# Generalized Qualitative Spatio-Temporal Reasoning: Complexity and Tableau Method

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**Résumé**

Dans cet article, nous traitons de la logique spatio-temporelle qui résulte de combinaison de la logique temporelle propositionnelle (PTL) avec le langage de contraintes spatiales qualitatives $\mathcal{L}_1$. Nous proposons une méthode de résolution à base de tableaux sémantiques basée sur la méthode des tableaux de Wolper pour PTL. Par ailleurs, nous étudions les effets de l'utilisation des propriétés de *compacité* et de *patchwork* pour le raisonnement spatio-temporelle. Nous étudions notamment l'effet de ces propriétés sur la complexité du problème de satisfiabilité de $\mathcal{L}_1$ en remplaçant notamment la propriété de cohérence globale utilisée habituellement dans la littérature. Ceci permet la généralisation à un plus grand nombre de langages de contraintes spatiales qualitatives. Enfin, les résultats obtenus permettent de prouver la validité de notre méthode des tableaux pour $\mathcal{L}_1$.

**Abstract**

We study the spatiotemporal logic that results by combining the propositional temporal logic (PTL) with a qualitative spatial constraint language, namely, the $\mathcal{L}_1$ logic, and present a first semantic tableau method that given a $\mathcal{L}_1$ formula $\phi$ systematically searches for a model for $\phi$. Our approach builds on Wolper's tableau method for PTL, while the ideas provided can be carried to other tableau methods for PTL as well. Further, we investigate the implication of the constraint properties of *compactness* and *patchwork* in spatiotemporal reasoning. We use these properties to strengthen results regarding the complexity of the satisfiability problem in $\mathcal{L}_1$, by replacing the stricter global consistency property used in literature and generalizing to more qualitative spatial constraint languages. Finally, the obtained strengthened results allow us to prove the correctness of our tableau method for $\mathcal{L}_1$.

## 1 Introduction

Time and space are fundamental cognitive concepts that have been the focus of study in many scientific disciplines, including Artificial Intelligence and, in particular, Knowledge Representation. Knowledge Representation has been quite successful in dealing with the concepts of time and space, and has developed formalisms that range from temporal and spatial databases [18], to quantitative models developed in computational geometry [14] and qualitative constraint languages and logical theories developed in qualitative reasoning [7, 21].

Towards constraint-based qualitative spatiotemporal reasoning, most of the work has relied on formalisms based on the propositional temporal logic (PTL), also known as linear temporal logic, and the qualitative spatial constraint language RCC-8 [21, 20]. PTL [9] is the well known temporal logic comprising operators $\mathcal{U}$ (until), $\bigcirc$ (next point in time), $\Box$ (always), and $\Diamond$ (eventually) over various flows in time, such as $\langle \mathbb{N}, < \rangle$. RCC-8 is a fragment of the Region Connection Calculus (RCC) [15] and is used to describe regions that are non-empty regular subsets of some topological space by stating their topological relations to each other. The topological relations comprise relations $DC$ (disconnected), $EC$ (externally connected), $EQ$ (equal), $PO$ (partially overlapping), $TPP$ (tangential proper part), $TPPi$ (tangential proper part inverse), $NTPP$ (non-tangential proper part), $NTPPi$ (non-tangential proper part inverse). These 8 relations are depicted in [15, Fig. 4]. One of the most important of such formalisms is the $\mathcal{ST}_1^-$ logic [5]. For example, one can have the following statement using that formalism: $\Diamond TPP(X, Y)$, which translates to "eventually region $X$ will be a tangential proper part of region $Y$".

In this paper, we consider a generalization of the

$\mathcal{ST}_1^-$ logic, denoted by $\mathcal{L}_1$, which is the product of the combination of PTL [9] with any qualitative spatial constraint language, such as RCC-8 [15], Cardinal Direction Algebra (CDA) [4, 11], and Block Algebra (BA) [6], and make the following contributions: (*i*) we show that satisfiability checking of a $\mathcal{L}_1$ formula is PSPACE-complete if the qualitative spatial constraint language considered has the constraint properties of *compactness* and *patchwork* [12] for atomic networks, thus, strengthening previous related results that required atomic networks to be *globally consistent* [2, 3], and (*ii*) we present a first semantic tableau method that given a $\mathcal{L}_1$ formula $\phi$ systematically searches for a model for $\phi$. This method builds on the tableau method for PTL of Wolper [19], and makes use of our strengthened results to ensure soundness and completeness. It is important to note, that Wolper's method serves as the basis to illustrate our line of reasoning, and that the techniques presented can be carried to other more efficient tableau methods for PTL as well.

As opposed to the $\mathcal{ST}_1^-$ logic [5], $\mathcal{L}_1$ does not rely on the semantics or a particular interpretation of the qualitative spatial constraint language used, but rather on constraint properties, namely, *compactness* and *patchwork* [12]. These properties have been found to hold for RCC-8, Cardinal Direction Algebra (CDA), Block Algebra (BA), and their derivatives [8].

The organization of the paper is as follows. In Section 2 we recall the definition of a qualitative spatial constraint language, along with the properties of compactness, patchwork, and global consistency. Section 3 introduces the $\mathcal{L}_1$ logic, and in Section 4 we explain its implication with compactness and patchwork. In Section 5 we present our tableau method for checking the satisfiability of a $\mathcal{L}_1$ formula. In Section 6 we conclude and give directions for future work.

## 2 Preliminaries

A (binary) qualitative temporal or spatial constraint language [17] is based on a finite set B of *jointly exhaustive and pairwise disjoint* (JEPD) relations defined on a domain D, called the set of base relations. The base relations of set B of a particular qualitative constraint language can be used to represent the definite knowledge between any two entities with respect to the given level of granularity. B contains the identity relation Id, and is closed under the inverse operation ($^{-1}$). Indefinite knowledge can be specified by disjunctions of possible base relations, and is represented by the set containing them. Hence, $2^{\mathsf{B}}$ represents the total set of relations. $2^{\mathsf{B}}$ is equipped with the usual set-theoretic operations (union and intersection), the inverse operation, and the weak composition operation denoted by $\diamond$ [17]. A network from any qualitative spatial constraint language, such as RCC-8 [15], Cardinal Direction Algebra (CDA) [4, 11], or Block Algebra (BA) [6], can be formulated as a qualitative constraint network (QCN) as follows (a RCC-8 example of which is shown in Figure 1).

