From Qualitative to Discrete Constraint Networks

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Abstract. In this paper, we present some preliminary results about the connections existing between qualitative and discrete constraint networks. We present a natural encoding of any qualitative network $\mathcal{N}$ into a discrete one $\mathcal{P}$ such that the constraints of $\mathcal{N}$ become the variables of $\mathcal{P}$ and the constraints of $\mathcal{P}$ are defined by the weak composition table of the used qualitative algebra. We then introduce some properties about the (global) consistency of networks, circumscribing conditions under which the two models are equivalent. We also relate some domain filtering consistencies (such as generalized arc consistency) of discrete networks encoding qualitative ones with $\circ$-consistency, where $\circ$ denotes the weak composition of the qualitative calculus.

1 Introduction

The need for reasoning about time and space arises in many areas of Artificial Intelligence, including computer vision, natural language understanding, geographic information systems (GIS), scheduling, planning, diagnosis and genetics. Numerous formalisms for representing and reasoning about time and space in a qualitative way have been proposed in the past two decades [1, 16, 13, 5, 15, 11, 4].

Those formalisms involve a finite set of basic relations denoting qualitative relationships between temporal or spatial entities. Intersection, overlapping, containment, precedence are examples of such qualitative relationships. For instance, in the field of qualitative reasoning about temporal data, there is a well known formalism called Allen's calculus [1]. It is based on intervals of the rational line for representing temporal entities and thirteen basic relations between such intervals are used to represent the qualitative situations between temporal entities: an interval can follow another one, meet another one, and so on.

Typically, Qualitative Constraint Networks (QCNs) are used to express information on a spatial or temporal situation. Each constraint of a QCN represents a set of acceptable qualitative configurations between some temporal or spatial entities and is defined by a set of basic relations.

On the other hand, the discrete Constraint Satisfaction Problem (CSP) is at the heart of Constraint Programming. Its task is to determine the satisfiability of a Discrete Constraint Network (DCN), i.e. a network such that each variable takes its values in an associated discrete domain. For solving DCNs, tree search algorithms are commonly used. To limit their combinatorial explosion, various improvements have been proposed. Such improvements mainly concern ordering heuristics, filtering techniques and conflict analysis, and can be conveniently classified as look-ahead and look-back schemes [8].

In this paper, we report on current work concerning the representation of qualitative networks by discrete ones. More particularly, we define and study a transformation that allows for translating a QCN into a DCN. We show that satisfiability (unlike unsatisfiability) is preserved by this transformation. Moreover, we study the links between local consistency concepts of qualitative and discrete models. The final objective of this work is to detect and import into the qualitative domain the most efficient inference and search methods of the discrete model.

This paper is organized as follows. After introducing some technical background about discrete and qualitative constraint networks, we introduce an encoding of qualitative networks into discrete ones while addressing the issue of satisfiability. Then, we relate local consistencies from the two qualitative and discrete paradigms. Finally, we conclude with some perspectives.

2 Background on Discrete Constraint Networks

Definition 1. A Discrete Constraint Network (DCN) $\mathcal{P}$ is a triple (X,D,C) where:

- X is a finite set of variables;
- D is a mapping which associates to each variable $x \in X$ a finite set of values $D(x)$ called domain;
- C is a finite set of constraints such that each constraint $c \in C$ involves a subset of variables of X, called scope and denoted by $vars(c)$, and has an associated relation, denoted $rel(c)$, which contains the set of tuples allowed for the variables of its scope.

A solution to a discrete constraint network is an assignment of values to all the variables such that all the constraints are satisfied. A constraint network is said to be satisfiable or consistent iff it admits at least one solution. Two discrete constraint networks are equivalent iff they admit the same set of solutions.

Arc Consistency (AC) remains the central property of discrete constraint networks and establishing AC on a given network $\mathcal{P}$ involves removing all values that are not arc-consistent.

Definition 2. Let $\mathcal{P} = (X,D,C)$ be a DCN. A pair (x,a), with $x \in X$ and $a \in D(x)$, is arc-consistent iff $\forall c \in C \mid x \in vars(C)$, there exists a support of (x,a) in C, i.e. a tuple $t \in rel(c)$ such that $t[x] = a$ and $t[y] \in D(y)$ $\forall y \in vars(c)$[1]. $\mathcal{P}$ is arc consistent iff $\forall x \in X$, $D(x) \neq \emptyset$ and $\forall a \in D(x)$, (x,a) is arc-consistent.

