each variable $v^t \in V$ represents the occurrence of the spatial component of entity v at point of time t, with $t \in \mathbb{N}$. $C(u^t, v^{t'})$ is a constraint on the relative positions of the occurrence of u at point of time t and that of v at point of time t'. The constraints expressed by C are twofold: firstly, all constraints from 0 up to t_{max} have to be satisfied; secondly, all constraints from t_{min} to t_{max} have to be satisfied up to t_{max}, but also on all subsequent periods $\{t_{min} + t, \ldots, t_{max} + t\}$, where $t \in \mathbb{N}$. In other words, the structure defines both initial constraints (up to t_{max}), and a recurrent pattern, or *motif*, of constraints (from t_{min} to t_{max}) which repeats itself indefinitely. The *motif* of a UPQCN $\mathcal{U}$ is defined as follows.

Definition 8 ([Condotta *et al.*, 2005]) *Let* $\mathcal{U} = (V, C, t_{min}, t_{max})$ *be a* UPQCN *over* n *entities. The* motif *of* $\mathcal{U}$*, denoted by* $motif(\mathcal{U})$*, is the* QCN $\mathcal{N}_m = (V_m, C_m)$*, where* $V_m = \{v^t \mid v^t \in V$ *and* $t \leq lg\}$*, with* $lg = t_{max} - t_{min}$*, and* $\forall\ m, m' \in \{0, \ldots, n\}$ *and* $\forall\ k, k' \in \{0, \ldots, lg\}$*,* $C_m(v_m^k, v_{m'}^{k'}) = C(v_m^{k+t_{min}}, v_{m'}^{k'+t_{min}})$*.*

As noted earlier, a UPQCN is a QCN that extends over a fixed-width window of time, and contains a smaller QCN as a recurrent pattern evolving over time. We can define fragment $\mathcal{L}_{\mathsf{UPQCN}}$ to capture periodicity as follows.

Definition 9 *Given a* UPQCN $\mathcal{U} = (V, C, t_{min}, t_{max})$*, fragment* $\mathcal{L}_{\mathsf{UPQCN}}$ *comprises formulas of the following form:*

$$\phi \wedge \bigcirc^m \Box \phi_{motif(\mathcal{U})}$$

where ϕ *is a* $\mathcal{L}_{\mathsf{QCN}}$ *formula that corresponds to* $\mathcal{U}$*,* $\phi_{motif(\mathcal{U})}$ *is a* $\mathcal{L}_{\mathsf{QCN}}$ *formula that corresponds to the periodical part of* $\mathcal{U}$*, namely,* $motif(\mathcal{U})$*, and* $m = t_{min}$ *defines the beginning of the recurrent pattern.*

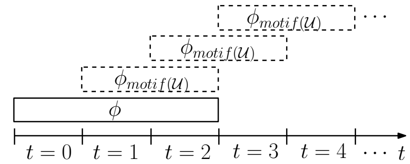

Figure 4: A $\mathcal{L}_{\mathsf{UPQCN}}$ formula ϕ over timeline t

Clearly, a UPQCN $\mathcal{U}$ is satisfiable if and only if the $\mathcal{L}_{\mathsf{UPQCN}}$ formula representing it is satisfiable. Given a UPQCN $\mathcal{U}$, the relationship between $\mathcal{L}_{\mathsf{QCN}}$ formulas ϕ and $\phi_{motif(\mathcal{U})}$, as provided in the aforementioned definition, is depicted in Figure 4. As ϕ stretches over the timeline, it forms a pattern, denoted by $\phi_{\text{motif}(\mathcal{U})}$, which holds at every consecutive period of time after its first appearance, as clearly observed in the figure.