**Definition 1** *A* QCN *is a tuple* $(V, C)$ *where* $V$ *is a non-empty finite set of variables and* $C$ *is a mapping that associates a relation* $C(v, v') \in 2^{\mathsf{B}}$ *to each pair* $(v, v')$ *of* $V \times V$*. Mapping* $C$ *is such that* $C(v, v) = \{\mathsf{Id}\}$ *and* $C(v, v') = (C(v', v))^{-1}$ *for every* $v, v' \in V$*.*

If $b$ is a base relation, $\{b\}$ is a singleton relation. An *atomic* QCN is a QCN where each constraint is a singleton relation. Note that we always regard a QCN as a complete network. Given two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$, $\mathcal{N} \cup \mathcal{N}'$ denotes the QCN $\mathcal{N}'' = (V'', C'')$, where $V'' = V \cup V'$, $C''(u, v) = C''(v, u) = \mathsf{B}$ for all $(u, v) \in (V \setminus V') \times (V' \setminus V)$, $C''(u, v) = C(u, v) \cap C'(u, v)$ for every $u, v \in V \cap V'$, $C''(u, v) = C(u, v)$ for every $(u, v) \in (V \times V) \setminus (V' \times V')$, and $C''(u, v) = C'(u, v)$ for every $(u, v) \in (V' \times V') \setminus (V \times V)$. Given a QCN $\mathcal{N} = (V, C)$ and $u, v \in V$, $C(u, v)$ will be also denoted by $\mathcal{N}[u, v]$.

We can interpret any QCN $\mathcal{N} = (V, C)$ using a structure of the form $\mathcal{M}_{\mathsf{S}} = (\mathsf{D}, \alpha)$, where $\alpha$ is a mapping that associates elements of D to elements of $V$. For the case of RCC-8 for example, if $\mathcal{T}$ is some topological space [13], let $\mathcal{R}(\mathcal{T})$ denote the set of all non-empty regular closed subsets in $\mathcal{T}$. Then, the domain D of RCC-8 is the set $\mathcal{R}(\mathcal{T})$, which can be infinite. A structure $\mathcal{M}_{\mathsf{S}} = (\mathsf{D}, \alpha)$ is a model for a QCN $\mathcal{N} = (V, C)$, also called a *solution*, if mapping $\alpha$ can yield a spatial configuration where the relations between the spatial variables can be described by $C$. It follows that a QCN is satisfiable if there exists a model for it. A *partial solution* for $\mathcal{N}$ on $V' \subseteq V$ is the mapping $\alpha$ restricted to $V'$.

Checking the satisfiability of a RCC-8, CDA, or BA network is $\mathcal{NP}$-complete in the general case [16, 11, 6]. However, there exist large maximal tractable subclasses of RCC-8, CDA, and BA, which allow for practical and efficient reasoning. In particular, checking the satisfiability of a QCN $(V, C)$ of RCC-8, CDA, or BA comprising only relations from one of its maximal tractable subclasses containing all singleton relations and the universal relation B, can be done in $O(|V|^3)$ time using the $\diamond$-consistency algorithm (also called *algebraic closure*), that iteratively performs the following operation until a fixed point $\overline{C}$ is reached: $\forall v, v', v'' \in V,\ C(v, v') \leftarrow C(v, v') \cap (C(v, v'') \diamond C(v'', v'))$ [17].
Let us recall the definition of global consistency.

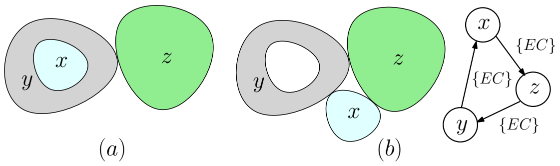


Figure 1: RCC-8 configurations

**Definition 2** *A* QCN $\mathcal{N} = (V, C)$ *is* globally consistent *if and only if, for any $V' \subset V$, every partial solution on $V'$ can be extended to a partial solution on $V' \cup \{v\} \subseteq V$, for any $v \in V \setminus V'$.*

We now recall the definitions of the constraint properties of *patchwork* and *compactness* in the context of qualitative reasoning and give an example of how the former properties combined are less strict than global consistency alone. (To be precise, [12] introduced patchwork for atomic QCNs, and [8] generalized it also for non-atomic ones).

**Definition 3 ([8, 12])** *A qualitative temporal or spatial constraint language has* patchwork*, if for any finite satisfiable constraint networks $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V', C')$ defined in this language where for any $u, v \in V \cap V'$ we have that $C(u, v) = C'(u, v)$, the constraint network $\mathcal{N} \cup \mathcal{N}'$ is satisfiable.*

In light of patchwork, which concerns finite networks, compactness ensures satisfiability of an infinite sequence of finite satisfiable extensions of a network.

**Definition 4 ([8])** *A qualitative temporal or spatial constraint language has* compactness*, if any infinite set of constraints defined in this language is satisfiable whenever all its finite subsets are satisfiable.*

Intuitively, patchwork ensures that the combination of two satisfiable constraint networks that agree on their common part, i.e., on the constraints between their common variables, continues to be satisfiable, while compactness allows for defining satisfiable networks of infinite size. Global consistency implies patchwork, but the opposite is not true. Even though RCC-8 has patchwork [8], it does not have global consistency [17].

**Example**. Let us consider the spatial configuration shown in Figure 1(a). Region $y$ is a doughnut, and region $x$ is externally connected to it, by occupying its hole. Further, region $z$ is externally connected to region $y$. For RCC-8 we know that the constraint network $\{EC(x, y), EC(y, z), EC(x, z)\}$ is satisfiable as it is $\diamond$-consistent. However, the valuation of region variables $x$ and $y$ is such that it is impossible to extend it with a valuation of region variable $z$ so that $EC(x, z)$ may hold. Patchwork allows us to disregard any partial valuations and focus on the satisfiability of the network. Then, we can consider a valuation that respects the constraint network. Such a valuation is, for example, the one presented in Figure 1(b) along with its atomic QCN on the right.

## 3 The $\mathcal{L}_1$ spatiotemporal logic

In general, a spatial QCN, as described in Section 2, constitutes a static spatial configuration in some domain, over a set of spatial variables $V$. To be able to describe a spatial configuration that changes over time, we can combine PTL [9] with a qualitative spatial constraint language in a unique formalism. The domain D of a QCN will always remain the same, but the spatial variables in it may spatially change with the passing time (e.g., in shape, size, or orientation). We can interpret formulas of such a spatiotemporal formalism using a spatiotemporal structure defined as follows.

**Definition 5** *A* ST-structure *is a tuple $\mathcal{M}_{\mathsf{ST}} = (\mathsf{D}, \mathbb{N}, \alpha)$, where $\alpha$ is a mapping that associates elements of* D *to the spatial variables of a set $V$ at a point of time $i \in \mathbb{N}$. Thus, $\alpha(i)$ denotes the set of elements of* D *that are associated with the spatial variables of $V$ at point of time $i$. By extending notation, $\alpha(v, i)$, where $v \in V$, denotes the element of* D *that is associated with spatial variable $v$ at point of time $i$.*

For example, in the case of RCC-8, $\alpha$ would be a mapping associating elements of $\mathcal{R}(\mathcal{T})$ to spatial region variables at a point of time $i \in \mathbb{N}$. The set of atomic propositions AP in the case of standalone PTL [9] is replaced by the set of base relations B of the qualitative spatial constraint language considered. We will call such a spatiotemporal formula over B a $\mathcal{L}_0$ formula. Thus, the set of $\mathcal{L}_0$ formulas over B is inductively defined as follows: if $P \in \mathsf{B}$ then $P$ is a $\mathcal{L}_0$ formula, and if $\psi$ and $\phi$ are $\mathcal{L}_0$ formulas then $\neg\phi$, $\phi \vee \psi$, $\bigcirc\phi$, $\Box\phi$, $\Diamond\phi$, and $\phi\,\mathcal{U}\,\psi$ are $\mathcal{L}_0$ formulas.