The definition above is given in the general case, that is to say for instances involving constraints of any arity. Then, one usually talks about Generalized Arc Consistency (GAC) (e.g. see [6]) or hyper-arc consistency (e.g. see [3]). We will say that an assignment of a value to each variable of a set $S \subseteq X$ of variables is consistent iff any

[1] $t[x]$ denotes the value assigned to x in t

constraint $c \in C$ only involving assigned variables of S (i.e. $vars(c) \subseteq S$) is satisfied. $\mathcal{P}$ is said to be (i,j)-consistent iff any consistent assignment to i variables can be extended to a consistent assignment to j additional variables. Also, the k-consistency concept (with $k \geq 1$) is defined [9] as being equivalent to $(k-1,1)$-consistency. Finally, a DCN is strong k-consistent iff it is j-consistent, for any j in $\{1,\ldots,k\}$.

To solve a discrete constraint network, one can apply inference or search methods [8]. Usually, domains of variables are reduced by removing inconsistent values, i.e. values that can not occur in any solution. We can then compare the different states of a network during inference or search by focusing on domains as follows:

Definition 3. Let $\mathcal{P} = (X,D,C)$ and $\mathcal{P}' = (X,D',C)$ be two DCNs. $\mathcal{P}' \subseteq \mathcal{P}$ iff $\forall x \in X$, $D'(x) \subseteq D(x)$.

3 Background on Qualitative Calculi

3.1 Relations and Operations

A qualitative calculus involves a finite set B of binary[2] relations, called basic relations, defined on a domain D. The elements of D represent temporal or spatial entities. Each basic relation of B corresponds to a particular possible configuration between two temporal or spatial entities. The relations of B are jointly exhaustive and pairwise disjoint, which means that any pair of elements of D belongs to exactly one basic relation in B. Moreover, for each basic relation $B \in \mathsf{B}$ there exists a basic relation of B, denoted by $B^{\sim}$, corresponding to the transposition of B. Moreover, we suppose that a particular relation of B is the identity relation on D, we denote this basic relation by Id. The set A is defined as the set of relations corresponding to all unions of the basic relations: $\mathsf{A} = \{\bigcup E : E \subseteq \mathsf{B}\}$. It is customary to represent an element $B_1 \cup \ldots \cup B_m$ (with $B_i \in \mathsf{B}$ for each i such that $1 \leq i \leq m$) of A by the set $\{B_1,\ldots,B_m\}$ belonging to 2^{B}. Hence, we make no distinction between A and 2^{B} in the sequel.

As an example, consider the well known temporal qualitative formalism called Allen's calculus [2]. It uses intervals of the rational line for representing temporal entities. Hence, D is the set $\{(x^-,x^+) \in \mathbb{Q} \times \mathbb{Q} : x^- < x^+\}$. The set of basic relations consists of a set of thirteen binary relations $\mathsf{B} = \{eq,b,bi,m,mi,o,oi,s,si,d,di,f,fi\}$ corresponding to all possible configurations between two intervals. These basic relations are depicted in Figure 1. We have $\mathsf{Id} = eq$.

As a set of subsets, A is equipped with the usual set-theoretic operations including intersection ($\cap$) and union ($\cup$). As a set of binary relations, it is also equipped with the operation of converse ($\sim$) and an operation of composition ($\circ$) sometimes called weak composition or qualitative composition. The converse of a relation R in A is the union of the transpositions of the basic relations contained in R. The composition $A \circ B$ of two basic relations A and B is the relation $R = \{C \in \mathsf{B} \mid \exists x,y,z \in \mathsf{D}, x\ A\ y, y\ B\ z \text{ and } x\ C\ z\}$. The composition $R \circ S$ of $R,S \in \mathsf{A}$ is the relation $T = \bigcup_{A \in R, B \in S}\{A \circ B\}$. Computing the results of these various operations for relations of 2^{B} can be done efficiently by using tables giving the results of these operations for the basic relations of B. For instance,

[2] In this paper, we focus on binary relations but this work can be extended to n-ary relations with $n > 2$.