Example. Let us consider the $\mathcal{L}_{\mathsf{UPQCN}}$ formula $PO(X, Y) \wedge EC(X, \bigcirc X) \wedge \bigcirc\Box DC(X, \bigcirc Y)$. Assuming a timeline t, formula ϕ corresponds to a UPQCN $\mathcal{U}$ that extends over three consecutive points of time $t = 0$, $t = 1$, and $t = 2$. At $t = 0$ we have the constraint $PO(X, Y)$, between points of time $t = 0$ and $t = 1$ we have the constraint $EC(X, \bigcirc X)$, and between points of time $t = 1$ and $t = 2$ we have the constraint $DC(X, \bigcirc Y)$. It is easy to see that due to the $\Box$ operator, constraint $DC(X, \bigcirc Y)$ must hold over the period defined by points of time $t = 1$ and $t = 2$, but also over all consecutive periods of time in t. In fact, $DC(X, \bigcirc Y)$ is the motif of $\mathcal{U}$, captured by $\phi_{\text{motif}(\mathcal{U})}$, where $m = 1$ in our example $\mathcal{L}_{\mathsf{UPQCN}}$ formula.

The main result of [Condotta *et al.*, 2005] concerns the satisfiability problem for a UPQCN where the qualitative constraints belong to a class for which all $\diamond$-consistent and not trivially inconsistent QCNs are globally consistent. More precisely, it was shown that the satisfiability of a UPQCN $\mathcal{U}$ can be checked by characterizing a particular infinite sequence of finite $\diamond$-consistent and not trivially inconsistent QCNs representing an infinite consistent valuation of $\mathcal{U}$. Each of the QCNs of such a sequence represents a set of spatial constraints in a fixed-width window of time. The set of spatial constraints at point of time i, is given by the i-th QCN in the infinite sequence, and shares spatial constraints with the next QCN. Moreover, in such a sequence, there exists a point of time after which every QCN replicates the same set of spatial constraints with the previous QCN in the sequence. Global consistency was then used to prove that by considering all the QCNs of the aforementioned sequence we obtain a consistent set of constraints. We can generalize the result of [Condotta *et al.*, 2005] with the following proposition:

Proposition 4 *Let* $V = \{v_0, \ldots, v_n\}$ *be a set of variables,* $w \geq 0$*,* $t \geq 0$ *two integers, and* $\mathcal{S} = (\mathcal{N}_0 = (V_0, C_0), \mathcal{N}_1 = (V_1, C_1), \ldots)$ *a countably infinite sequence of not trivially inconsistent and* $\diamond$*-consistent* QCN*s, as shown in Figure 5, such that:*

- *for each* $i \geq 0$*,* V_i *is defined by the set of variables* $\{v_0^0, \ldots, v_n^0, \ldots, v_0^w, \ldots, v_n^w\}$*,*

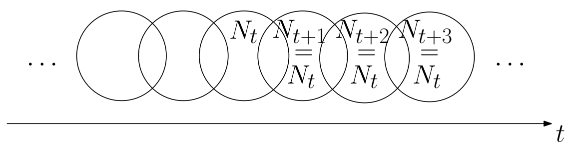

Figure 5: A countably infinite sequence of not trivially inconsistent and $\diamond$-consistent QCNs, where there exists a point of time t after which the QCNs in the sequence represent the same set of constraints

- *for each* $i \geq 0$*, for all* $m, m' \in \{0, \ldots, n\}$*, and for all* $k, k' \in \{1, \ldots, w\}$, $C_i(v_m^k, v_{m'}^{k'}) = C_{i+1}(v_m^{k-1}, v_{m'}^{k'-1})$,
- *for all* $m, m' \in \{0, \ldots, n\}$*, all* $k, k' \in \{0, \ldots, w\}$*, and all* $t' > t$, $C_{t'}(v_m^k, v_{m'}^{k'}) = C_t(v_m^k, v_{m'}^{k'})$.