A simple example of a $\mathcal{L}_0$ formula is $\Box NTPP(Athens, Greece)$, stating that Athens will always be located in Greece. To increase the expressiveness of the $\mathcal{L}_0$ logic we can allow the application of operator $\bigcirc$ to spatial variables, i.e., we can have the following statement in RCC-8: $\Box EQ(Greece, \bigcirc Greece)$, which translates to "Greece will never change its borders". We call the enriched logic the $\mathcal{L}_1$ logic.

**Definition 6** *Given a $\mathcal{L}_1$ formula $\phi$ over* B*, we write $\langle \mathcal{M}_{\mathsf{ST}}, i \rangle \models \phi$ for the fact that $\mathcal{M}_{\mathsf{ST}}$ satisfies $\phi$ at point* *of time $i$, with $i \in \mathbb{N}$ (or formula $\phi$ is true in $\mathcal{M}_{\mathsf{ST}}$ at point of time $i$). The semantics is then defined as follows:*

- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models P(\bigcirc^n v, \bigcirc^m v')$ *iff the relation that holds between $\alpha(v, i+n)$ and $\alpha(v', i+m)$ is the relation $P$, with $P \in \mathsf{B}$*
- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \neg\phi$ *iff* $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \not\models \phi$
- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \phi \vee \psi$ *iff* $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \phi$ *or* $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \psi$
- $\langle \mathcal{M}_{\mathsf{ST}}, i\rangle \models \phi\,\mathcal{U}\,\psi$ *if there exists a $k \in \mathbb{N}$ such that $i \leq k$, $\langle \mathcal{M}_{\mathsf{ST}}, k\rangle \models \psi$, and for all $j \in \mathbb{N}$, if $i \leq j$ and $j < k$ then $\langle \mathcal{M}_{\mathsf{ST}}, j\rangle \models \phi$*

Formulas of the form $\Diamond\phi$ and $\Box\phi$ are abbreviations for $\top\,\mathcal{U}\,\phi$ and $\neg(\top\,\mathcal{U}\,\neg\phi)$ respectively. A structure $\mathcal{M}_{\mathsf{ST}} = (\mathsf{D}, \mathbb{N}, \alpha)$, for which $\langle \mathcal{M}_{\mathsf{ST}}, 0\rangle \models \phi$, is a model for $\phi$. It follows that a $\mathcal{L}_1$ formula $\phi$ is satisfiable if there exists a model for it. Note that a formula of the form $\bigcirc^k P(\bigcirc^l v, \bigcirc^m v')$ is equivalent to formula $P(\bigcirc^{l+k} v, \bigcirc^{m+k} v')$. The size of $P(\bigcirc^{l+k} v, \bigcirc^{m+k} v')$ is then defined to be equal to $\max\{l+k, m+k\}$. Like in [2], we define the size of any $\mathcal{L}_1$ formula $\phi$, denoted by $|\phi|$, inductively as follows: $P(\bigcirc^l v, \bigcirc^m v') = \max\{l, m\}$; $|\neg\phi| = |\phi|$; $|\phi \vee \psi| = |\phi\,\mathcal{U}\,\psi| = \max\{|\phi|, |\psi|\}$. The size of a set of $\mathcal{L}_1$ formulas $\chi = \{\phi, \psi, \ldots\}$, will be the maximum size among its formulas, i.e., $|\chi| = \max\{|\phi|, |\psi|, \ldots\}$. The number of occurrences of symbols in a $\mathcal{L}_1$ formula $\phi$ will be denoted by *length*$(\phi)$.

## 4 Revisiting the satisfiability problem in $\mathcal{L}_1$

In this section, we revisit a result regarding the satisfiability of $\mathcal{L}_1$ formulas in a ST-structure, using patchwork and compactness. These properties strengthen previous results, in that we do not longer need to restrict atomic QCNs to being globally consistent as in [2, 3], but we can consider atomic QCNs that have compactness and patchwork. As explained in Section 2, compactness and patchwork combined are less strict than global consistency alone.

Given a $\mathcal{L}_1$ formula $\phi$, Balbiani and Condotta in [2] show that the satisfiability of formula $\phi$ can be checked by characterizing a particular infinite sequence of finite satisfiable atomic QCNs representing an infinite consistent valuation of $\phi$. Each of the QCNs of such a sequence represents a set of spatial constraints in a fixed-width window of time. The set of spatial constraints at point of time $i$, is given by the $i$-th QCN in the infinite sequence, and shares spatial constraints with the next QCN. Moreover, in such a sequence, there exists a point of time after which the corresponding QCNs replicate the same set of spatial constraints. The global consistency property is then used for the following two tasks:

(i) to prove that by considering all the QCNs of the aforementioned sequence we obtain a consistent set of constraints;

(ii) to prove that in such an infinite sequence, a subsequence which begins and ends with two QCNs representing the same set of spatial constraints can be reduced to just considering the first QCN.

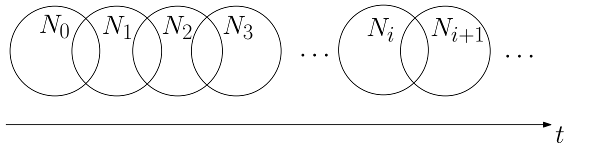


Figure 2: A countably infinite sequence of satisfiable atomic QCNs that agree on their common part

In the sequel, we formally show that tasks $(i)$ and $(ii)$ can be performed using the properties of patchwork and compactness instead. As a consequence, we can generalize a result regarding the satisfiability of a $\mathcal{L}_1$ formula $\phi$ to a larger class of calculi than the previously considered in literature. We now introduce the two aforementioned tasks in the form of two propositions.