Relation	Symbol	Converse	Meaning
precedes	b	bi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

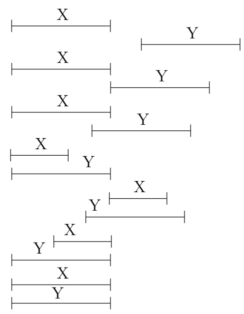


Fig. 1: The basic relations of Allen's calculus.

consider the relations $R = \{eq, b, o, si\}$ and $S = \{d, f, s\}$ of Allen's calculus, we have $R^{\smile} = \{eq, bi, oi, s\}$. The relation $R \circ S$ is $\{d, f, s, b, o, m, eq, si, oi\}$.

3.2 Qualitative Constraint Networks

A qualitative constraint network (QCN) is a pair composed of a set of variables and a set of constraints. The set of variables represents spatial or temporal entities of the system. A constraint consists of a set of acceptable basic relations (the possible configurations) between two variables. Formally, a QCN is defined in the following way:

Definition 4. A QCN is a pair $\mathcal{N} = (V, C)$ where $V = \{v_1, \ldots, v_n\}$ is a finite set of n variables and C is a map that assigns to each pair (v_i, v_j) of $V \times V$ a set $C(v_i, v_j) \in 2^{\mathsf{B}}$ of basic relations. In the sequel, $C(v_i, v_j)$ will be also denoted by C_{ij}. C is such that $C_{ii} \subseteq \{\mathsf{Id}\}$ and $C_{ij} = C_{ji}^{\smile}$ for all $v_i, v_j \in V$.

With regard to a QCN $\mathcal{N} = (V, C)$, we have the following definitions. A *solution* of $\mathcal{N}$ is a map σ from V to D such that $(\sigma(v_i), \sigma(v_j))$ satisfies C_{ij} for all $v_i, v_j \in V$. $\mathcal{N}$ is consistent iff it admits a solution. A QCN $\mathcal{N}' = (V', C')$ is a *sub*-QCN of $\mathcal{N}$ (denoted by $\mathcal{N}' \subseteq \mathcal{N}$) if and only if $V = V'$ and $C'_{ij} \subseteq C_{ij}$ for all $v_i, v_j \in V$. A QCN $\mathcal{N}' = (V', C')$ is *equivalent* to $\mathcal{N}$ if and only if $V = V'$ and both networks $\mathcal{N}$ and $\mathcal{N}'$ have the same solutions. The *minimal* QCN of $\mathcal{N}$ is the smallest (for $\subseteq$) sub-QCN of $\mathcal{N}$ equivalent to $\mathcal{N}$. An *atomic* QCN is a QCN such that each C_{ij} contains exactly one basic relation. A *scenario* of $\mathcal{N}$ is an atomic sub-QCN of $\mathcal{N}$.

Given a QCN $\mathcal{N}$, the main issue to be addressed is the consistency problem: decide whether or not $\mathcal{N}$ admits (at least) a solution. Most of the algorithms used for solving this problem are based on a method which we call the $\circ$-closure method. The $\circ$-closure method is a constraint propagation method allowing to enforce the $(0,3)$-consistency of a QCN $\mathcal{N} = (V, C)$, which means that all restrictions of $\mathcal{N}$ to 3-variables are consistent. The $\circ$-closure method consists in iteratively performing the following operation: $C_{ij} := C_{ij} \cap (C_{ik} \circ C_{kj})$, for all v_i, v_j, v_k of V, until a fix-point is reached. The QCN obtained in this way is a sub-QCN of $\mathcal{N}$ which is equivalent to it, and such that $C_{ij} \subseteq C_{ik} \circ C_{kj}$, for all v_i, v_j, v_k of V.

This latter property is expressed by saying that this sub-network is ∘-closed (to simplify, in the sequel, we will assume that a ∘-closed QCN does not contain the empty relation associated with a constraint). When the QCN obtained in this way contains the empty relation as a constraint, we can assert that the initial QCN is not consistent. However, when it is not the case, we cannot (in the general case) infer the consistency of the network. Despite this, the ∘-closure method is the main constraint propagation method used for qualitative constraint networks.