We have that if the qualitative spatial constraint language considered has compactness, patchwork for not trivially inconsistent and $\diamond$*-consistent* QCN*s, and* $\diamond$*-consistency which implies satisfiability, then* $\mathcal{S}$ *defines a consistent set of qualitative constraints.*

Proof. Since $\diamond$-consistency implies satisfiability, for each $i \geq 0$ we have that $\mathcal{N}_i$ is a satisfiable QCN. Given $\mathcal{N}_i$, we rewrite its set of variables to $\{v_0^i, \ldots, v_n^i, \ldots, v_0^{w+i}, \ldots, v_n^{w+i}\}$. Then, by patchwork we can assert that for each integer $k \geq 0$, $\bigcup_{k \geq i \geq 0} \mathcal{N}_i$ is a consistent set of qualitative constraints. Suppose though, that $\bigcup_{i \geq 0} \mathcal{N}_i$ is an inconsistent set. By compactness we know that there exists an integer $k' \geq 0$ for which $\bigcup_{k' \geq i \geq 0} \mathcal{N}_i$ is inconsistent. This is a contradiction. Thus, $\mathcal{S}$ defines a consistent set of qualitative constraints. $\dashv$

Using Proposition 4 and the line of reasoning followed in [Condotta *et al.*, 2005] we can prove the following theorem:

Theorem 3 *The satisfiability problem for a* $\mathcal{L}_{\mathsf{UPQCN}}$ *formula that corresponds to a* UPQCN *defined on a qualitative spatial constraint language having compactness, patchwork for not trivially inconsistent and* $\diamond$*-consistent* QCN*s, and* $\diamond$*-consistency which implies satisfiability, is in* PTIME*.*

By Proposition 1 and Theorem 3 we have the following result:

Corollary 2 *The satisfiability problem for a* $\mathcal{L}_{\mathsf{UPQCN}}$ *formula that corresponds to a* UPQCN *defined on one of the classes* $\hat{\mathcal{H}}_8, \mathcal{C}_8$*, or* $\mathcal{Q}_8$ *for* RCC-8*,* $\mathcal{B}_{\mathsf{CDA}}$ *for* CDA*,* $\mathcal{H}_{\mathsf{IA}}^n$ *for* BA*, and* $\mathcal{H}_{\mathsf{IA}}$ *for* IA*, is in* PTIME*.*

Theorem 3 is a significant strengthening of the main result obtained in [Condotta *et al.*, 2005], as we no longer need to restrict ourselves to a small class of relations satisfying global consistency (if such a class exists), but we can use a maximal tractable subclass of relations for the considered calculi here. For example, for RCC-8 there does not exist a class of relations containing all singleton relations that satisfies global consistency (as explained in Section 2 and stated in [Renz and Ligozat, 2005]), but the class of relations satisfying patchwork and compactness can be any of its maximal tractable subclasses $\hat{\mathcal{H}}_8, \mathcal{C}_8$, and $\mathcal{Q}_8$ [Renz, 1999], which comprise up to $\sim 60\%$ of the whole set of RCC-8 relations.

5.2 Spatiotemporal smoothness and continuity

In [Westphal *et al.*, 2013] the authors study the problem of transition constraints in Point Algebra (PA) [van Beek, 1992]. In particular, they take a relational approach to the problem and define global constraints that capture smoothness and continuity. Here, we will make a similar contribution for spatiotemporal logics, as we will define statements that can capture smoothness and continuity within the context of $\mathcal{L}_1$.

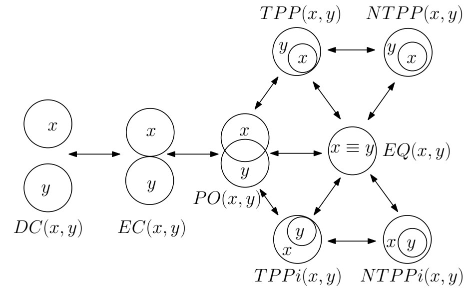

Figure 6: A transition graph of RCC-8

Smoothness and continuity in a qualitative spatial constraint language can be encoded by a *conceptual neighborhood graph* [Freksa, 1991], or *transition graph* for short, which is defined as follows.