**Proposition 1** *Let $V = \{v_0, \ldots, v_n\}$ be a set of variables, $w \geq 0$ an integer, and $\mathcal{S} = (\mathcal{N}_0 = (V_0, C_0), \mathcal{N}_1 = (V_1, C_1), \ldots)$ a countably infinite sequence of satisfiable atomic* QCN*s, as shown in Figure 2, such that:*

- *for each $i \geq 0$, $V_i$ is defined by the set of variables $\{v_0^0, \ldots, v_n^0, \ldots, v_0^w, \ldots, v_n^w\}$,*
- *for each $i \geq 0$, for all $m, m' \in \{0, \ldots, n\}$, and for all $k, k' \in \{1, \ldots, w\}$, $C_i(v_m^k, v_{m'}^{k'}) = C_{i+1}(v_m^{k-1}, v_{m'}^{k'-1})$.*

*We have that if the constraint language considered has compactness and patchwork for atomic* QCN*s, then $\mathcal{S}$ defines a consistent set of qualitative constraints.*

**Proof**. Given $\mathcal{N}_i$, we rewrite its set of variables to $\{v_0^i, \ldots, v_n^i, \ldots, v_0^{w+i}, \ldots, v_n^{w+i}\}$. Then, by patchwork we can assert that for each integer $k \geq 0$, $\bigcup_{k \geq i \geq 0} \mathcal{N}_i$ is a consistent set of qualitative constraints. Suppose though, that $\bigcup_{i \geq 0} \mathcal{N}_i$ is an inconsistent set. By compactness we know that there exists an integer $k' \geq 0$ for which $\bigcup_{k' \geq i \geq 0} \mathcal{N}_i$ is inconsistent. This is a contradiction. Thus, $\mathcal{S}$ defines a consistent set of qualitative constraints. $\dashv$

The second proposition follows.

**Proposition 2** *Let $V = \{v_0, \ldots, v_n\}$ be a set of variables, $w \geq 0, t > t' \geq 0$ three integers, and $\mathcal{S} = (\mathcal{N}_0 = (V_0, C_0), \mathcal{N}_1 = (V_1, C_1), \ldots)$ a countably infinite se-*

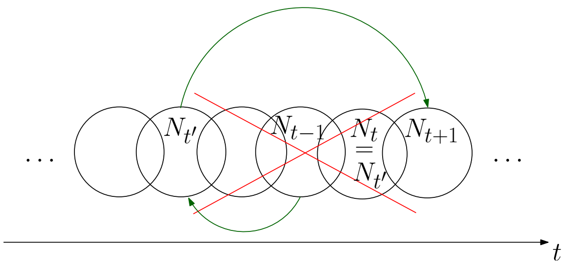


Figure 3: A countably infinite sequence of satisfiable atomic QCNs that contains a sub-sequence which begins and ends with two QCNs representing the same set of spatial constraints; we can reduce the sub-sequence to just considering the first QCN and patch it with the QCN following the sub-sequence

*quence of satisfiable atomic* QCN*s, as shown in Figure 3, such that:*

- *for each* $i \geq 0$, $V_i$ *is defined by the set of variables* $\{v_0^0, \ldots, v_n^0, \ldots, v_0^w, \ldots, v_n^w\}$,
- *for each* $i \geq 0$, *for all* $m, m' \in \{0, \ldots, n\}$, *and for all* $k, k' \in \{1, \ldots, w\}$, $C_i(v_m^k, v_{m'}^{k'}) = C_{i+1}(v_m^{k-1}, v_{m'}^{k'-1})$,
- *for all* $m, m' \in \{0, \ldots, n\}$ *and all* $k, k' \in \{0, \ldots, w\}$, $C_{t'}(v_m^k, v_{m'}^{k'}) = C_t(v_m^k, v_{m'}^{k'})$.

*Let* $\mathcal{S}' = (\mathcal{N}'_0 = (V'_0, C'_0), \mathcal{N}'_1 = (V'_1, C'_1), \ldots)$ *be the infinite sequence defined by:*

- *for all* $i \in \{0, \ldots, t'\}$, $\mathcal{N}'_i = \mathcal{N}_i$,
- *for all* $i > t'$, $V'_i = V_i$, *and for all* $m, m' \in \{0, \ldots, n\}$ *and all* $k, k' \in \{0, \ldots, w\}$, $C'_i(v_m^k, v_{m'}^{k'}) = C_{i+(t-t')}(v_m^k, v_{m'}^{k'})$.

*We have that if the constraint language considered has compactness and patchwork for atomic* QCN*s, then* $\mathcal{S}'$ *defines a consistent set of qualitative constraints.*

**Proof.**We have $\mathcal{N}_i$ which is a satisfiable QCN for all $i \geq 0$. From this, we can deduce that $\mathcal{N}'_i$ is a satisfiable QCN for all $i \geq 0$. By Proposition 1 we can deduce that $\mathcal{S}'$ defines a consistent set of qualitative constraints. ⊣

We now can obtain the following result:

**Theorem 1** *Checking the satisfiability of a* $\mathcal{L}_1$ *formula* $\phi$ *in a* ST*-structure is* PSPACE*-complete in length*($\phi$) *if the qualitative spatial constraint language considered has compactness and patchwork for atomic* QCN*s.*

**Proof.**(Sketch) Consider the approach in [2] where a proof of PSPACE-completeness is given for a logic that considers qualitative constraint languages for which satisfiable atomic QCNs are globally consistent (see Theorem 1 in [2]). To be able to replace the use of global consistency with the use of patchwork and compactness, we need to use Propositions 1 and 2 in the proofs of Lemmas 3 and 4 in [2]. The interested reader can verify that the aforementioned proofs make use of global consistency to perform exactly the tasks described by Propositions 1 and 2. Since these propositions build on compactness and patchwork, we can prove PSPACE-completeness using these properties instead. ⊣

Theorem 1 allows us to consider more calculi than the ones considered in literature for which the combination with PTL yields PSPACE-completeness. Due to the lack of global consistency for RCC-8 [17], in [5] the authors restrict themselves to a very particular domain interpretation of RCC-8 to prove that the $\mathcal{ST}_1^-$ logic is PSPACE-complete. As already noted in Section 1, the $\mathcal{ST}_1^-$ logic is the $\mathcal{L}_1$ logic when the considered qualitative constraint language is RCC-8. $\mathcal{L}_1$ does not rely on the semantics of the qualitative constraint language used, but rather on the constraint properties of *compactness* and *patchwork* [12]. Therefore, $\mathcal{L}_1$ is by default able to consider all calculi that have these properties, such as RCC-8 [15], Cardinal Direction Algebra (CDA) [4, 11], Block Algebra (BA) [6], and even Interval Algebra (IA) [1] when viewed as a spatial calculus. The most notable languages that have patchwork and compactness are listed in [8].

## 5 Semantic tableau for $\mathcal{L}_1$

In this section, we present a semantic tableau method that given a $\mathcal{L}_1$ formula $\phi$ systematically searches for a model for $\phi$. The method builds on the tableau method for PTL of Wolper [19], and makes use of the results of Section 4 to ensure soundness and completeness.