4 Encoding Qualitative Networks into Discrete Ones

The idea of mapping qualitative networks into discrete ones is quite natural, but, to the best of our knowledge, it has not been formalized and studied in the general case (i.e. for any qualitative algebra). However, we can cite the work of Pham et al. [14] who propose such a transformation for the Interval Algebra (IA). More precisely, any IA network $\mathcal{N}$ can be encoded into a discrete network $\mathcal{P}$ as follows. First, each constraint of $\mathcal{N}$ is mapped to a variable of $\mathcal{P}$ whose domain corresponds to the atomic relations of the constraint (and, as a consequence, a subset of B). Second, each triple of constraints of $\mathcal{N}$ is mapped to a ternary constraint of $\mathcal{P}$ such that the associated relation contains all valid 3-tuples satisfying the weak composition.

In this section, we propose a more preservative encoding of qualitative networks into discrete ones. In our case, a QCN $\mathcal{N}$ is transformed into a ternary DCN $\mathcal{P}$ where the constraints of $\mathcal{N}$ become the variables of $\mathcal{P}$ and the constraints of $\mathcal{P}$ are such that their associated relations are defined by the entire table of weak composition. More formally, we define such a transformation, denoted $\mathsf{T}_{\mathsf{DCN}}$, as follows:

Definition 5. Let $\mathcal{N} = (V,C)$ be a QCN. $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is the DCN $\mathcal{P} = (X,D,C')$ defined by:

- for each pair of variables $v_i, v_j \in V$ with $0 < i \leq j \leq n$, X contains a variable x_{ij}. The domain of x_{ij} is defined by C_{ij};
- for each triple of variables $v_i, v_j, v_k \in V$ with $0 < i < k < j \leq n$, C' contains a ternary constraint C'_{ijk} involving the three variables x_{ij}, x_{ik}, x_{kj} and defined by $C'_{ijk} = TC$ with $TC = \{(a,b,c) \in \mathsf{B}^3 : a \in b \circ c\}$.

Remark that the main difference between the approach that we describe below and the approach of [14] is that the ternary constraints of the discrete network are not reduced by weak composition. Hence, we remain closer to the initial qualitative networks.

Firstly, we can prove that this transformation is sound for the consistency problem:

Proposition 1. *Let $\mathcal{N} = (V,C)$ be a* QCN*. If $\mathcal{N}$ is consistent then $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is consistent.*

Proof. Let $\mathcal{N} = (V,C)$ be a QCN and $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N}) = (X,D,C')$ be the DCN obtained from $\mathcal{N}$. If $\mathcal{N}$ is consistent then there exists a consistent scenario $\mathcal{S} = (V,C'')$ of $\mathcal{N}$. As $\mathcal{S}$ is consistent and atomic, $\mathcal{S}$ is ∘-closed. Now, let us consider the assignment I of the variables X defined by $I(x_{ij}) = b_{ij}$ with $C''_{ij} = \{b_{ij}\}$ for all $x_{ij} \in X$. $\mathcal{S}$ is a

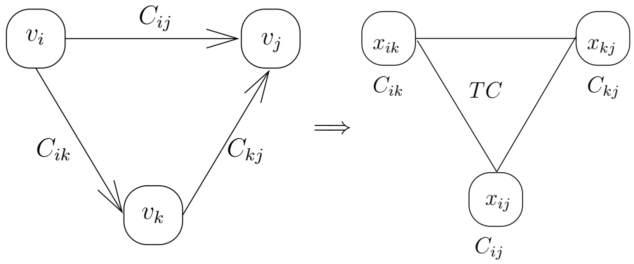


Fig. 2: The transformation $\mathsf{T}_{\mathsf{DCN}}$.

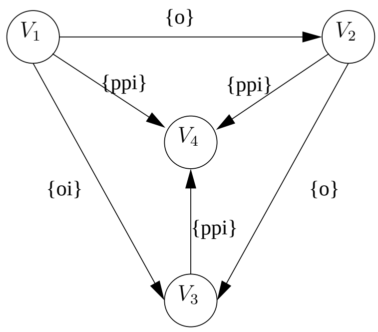
subnetwork of $\mathcal{N}$, hence $I(x_{ij}) \in D(x_{ij})$. Let $x_{ij}, x_{ik}, x_{kj} \in X$ with $0 < i < k < j \leq n$, $C''_{ij} \subseteq C''_{ik} \circ C''_{kj}$ as $\mathcal{S}$ is $\circ$-closed. As a consequence, $(I(x_{ij}), I(x_{ik}), I(x_{kj})) \in TC$. Hence, $(I(x_{ij}), I(x_{ik}), I(x_{kj})) \in C'_{ijk}$. We can conclude that I is a solution of $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$. So, $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is consistent. □