Definition 10 *A transition graph is a graph* $\Gamma = (\mathsf{B}, E)$ *where* $E = \{(b, b') \mid (b, b') \in \mathsf{B} \times \mathsf{B};$ *and* b *and* b' *are* conceptually proximal *[Freksa, 1991]}* *(and* B *is the set of base relations of a qualitative constraint language, as a reminder).*

As an example, a transition graph of RCC-8 is depicted in Figure 6. (Self loops are omitted in Figure 6 for convenience; clearly, every base relation is conceptually proximal to itself.) In RCC-8 the base relations $DC(v, v')$ and $PO(v, v')$ are not conceptually proximal since a transition between those relations must go through relation $EC(v, v')$. Transition graphs can be established for all qualitative spatial constraint languages, a subset of which are found in [Freksa, 1991; Santos and Moreira, 2009; Egenhofer, 2010].

Definition 11 *Given a transition graph* $\Gamma = (\mathsf{B}, E)$*, the* conceptual neighborhood *of a vertex* $b \in \mathsf{B}$ *is the set* $N_C(b) = \{b' \mid (b, b') \in E\}$*.*

We can capture transition constraints in the $\mathcal{L}_{\mathsf{QCN}}$ logic, by defining a particular formula ϕ_Γ comprising certain statements as follows.

Definition 12 *Given the set of spatial variables* V *of a* $\mathcal{L}_{\mathsf{QCN}}$ *formula, and a transition graph* $\Gamma = (\mathsf{B}, E)$*, formula* ϕ_Γ *is defined for all* $(v, v') \in V \times V$ *as a conjunction of the following* $\mathcal{L}_1$ *statements:*

$$\bigwedge_{b \in \mathsf{B}} \Box(b(v, v') \rightarrow \bigcirc(\bigvee_{b' \in N_C(b)} (b'(v, v'))))$$

We can obtain the following theorem:

Theorem 4 *Given a* $\mathcal{L}_{\mathsf{QCN}}$ *formula* ϕ *defined on a qualitative spatial constraint language, and a transition graph* Γ *of that language, checking the satisfiability of formula* $\phi \wedge \phi_\Gamma$ *in a* ST*-structure has the same complexity as the satisfiability problem for the corresponding* QCN *of* ϕ *in that language.*

Proof. (Sketch) Formula ϕ_Γ can be seen as a set of integrated constraints on each pair of the same spatial entities appearing at adjacent points of time. We can check all such constraints in a candidate solution in polynomial time, as the timeline is upper bounded by $|\phi|$. In particular, the $\Box$ operator propagates the transition constraints indefinitely, but, due to compactness and its implication regarding infinite sequences of finite satisfiable extensions of a network, we only need to propagate the constraints up to point of time $|\phi|$, and assume to always have the same satisfiable valuation afterwards. $\dashv$

By Theorem 4 and the discussion in Section 2 we can obtain the following result:

Corollary 3 *Given a $\mathcal{L}_{\mathsf{QCN}}$ formula ϕ defined on* RCC-8*,* CDA*,* BA*, or* IA*, and a transition graph Γ of the considered language, checking the satisfiability of formula $\phi \wedge \phi_\Gamma$ in a* ST*-structure is* NP*-complete.*

6 Conclusion and future work

In this paper, we considered a generalized qualitative spatiotemporal formalism, namely, the $\mathcal{L}_1$ logic, which is the product of the combination of PTL with any qualitative spatial constraint language, such as RCC-8, Cardinal Direction Algebra, and Block Algebra, and showed that satisfiability checking of a $\mathcal{L}_1$ formula is PSPACE-complete if the constraint language considered has the constraint properties of *compactness* and *patchwork* for atomic networks, thus, strengthening previous results that required atomic networks to be *globally consistent* and, consequently, generalizing to a larger class of calculi. Further, we defined fragments of the $\mathcal{L}_1$ logic to capture properties that deal with spatial behaviour in a temporal universe, such as periodicity, and continuity and smoothness, formally introduced them, and investigated their computational properties. Regarding periodicity, we used again the properties of compactness and patchwork to obtain a much stronger result than the one existing in literature. To summarize, we revisited, re-established, and strengthened results on the computational complexity of the $\mathcal{L}_1$ logic and on important aspects of spatial change in a temporal universe, such as periodicity. A promising line of future research would be to consider SAT-encodings of the formalism and its particular fragments that we studied in this paper.

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