### 5.1 Rules for constructing a semantic tableau

The decomposition rules of the temporal operators are based on the following identities, which are called *eventualities* (where $\Box$ abbreviates $\neg\Diamond\neg$):

- $\Diamond\phi \equiv \phi \vee \bigcirc\Diamond\phi$
- $\phi\, \mathcal{U}\, \psi \equiv \psi \vee (\phi \wedge \bigcirc(\phi\, \mathcal{U}\, \psi))$

Note that decomposing eventualities can lead to an infinite tableau. However, we will construct a finite tableau by identifying nodes that are labeled by the same set of formulas, thus, ensuring that infinite periodicity will not exist. To test a $\mathcal{L}_1$ formula $\phi$ for satisfiability, we will construct a directed graph. Each node $n$ of the graph will be labeled by a set of formulas, and initially the graph will contain a single node, labeled by $\{\phi\}$. Similarly to Wolper [19], we distinguish between *elementary* and *non-elementary* formulas:

**Definition 7** *A $\mathcal{L}_1$ formula is elementary if its main connective is $\bigcirc$ (viz., $\bigcirc$-formula), or if it corresponds to a base relation $P \in \mathsf{B}$.*

Then, the construction of the graph proceeds by using the following decomposition rules which map each non-elementary formula $\phi$ into a set of sets of formulas:

- $\neg P(\bigcirc^n v, \bigcirc^m v') \rightarrow \{\{P'(\bigcirc^n v, \bigcirc^m v')\} \mid P' \in \mathsf{B} \setminus \{P\}\}$
- $\neg\neg\phi \rightarrow \{\{\phi\}\}$
- $\neg \bigcirc \phi \rightarrow \{\{\bigcirc\neg\phi\}\}$
- $\phi \wedge \psi \rightarrow \{\{\phi, \psi\}\}$
- $\neg(\phi \wedge \psi) \rightarrow \{\{\neg\phi\}, \{\neg\psi\}\}$
- $\Diamond\phi \rightarrow \{\{\phi\}, \{\bigcirc\Diamond\phi\}\}$
- $\neg\Diamond\phi \rightarrow \{\{\neg\phi, \neg\bigcirc\Diamond\phi\}\}$
- $\phi\,\mathcal{U}\,\psi \rightarrow \{\{\psi\}, \{\phi, \bigcirc(\phi\,\mathcal{U}\,\psi)\}\}$
- $\neg(\phi\,\mathcal{U}\,\psi) \rightarrow \{\neg\psi, \neg\phi \vee \neg\bigcirc(\phi\,\mathcal{U}\,\psi)\}$

During the construction, we *mark* formulas to which a decomposition rule has been applied to avoid decomposing the same formula twice. If $\psi$ is a formula, $\psi*$ denotes $\psi$ marked.

### 5.2 Systematic construction of a semantic tableau

A tableau $\mathcal{T}$ can be seen as a directed graph where each of its nodes $n$ is labeled with a set of formulas $\mathcal{T}(n)$. The root node is labeled with the singleton set $\{\phi\}$ for the $\mathcal{L}_1$ formula $\phi$ whose satisfiability we wish to check. The children of the nodes are obtained by applying the rules presented in Section 5.1.

Given a set of $\mathcal{L}_1$ formulas $\chi$ over the set of variables $\{x_0, \ldots, x_l\}$, we denote by $expandVars(\chi)$ the set $\{\bigcirc^0 x_0,\ldots,\bigcirc^0 x_l,\ldots,\bigcirc^{|\chi|} x_0,\ldots,\bigcirc^{|\chi|} x_l\}$. We first define a translation of a node of a tableau to a $\mathsf{QCN}$.

**Definition 8** *Let $n$ be a node of a tableau $\mathcal{T}$ for a $\mathcal{L}_1$ formula $\phi$, and $\{x_0, \ldots, x_l\}$ the set of variables in $\phi$. Then, $\mathcal{N}(n)$ will denote the* $\mathsf{QCN} = (V, C)$, *where $V = \{v_0^0, \ldots, v_l^0, \ldots, v_0^{|\phi|}, \ldots, v_l^{|\phi|}\}$, and $C(v_m^k, v_{m'}^{k'}) = \{P(\bigcirc^k x_m, \bigcirc^{k'} x_{m'})\}$ if $P(\bigcirc^k x_m, \bigcirc^{k'} x_{m'}) \in \mathcal{T}(n)$, and $C(v_m^k, v_{m'}^{k'}) = (\mathsf{B}$ if $v_m^k \neq v_{m'}^{k'}$, else $\{\mathsf{Id}\})$ otherwise, $\forall\, m, m' \in \{0, \ldots, l\}$ and $\forall\, k, k' \in \{0, \ldots, |\phi|\}$.*

Let us also define the notions of a *state* and a *pre-state*, which we will be referring to a lot in what follows.

**Definition 9** *A node $n$ that contains only elementary and marked formulas* and *for which we have that $\mathcal{N}(n)$ is atomic is called a* state, *and a node $m$ that is either the root node or the direct child node of a state (which leaps to the next point of time) is called a* pre-state.

We give a definition of eventuality fulfillment that will be of use later on.

**Algorithm 1:** $\mathsf{Clotho}(\phi)$

**in** : A $\mathcal{L}_1$ formula $\phi$.
**output** : A semantic tableau $\mathcal{T}$ for $\phi$.
1 **begin**
2 create root node $\{\phi\}$ and mark it unprocessed;
3 **while** $\exists$ *unprocessed node* $n$ **do**
4 **if** $\mathcal{T}(n)$ *contains an unmarked non-elementary formula* $\psi$ **then**
5 mark node $n$ processed;
6 **foreach** $\gamma \in \Gamma$, *where* $\Gamma$ *is the result of applying a decomposition rule to* $\psi$ **do**
7 create a child node $m$;
8 $\mathcal{T}(m) \leftarrow (\mathcal{T}(n) - \{\psi\}) \cup \gamma \cup \{\psi*\}$;
9 mark node $m$ unprocessed;
10 **else if** $\mathcal{T}(n)$ *contains only elementary and marked formulas* **then**
11 mark node $n$ processed;
12 $\mathsf{filling} \leftarrow \emptyset$;
13 **foreach** $u, v \in expandVars(\phi)$ **do**
14 **if** $\nexists\, P(u, v) \in \mathcal{T}(n)$ **then**
15 $\mathsf{filling} \leftarrow \mathsf{filling} \cup \{\mathsf{B}(u, v)\}$;
16 **if** $\mathsf{filling} \neq \emptyset$ **then**
17 create a child node $m$;
18 $\mathcal{T}(m) \leftarrow \mathcal{T}(n) \cup \mathsf{filling}$;
19 mark node $m$ unprocessed;
20 **else if** $\mathcal{T}(n)$ *contains* $\bigcirc$*-formulas* **then**
21 create a child node $m$;
22 $\mathcal{T}(m) \leftarrow \{\psi \mid \bigcirc\psi \in \mathcal{T}(n)\}$;
23 $\mathcal{T}(m) \leftarrow \mathcal{T}(m) \cup \{P(\bigcirc^{i-1}u, \bigcirc^{j-1}v) \mid P(\bigcirc^i u, \bigcirc^j v) \in \mathcal{T}(n)$ if $i, j \geq 1\}$;
24 mark node $m$ unprocessed;