Unfortunately, the encoding is not complete for some qualitative calculi. As an illustration, let us consider the QCN $\mathcal{N}$ depicted in Figure 3 which is defined in the cyclic interval algebra [10, 4]. This qualitative network $\mathcal{N}$ is inconsistent whereas the discrete network $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is consistent. A solution of this DCN is given by instantiating each variable by the value of its domain. Despite this, we have the following weaker property:

Proposition 2. *Let $\mathcal{N} = (V, C)$ be a* QCN. *If $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is consistent then $\mathcal{N}$ admits a $\circ$-closed scenario.*

Proof. Let $\mathcal{N} = (V, C)$ be a QCN and $\mathcal{P} = \mathsf{T}_{\mathsf{DCN}}(\mathcal{N}) = (X, D, C')$. If $\mathcal{P}$ is consistent then there exists a consistent instantiation I for $\mathcal{P}$. Let $\mathcal{S} = (V, C'')$ be the QCN defined by : $C''_{ij} = \{I(x_{ij})\}$ for all $0 < i \leq j \leq n$, $C''_{ij} = (C''_{ji})^{\smile}$ for all $0 < j < i \leq n$. Remark that $C''_{ij} \neq \{\}$ for all $0 < j < i \leq n$. Let $i, j, k \in \{1, \ldots, n\}$. Firstly, consider the case where i, j, k are distinct numbers. Suppose without any loss of generality than $i < k < j$. We have $(I(x_{ij}), I(x_{ik}), I(x_{kj})) \in TC$, as a consequence there exists $d_i, d_j, d_k \in \mathsf{D}$ such that $d_i\ C''_{ij}\ d_j$, $d_i\ C''_{ik}\ d_k$, $d_k\ C''_{kj}\ d_j$, $d_j\ C''_{ji}\ d_i$, $d_k\ C''_{ki}\ d_i$ and $d_j\ C''_{jk}\ d_k$. Moreover we can remark that $d_i\ C''_{ii}\ d_i$, $d_j\ C''_{jj}\ d_j$ and $d_k\ C''_{kk}\ d_k$ since $C''_{ii} = C''_{jj} = C''_{kk} = \{\mathsf{Id}\}$. From all this we know that $\mathcal{S}$ is an atomic QCN and is consistent on all triples of distinct variables $v_i, v_j, v_k \in V$. It results that $C''_{ij} \subseteq C''_{ik} \circ C''_{kj}$ for all distinct variables $v_i, v_j, v_k \in V$. Now, consider $i, j, k \in \{1, \ldots, n\}$ with $i = j$. We have $C''_{ij} = \{\mathsf{Id}\}$. By definition of the weak composition and the converse we know that $\{\mathsf{Id}\} \in b \circ b^{\smile}$ for all $b \in \mathsf{B}$. It results that $C''_{ij} \subseteq C''_{ik} \circ C''_{kj}$ for all $k \in \{1, \ldots, n\}$ since $C''_{kj} = C''_{ki} = (C''_{ik})^{\smile}$. Now suppose that $i, j, k \in \{1, \ldots, n\}$ with $i = k$ (resp. $j = k$). We have $C''_{ij} \subseteq C''_{ik} \circ C''_{kj}$ since $C''_{ik} = \{\mathsf{Id}\}$ and $C''_{kj} = C''_{ij}$ (resp. $C''_{kj} = \{\mathsf{Id}\}$ and $C''_{ik} = C''_{ij}$). We can conclude that $\mathcal{S}$ is $\circ$-closed. □

A qualitative calculus will be said to be *nice* iff it satisfies the following property: a scenario is consistent if and only if it is $\circ$-closed. In fact, many qualitative calculi are nice, and in particular the well known Allen's calculus. From Propositions 1 and 2, we can establish the following property (whose proof is immediate):



Variable x	$D(x)$
x_{11}	$\{eq\}$
x_{12}	$\{o\}$
x_{13}	$\{oi\}$
x_{14}	$\{ppi\}$
x_{22}	$\{eq\}$
x_{23}	$\{o\}$
x_{24}	$\{ppi\}$
x_{33}	$\{eq\}$
x_{34}	$\{ppi\}$
x_{44}	$\{eq\}$

Constraints C_{ijk} for $0 < i \le j \le 4$ and $0 < k \le 4$
$\{(eq,eq,eq),(eq,o,o),(eq,s,s),(eq,d,d),$ $(eq,fi,fi),(eq,ppi,ppi),(o,ppi,ppi),(o,o,oi),$ $\ldots\}$

Fig. 3: A QCN $\mathcal{N}$ of the cyclic interval algebra. Although $\mathcal{N}$ is inconsistent, $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N}) = (X,D,C)$ is consistent.

Proposition 3. *Let $\mathcal{N}$ be a* QCN *defined in a* nice *qualitative calculus. $\mathcal{N}$ is consistent iff $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is consistent.*

We can also show that the transformation $\mathsf{T}_{\mathsf{DCN}}$ preserves minimality and equivalence.

Proposition 4. *Let $\mathcal{N}$ be a* QCN*. $\mathcal{N}$ is minimal iff $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is minimal.*

Proposition 5. *Let $\mathcal{N}$ and $\mathcal{N}'$ be two* QCN*s. If $\mathcal{N}$ and $\mathcal{N}'$ are equivalent then $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ and $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N}')$ are equivalent.*

To close this section, we define the converse transformation of $\mathsf{T}_{\mathsf{DCN}}$, namely the transformation $\mathsf{T}_{\mathsf{QCN}}$.

Definition 6. Let $\mathcal{N} = (V,C)$ be a QCN and $\mathcal{P} = (X,D,C')$ be a DCN such that $\mathcal{P} \subseteq \mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$. $\mathsf{T}_{\mathsf{QCN}}(\mathcal{P})$ is the QCN (V,C'') defined by $C''_{ij} = D(x_{ij})$ and $C''_{ji} = (C''_{ij})^{\sim}$ for all $0 < i \le j \le n$.

We have the following properties :

Proposition 6. *Let $\mathcal{N}$ be a* QCN.

(a) $\mathcal{N} = \mathsf{T}_{\mathsf{QCN}}(\mathsf{T}_{\mathsf{DCN}}(\mathcal{N}))$*;*
(b) *if $\mathcal{P} \subseteq \mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ then $\mathsf{T}_{\mathsf{QCN}}(\mathcal{P}) \subseteq \mathcal{N}$;*
(c) *if $\mathcal{N} \subseteq \mathcal{N}'$ then $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N}) \subseteq \mathsf{T}_{\mathsf{DCN}}(\mathcal{N}')$.* □

5 Equivalence between Local Consistencies

In this section we study the relationships of qualitative and discrete constraint networks in terms of (local) consistencies.

Proposition 7. *Let $\mathcal{N}$ be a $\circ$-closed* QCN. *$\mathcal{N}$ and* $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ *are* $(0,3)$*-consistent.*

Proof. For the first claim, let $\mathcal{N} = (V, C)$ be a $\circ$-closed QCN. We know that $C_{ij} \subseteq C_{ik} \circ C_{kj}$ for all $0 < i, j, k \leq n$. There exists $b_{ij} \in C_{ij}$, $b_{ik} \in C_{ij}$ and $b_{kj} \in C_{kj}$ such that $b_{ij} \in b_{ik} \circ b_{kj}$. By definition of the weak composition, there exist $y_i, y_j, y_k \in \mathsf{D}$ such that $y_i\ b_{ij}\ y_j$, $y_i\ b_{ik}\ y_k$ and $y_k\ b_{kj}\ y_j$. Moreover, by definition of QCNs we know that $b_{ij}^{\smile} \in C_{ji}$, $b_{ik}^{\smile} \in C_{ki}$ and $b_{kj}^{\smile} \in C_{jk}$. Hence, by definition of the inverse we have: $y_j\ b_{ij}^{\smile}\ y_i$, $y_k\ b_{ik}^{\smile}\ y_i$ and $y_j\ b_{kj}^{\smile}\ y_k$. Moreover $y_i\ C_{ii}\ y_i$, $y_j\ C_{jj}\ y_j$ and $y_k\ C_{kk}\ y_k$ since $C_{ii} = C_{jj} = C_{kk} = \{\mathsf{Id}\}$. It results that the restriction of $\mathcal{N}$ on v_i, v_j, v_k is consistent for all $0 < i, j, k \leq n$. We can conclude that $\mathcal{N}$ is $(0,3)$-consistent.