**Definition 10** *Let $\mathcal{T}$ be a tableau, and $\pi$ a path in $\mathcal{T}$ defined from nodes $n_1, n_2, \ldots, n_j$. Any eventuallity $\Diamond\epsilon_2$ or $\epsilon_1\,\mathcal{U}\,\epsilon_2 \in \mathcal{T}(n_i)$, with $1 \leq i \leq j$, is* fulfilled *in $\pi$ if there exists $k$, with $i \leq k \leq j$, such that $\epsilon_2 \in \mathcal{T}(n_k)$.*

We now present $\mathsf{Clotho}$, an algorithm that constructs a semantic tableau $\mathcal{T}$ for a given formula $\phi$, as shown in Algorithm 1. At any given point of time, we construct all the possible atomic $\mathsf{QCN}$s comprising base relations that extend from the given point of time to a future point of time. This is achieved by repeatedly applying the decomposition rules to a node comprising unmarked non-elementary formulas (lines 4 to 9), and sequentially populating a node comprising only elementary and marked formulas with the universal relation $\mathsf{B}$ (lines 10 to 19) so that it may lead to a state. The universal relation $\mathsf{B}$ is only introduced on a pair of variables, if there does not exist any base relation on that same pair. The universal relation $\mathsf{B}$, as well as any other relation $r \in 2^{\mathsf{B}}$, is essentially the disjunction of base relations, as noted in Section 2. In particular, $\mathsf{B}$ is the disjunction of all the base relations of a given qualitative constraint language. As such, by decomposing $\mathsf{B}$ into base relations using the

**Algorithm 2:** Atropos($\mathcal{T}$)

```
in     : A semantic tableau 𝒯.
output : True or False.
begin
    do
        flag ← False;
        if there is a node n such that 𝒩(n) is an
        unsatisfiable QCN then
            eliminate node n; flag ← True;
        if all the children of a node n have been
        eliminated then
            eliminate node n; flag ← True;
        if a node n is a pre-state and not Lachesis(𝒯, n)
        then
            eliminate node n; flag ← True;
    while flag;
    if ∄ node n ∈ 𝒯 then return False else return
    True;
```

**Function** Lachesis($\mathcal{T}, n$)

```
in     : A semantic tableau 𝒯, and a node n.
output : True or False.
begin
    foreach eventuality ε ∈ 𝒯(n) do
        if ε is not fulfilled in any path π = ⟨n, ...⟩ then
            return False;
    return True;
```

disjunctive tableau rule, this approach allows us to obtain one or more nodes harboring atomic QCNs for a given point of time (viz., states), that represent a set of atomic spatial constraints in a fixed-width window of time. Once we have obtained our atomic QCNs for a given point of time, and assuming that the states that harbor them contain $\bigcirc$-formulas, we can leap to the next point of time and create pre-states, including all the atomic spatial constraints of the aforementioned QCNs that extend from the new point of time to a future point of time (lines 20 to 24). This can be seen as making a +1 time shift and maintaining all possible knowledge offered by previous states that extends from the new point of time to a future point of time. It is important to note that when we create a child node $m$ of a node $n$ (lines 7, 17, and 21), we only create a new node if there does not already exist a node in the graph labeled by $\mathcal{T}(m)$. Otherwise, we just create an arc from node $n$ to the existing node.

**Lemma 1** *Let $\mathcal{T}$ be a tableau for a $\mathcal{L}_1$ formula $\phi$ that has resulted after the application of algorithm* Clotho. *Then, $\mathcal{T}$ is finite. Actually, if $\phi$ is over a set of $l$ variables, then $\mathcal{T}$ has at most $O(|B|^{l^2\cdot(|\phi|+1)^3}\cdot 2^{length(\phi)})$ nodes.*

To decide the satisfiability of a $\mathcal{L}_1$ formula $\phi$ using the tableau that is generated by Clotho, we have to eliminate unsatisfiable nodes inductively, until a fixed point is reached. We present Atropos, an algorithm that achieves this goal, shown in Algorithm 2. If the root node is eliminated after the application of Atropos, we call the tableau *closed*, and *open* otherwise. Note that function Lachesis essentially searches for a path from a given pre-state to a node that fulfills an eventuality of the pre-state, as defined in Definition 10.