For the second claim let $\mathcal{P} = \mathsf{T}_{\mathsf{DCN}}(\mathcal{N}) = (X, D, C')$. Consider three variables x_{ij}, $x_{ik}, x_{kj} \in X$ with $0 < i < k < j \leq n$ (we consider these triples of variables since there are no constraint on other triples of variables). We have $C_{ij} \subseteq C_{ik} \circ C_{kj}$. As a consequence, there exists $b_{ij} \in C_{ij}$, $b_{ik} \in C_{ik}$ and $b_{kj} \in C_{kj}$ such that $b_{ij} \in b_{ik} \circ b_{kj}$. We have $b_{ij} \in D(x_{ij})$, $b_{ik} \in D(x_{ik})$, $b_{kj} \in D(x_{kj})$ and $(b_{ij}, b_{ik}, b_{kj}) \in C'_{ijk}$. We can conclude that $\mathcal{P}$ is $(0,3)$-consistent. □

Moreover, we have the following properties.

Proposition 8. *Let $\mathcal{N}$ be a $\circ$-closed* QCN. $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ *is strongly* 3*-consistent.*

Proof. Let $\mathcal{P} = \mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ where $\mathcal{N} = (V, C)$ is a $\circ$-closed QCN. From the fact that each domain of $\mathcal{P}$ is not empty and each constraint is a ternary constraint we can assert that $\mathcal{P}$ is $(0,1)$-consistent and $(1,1)$-consistent. Now, let us prove that $\mathcal{P}$ is also $(2,1)$-consistent. Let us consider three variables $x_{ij}, x_{ik}, x_{kj} \in X$ with $0 < i < k < j \leq n$ (we just consider triples of variables corresponding to the scope of a constraint). Let I be a partial consistent assignment on x_{ij} and x_{ik}. We know that $I(x_{ij}) \in C_{ij}$ and $I(x_{ik}) \in C_{ik}$. Moreover, $C_{ij} \subseteq C_{ik} \circ C_{kj}$. It results that there exists $b_{kj} \in C_{kj}$ such that $I(x_{ij}) \in I(x_{ik}) \circ b_{kj}$. Hence, $(I(x_{ij}), I(x_{ik}), b_{kj}) \in TC$, and besides, $b_{kj} \in D(x_{kj})$. As a consequence, by defining $I(x_{kj})$ with b_{kj} we extend I in a partial consistent assignment to x_{kj}. In a similar way of reasoning, we can extend a partial consistent assignment on x_{ij} and x_{kj} to the variable x_{ik} and extend a partial consistent assignment on x_{ik} and x_{kj} to the variable x_{ij}. Hence $\mathcal{P}$ is $(2,1)$-consistent. We can conclude that $\mathcal{P}$ is a strongly 3-consistent DCN. □

Proposition 9. *Let $\mathcal{N}$ be a $\circ$-closed* QCN. $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ *is generalized arc-consistent.*

Proof. Let $\mathcal{P} = \mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ where $\mathcal{N}$ is a $\circ$-closed QCN. From Proposition 8, we know that $\mathcal{P}$ is strongly 3-consistent. Since $\mathcal{P}$ only involves ternary constraints, it results that $\mathcal{P}$ is $(1,2)$-consistent and also generalized arc-consistent. □

A corollary of these propositions is that if $\mathcal{N}$ is a $\circ$-closed atomic QCN then $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ is a consistent DCN.

Proposition 10. *Let $\mathcal{N}$ be a* QCN. *If* $\mathsf{T}_{\mathsf{DCN}}(\mathcal{N})$ *is a generalized arc-consistent then* $\mathcal{N}$ *is* $\circ$*-closed.*