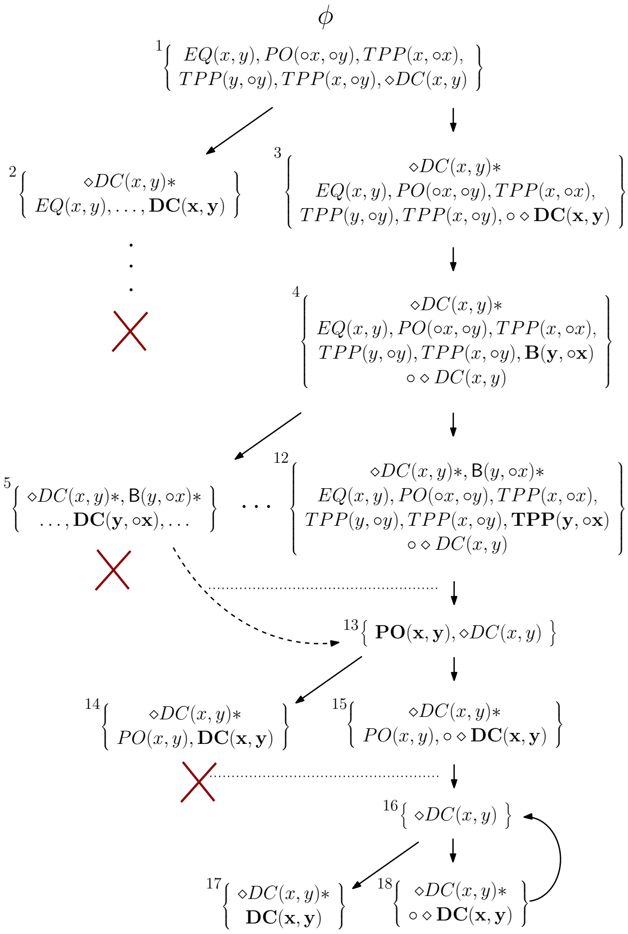


Figure 4: A $\mathcal{L}_1$ formula and its simplified tableau

**Example.** Let us consider formula $\phi = \{EQ(x\ ,y),\ PO(\bigcirc x,\ \bigcirc y),\ TPP(x,\ \bigcirc x),\ TPP(y,\ \bigcirc y),\ TPP(x,\ \bigcirc y),\ \Diamond DC(x,\ y)\}$. (For simplicity we assume that the decomposition rule for $\wedge$ has already been applied and resulted in the current set form for formula $\phi$.) The tableau obtained by the application of algorithms Clotho and Atropos for this formula is shown in Figure 4. Horizontal dotted lines distinguish between different points in time, thus, our tableau extends over three points of time. The root node is 1, the states are 5 to 12, 14, 15, 17, and 18, and the pre-states are 1, 13, and 16. By decomposing the initial formula using the tableau rules and populating it with universal relations where appropriate, we reach states 5 to 12, each one of which harbors a set of base relations that correspond to an atomic QCN. (Inverse relations are not shown to save space.) These atomic QCNs represent a set of atomic spatial constraints in a fixed-width window of time. After leaping to the next point of time and, consequently, obtaining pre-state 13, we include all the atomic spatial constraints of the aforementioned QCNs that extend from the new point of time to a future point of time. In this particular case, the atomic spatial constraints of interest narrow down to the single atomic constraint $PO(\bigcirc x, \bigcirc y)$, common for all states 5 to 12. Of course, since we are now at the next point of time, the constraint is rewritten to $PO(x, y)$. Again, we apply the rules and reach states 14 and 15, each one of which harbors an atomic QCN. We continue repeating the process until all our child nodes are labeled by sets of formulas already met in nodes of the tableau. In this case, the unique child node of state 18 would be labeled by the set of formulas of node 16, thus, we create an arc from 18 to 16. After having constructed our tableau, we delete unsatisfiable nodes 2, 5 to 11, and 14 using the $\diamond$-consistency operation on QCNs $\mathcal{N}(2)$, $\mathcal{N}(5)$ to $\mathcal{N}(11)$, and $\mathcal{N}(14)$ respectively. Inconsistencies stemming from nodes 2 and 14 are apparent, as there exist different base relations on a same pair of variables, whereas inconsistencies in nodes 5 to 11 stem from the fact that relation $TPP(y, \bigcirc x)$ is inferred by $\diamond$-consistency, which contradicts with the base relation that is defined on variables $y$ and $\bigcirc x$ in states 5 to 11. Formula $\phi$ is satisfiable, as the tableau is open, and a model can be constructed out of the sequence of states 12,15,17 which contains a self loop on 17 as relation $DC(x, y)$ repeats itself. These states harbor satisfiable atomic QCNs that completely agree on their common part due to our construction. In particular, we have the sequence $\mathcal{N}(12) \to \mathcal{N}(15) \to \mathcal{N}(17) \circlearrowleft$ that satisfies the prerequisites of Proposition 2, hence, satisfiability is met.

### 5.3 Soundness and completeness of our semantic tableau method

In this section, we prove that the tableau method as defined by algorithms Clotho and Atropos is sound and complete for checking the satisfiability of a $\mathcal{L}_\mathbf{1}$ formula $\phi$.

**Theorem 2 (soundness)** *If $\phi$ has a closed tableau, then $\phi$ is unsatisfiable.*

**Proof**. Let $\mathcal{T}$ be a closed tableau for $\phi$, that has resulted after the application of algorithms Clotho and Atropos. We prove by induction that if a node $n$ is eliminated, then $\mathcal{T}(n)$ is an unsatisfiable set of formulas. We distinguish three scenarios:

(i) a node $n$ is eliminated because $\mathcal{N}(n)$ is an unsatisfiable QCN (lines 4 to 5 in Atropos), thus, $\mathcal{T}(n)$ is an unsatisfiable set of formulas; unsatisfiability of $\mathcal{N}(n)$ can de detected by use of $\diamond$-consistency, which also disallows the conjunction of two or more base relations to be defined on a same pair of variables (base relations are jointly exhaustive and pairwise disjoint as noted in Section 2).

(ii) a node $n$ is eliminated because all of its child nodes are unsatisfiable and have been eliminated (lines 6 to 7 in Atropos). Child nodes can be created in the following three cases:

- (a) the decomposition rule $\psi \to \Gamma$, where $\psi \in \mathcal{T}(n)$, is applied and a child node is created for each $\gamma \in \Gamma$ (lines 4 to 9 in Clotho); we have that $\psi$ is satisfiable iff $\exists \gamma \in \Gamma$ that is satisfiable.
- (b) implicit knowledge in the parent node $n$ is made explicit in the child node $m$ through the introduction of the universal relation B (lines 10 to 19 in Clotho); by Definition 8 we have that $\mathcal{N}(n) = \mathcal{N}(m)$, thus, $\mathcal{N}(n)$ is satisfiable iff $\mathcal{N}(m)$ is satisfiable, and the same holds for the set of formulas $\mathcal{T}(n)$ and $\mathcal{T}(m)$.
- (c) node $n$ is a state and generates pre-state $m$ with $\mathcal{T}(m) = \{\psi \mid \bigcirc \psi \in \mathcal{T}(n)\} \cup \{P(\bigcirc^{i-1}u, \bigcirc^{j-1}v) \mid P(\bigcirc^{i}u, \bigcirc^{j}v) \in \mathcal{T}(n)$ if $i, j \geq 1\}$ (lines 20 to 24 in Clotho); clearly, $\mathcal{T}(n)$ is a satisfiable set of formulas iff $\{\psi \mid \bigcirc \psi \in \mathcal{T}(n)\}$ is a satisfiable set of formulas and iff $\mathcal{N}(m)$ is satisfiable.

(iii) a node $n$ is eliminated if it contains an eventuality that cannot be fulfilled in any path in the tableau (lines 8 to 9 in Atropos); since any model will correspond to a path in the tableau, we have that $\mathcal{T}(n)$ is an unsatisfiable set of formulas.

As we have considered all possible scenarios, at this point we conclude our proof. ⊣

Let us obtain a proposition that denotes that two successive states in a path of an open tableau harbor QCNs that completely agree on their common part.

**Proposition 3** *Let $\pi$ be a path going through an open tableau $\mathcal{T}$ for a $\mathcal{L}_\mathbf{1}$ formula $\phi$ that has resulted after the application of algorithms* Clotho *and* Atropos*, $s_t$ and $s_{t+1}$ two states of $\pi$ belonging to points of time $t$ and $t+1$ respectively, and $\{x_0, \ldots, x_l\}$ the set of variables in $\phi$. Then we have that $\mathcal{N}(s_t)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{t+1})[v_m^{k-1}, v_{m'}^{k'-1}]$ $\forall\, m, m' \in \{0, \ldots, l\}$ and $\forall\, k, k' \in \{1, \ldots, |\phi|\}$.*

**Proof**. State $s_t$ at point of time $t$ is followed by a pre-state $p$ at point of time $t+1$ in path $\pi$, whose set of base relations is $\{P(\bigcirc^{i-1}u, \bigcirc^{j-1}v) \mid P(\bigcirc^{i}u, \bigcirc^{j}v) \in \mathcal{T}(s_t)$ if $i, j \geq 1\} \cup \{P(\bigcirc^{i}u, \bigcirc^{j}v) \mid \bigcirc P(\bigcirc^{i}u, \bigcirc^{j}v) \in \mathcal{T}(s_t)\}$ by construction of our tableau (lines 20 to 24 in Clotho). The set of base relations of $\mathcal{T}(p)$ is carried over, possibly enriched, to state $s_{t+1}$ at point of time $t+1$. As such, let us assume that there exists an additional base relation $b(\bigcirc^{i-1}u, \bigcirc^{j-1}v)$ in the set of base relations of $s_{t+1}$, with $i, j \in \{1, \ldots, |\phi|\}$, such that $b(\bigcirc^{i}u, \bigcirc^{j}v) \notin \mathcal{T}(s_t)$. In this case, $\mathcal{N}(s_{t+1})$ is a QCN with two base relations defined on a same pair of variables. This QCN would have been deleted by the application of Atropos as specified also in the proof of Theorem 2. Thus, state $s_{t+1}$ could not have been in path $\pi$, resulting in a contradiction. Therefore, we have that $\mathcal{N}(s_t)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{t+1})[v_m^{k-1}, v_{m'}^{k'-1}]$ $\forall\ m, m' \in \{0, \ldots, l\}$ and $\forall\ k, k' \in \{1, \ldots, |\phi|\}$, and, as such, $\mathcal{N}(s_t)$ and $\mathcal{N}(s_{t+1})$ completely agree on their common part. ⊣

**Theorem 3 (completeness)** *If $\phi$ has an open tableau, then $\phi$ is satisfiable.*

**Proof**. Let $\mathcal{T}$ be an open tableau for $\phi$, that has resulted after the application of algorithms Clotho and Atropos. We need to show that there exists a path of nodes $\pi$ which defines a model for $\phi$. We distinguish two scenarios:

(i) if no eventualities need to be fulfilled, path $\pi$ can be simply a path starting from the root node and going through the tableau, defining a sequence of states $s_0, s_1, \ldots, s_t$, with $t \in \mathbb{N}$, and, consequently, yielding a sequence of QCNs as follows:

$$\mathcal{N}(s_0) \rightarrow \mathcal{N}(s_1) \ldots \rightarrow \mathcal{N}(s_t)$$

The sequence of QCNs is such that for all states $s_i$ and $s_{i+1}$, with $i \in \{0, \ldots, t-1\}$, along with a set of variables $\{x_0, \ldots, x_l\}$ in $\phi$, we have that $\mathcal{N}(s_i)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{i+1})[v_m^{k-1}, v_{m'}^{k'-1}]$ $\forall\ m, m' \in \{0, \ldots, l\}$ and $\forall\ k, k' \in \{1, \ldots, |\phi|\}$ by Proposition 3. Thus, the sequence of QCNs corresponds to the sequence shown in Figure 2, satisfies the prerequisites of Proposition 1, and is therefore satisfiable.

(ii) if eventualities need to be fulfilled, we show how we can construct a path $\pi$ that fulfills all eventualities as follows. For each pre-state $p \in \mathcal{T}$ containing an eventuality, we must find a path $\pi_p = \langle p, \ldots \rangle$ starting from $p$, such that all the eventualities contained in $p$ are fulfilled in $\pi_p$. We fulfill all the eventualities of $p$, one by one, as follows. For a selected eventuality $\epsilon \in \mathcal{T}(p)$, it is possible to find a path $\pi_p = \langle p, \ldots, p' \rangle$ in which $\epsilon$ is fulfilled and whose last node is a pre-state $p'$, as otherwise the node would have been deleted by the application of Atropos. By construction of our tableau, $p'$ will also contain the rest of the eventualities that need to be fulfilled (they are carried over from $p$ to $p'$), and it follows that we can extend path $\pi_p$ to fulfill a second one, and so on, until all the eventualities of $p$ are fulfilled. By linking together all paths $\pi_p$ $\forall$ pre-states $p \in \mathcal{T}$, we can obtain a path $\pi$ starting from the initial node and going through the tableau, defining a sequence of states $s_0, s_1, \ldots, s_{t-1}$, with $t \in \mathbb{N}$, with a final loop between state $s_{t-1}$ and a state $s_{t'}$, with $0 \leq t' \leq t-1$. The loop exists due to the fact that at point of time $t-1$ there exists a node $n$, whose child node $m$ is such that $\mathcal{T}(m) = \mathcal{T}(o)$, where $o$ is a node at point of time $t'$. In particular, we can view the sequence of states as a sequence of QCNs as follows:

$$\mathcal{N}(s_0) \rightarrow \mathcal{N}(s_1) \ldots \rightarrow \mathcal{N}(s_{t'}) \ldots \rightarrow \mathcal{N}(s_{t-1})$$

The sequence of QCNs is such that for all states $s_i$ and $s_{i+1}$, with $i \in \{0, \ldots, t-2\}$, along with a set of variables $\{x_0, \ldots, x_l\}$ in $\phi$, we have that $\mathcal{N}(s_i)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{i+1})[v_m^{k-1}, v_{m'}^{k'-1}]$ $\forall\ m, m' \in \{0, \ldots, l\}$ and $\forall\ k, k' \in \{1, \ldots, |\phi|\}$ by Proposition 3. Further, if we were to extend path $\pi$, we would obtain a state $s_t$ with $\mathcal{N}(s_t)[v_m^k, v_{m'}^{k'}] = \mathcal{N}(s_{t'})[v_m^k, v_{m'}^{k'}]$ $\forall\ m, m' \in \{0, \ldots, l\}$ and $\forall\ k, k' \in \{0, \ldots, |\phi|\}$ (i.e., $s_t$ replicates the same set of spatial constraints with $s_{t'}$, hence, the loop). Thus, the sequence of QCNs corresponds to the sequence shown in Figure 3, satisfies the prerequisites of Proposition 2, and is therefore satisfiable.

As we have considered all possible scenarios, at this point we conclude our proof. ⊣

## 6 Conclusion

In this paper, we considered a generalized qualitative spatiotemporal formalism, namely, the $\mathcal{L}_1$ logic, which is the product of the combination of PTL with any qualitative spatial constraint language, such as RCC-8, Cardinal Direction Algebra, and Block Algebra, and showed that satisfiability checking of a $\mathcal{L}_1$ formula is PSPACE-complete if the constraint language considered has the properties of *compactness* and *patchwork* for atomic networks, thus, strengthening previous results that required atomic networks to be *globally consistent* and generalizing to a larger class of calculi. Further, we presented a first semantic tableau method, that given a $\mathcal{L}_1$ formula $\phi$ systematically searches for a model for $\phi$. The method presented builds on the tableau method for PTL of Wolper, and makes use of our strengthened results to ensure soundness and completeness.

## 7 Future Work

In this paper, we implicitly considered that the satisfiability problem in a qualitative spatial constraint language relies on some canonical model, i.e., a structure that allows to model any (syntactically) consistent QCN. It is a very interesting future direction to consider domain interpretations that involve determined entities (constants) for a given qualitative spatial constraint language, and not just abstract infinite domains as it is normally the case. It has been shown that such domain intepretations do not always yield semantic truth when combined with the algorithms that are typically used in conjuction with some canonical model to derive the satisfiability of a QCN [10]. Therefore, we need to explore this implication in the satisfiability problem in $\mathcal{L}_\mathbf{1}$.

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