Proof. Let $\mathcal{P} = \mathsf{T}_{\mathsf{DCN}}(\mathcal{N}) = (X, D, C')$ with $\mathcal{N} = (V, C)$ a QCN. Suppose that $\mathcal{P}$ is generalized arc-consistent. Consider $0 < i, j, k \leq n$. Suppose that $i < k < j$ without any loss of generality. Let $b_{ij} \in C_{ij}$. We have $b_{ij} \in D(x_{ij})$. $\mathcal{P}$ is generalized arc-consistent, it results that there exist $b_{ik} \in D(x_{ik})$ and $b_{kj} \in D(x_{kj})$ with $(b_{ij}, b_{ik}, b_{kj}) \in C'_{ijk}$. As $b_{ik} \in C_{ik}$ and $b_{kj} \in C_{kj}$ we have $b_{ij} \in C_{ik} \circ C_{kj}$. Hence, $C_{ij} \subseteq C_{ik} \circ C_{kj}$. From this we also have $C_{ij}^{\smile} \subseteq (C_{ik} \circ C_{kj})^{\smile}$. Hence, $C_{ji} \subseteq C_{jk} \circ C_{ki}$. Now let $b_{ik} \in C_{ik}$. We have $b_{ik} \in D(x_{ik})$. $\mathcal{P}$ is generalized arc-consistent, it results that there exist $b_{ij} \in D(x_{ij})$ and $b_{kj} \in D(x_{kj})$ with $(b_{ij}, b_{ik}, b_{kj}) \in C'_{ijk}$. From the definition of the weak composition, since $(b_{ij}, b_{ik}, b_{kj}) \in TC$ we can assert that $(b_{ik}, b_{ij}, b_{kj}^{\smile}) \in TC$. As $b_{ij} \in C_{ij}$ and $b_{kj}^{\smile} \in C_{jk}$ we have $b_{ik} \in C_{ij} \circ C_{jk}$. Hence, $C_{ik} \subseteq C_{ij} \circ C_{jk}$. From this we also have $C_{ik}^{\smile} \subseteq (C_{ij} \circ C_{jk})^{\smile}$. Hence, $C_{ki} \subseteq C_{kj} \circ C_{ji}$. With a similar line of reasoning we can prove that $C_{kj} \subseteq C_{ki} \circ C_{ij}$ and $C_{jk} \subseteq C_{ji} \circ C_{ik}$. Now suppose that $i = j$. We have $C_{ij} = C_{ji} = \{\mathsf{Id}\}$. Moreover we know that $\{\mathsf{Id}\} \subseteq b \circ b^{\smile}$ for all $b \in \mathsf{B}$. It results that $C_{ij} \subseteq C_{ik} \circ C_{kj}$ and $C_{ji} \subseteq C_{jk} \circ C_{ki}$. Morevover it easy to see that $C_{ik} \subseteq C_{ij} \circ C_{jk}$, $C_{ki} \subseteq C_{kj} \circ C_{ji}$, $C_{jk} \subseteq C_{ji} \circ C_{ik}$ and $C_{kj} \subseteq C_{ki} \circ C_{ij}$. We obtain the same result with $i = k$ or $k = j$. Finally we can assert that $\mathcal{N}$ is a $\circ$-closed QCN. □

As a consequence, a way to obtain the $\circ$-closure of a QCN is to transform it into a DCN via $\mathsf{T}_{\mathsf{DCN}}$. Indeed, we can then apply a GAC algorithm and transform the obtained DCN into a QCN via $\mathsf{T}_{\mathsf{QCN}}$ (see Figure 4).

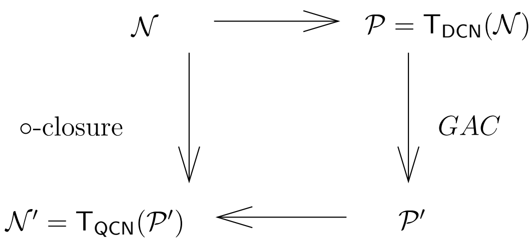


Fig. 4: The $\circ$-closure through the DCNs.

6 Future Work and Conclusions

Abscon [12] and QAT (Qualitative Algebra Toolkit) [7] are two JAVA constraint programming libraries developed at *CRIL-CNRS*. The first one is dedicated to discrete constraint networks. It can solve instances of any arity and implements state-of-the-art generic filtering (constraint propagation) and search algorithms. The second one is specialized in qualitative constraint networks. It aims to provide open and generic tools for defining and manipulating qualitative algebras and qualitative networks based on these algebras. QAT also provides several methods to tackle the main centers of interest when dealing with qualitative constraint networks, mainly the consistency problem, the problem of finding one or all solutions, and the minimal network problem.

Currently, using these libraries, we are studying the interest of mapping qualitative networks into discrete ones. One of our ultimate objective is to detect which (inference or search) methods from the discrete CSP community could be efficiently specialized to the qualitative algebras. For example, we project to experimentally determine whether exploiting GAC could be an efficient alternative $\circ$-closure for qualitative constraints. Another current line of research is the study of SAT encodings for QCNs.

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