A max-based merging of incommensurable ranked belief bases based on finite scales

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***Abstract*—Recently, several approaches have been proposed to merge possibly contradictory belief bases. This paper focuses on max-based merging operators applied to incommensurable ranked belief bases. We first propose a characterization of a result of merging using Pareto-like ordering on a set of possible solutions. Then we propose two equivalent ways to recover the result of merging. The first one is based on the notion of compatible rankings defined on finite scales. The second one is only based on total pre-orders induced by ranked bases to merge.**

Keywords: merging belief bases, ranked bases, incommensurability.

I. Introduction

The problem of merging multiple sources of information is an important issue in many applications. Recently, several approaches have been proposed for merging possibly contradictory belief bases [1]–[6]. Belief bases can be either flat (no priority relation is provided between different formulas of belief bases) or ranked. A ranked belief base (or a stratified belief base, a weighted belief base) is a set of well founded formulas, each with a rank (assumed here to be an integer). The higher is the rank associated with a formula, the more important is the formula. Each ranked belief base induces a ranking on set of possible interpretations (or solutions). Interpretations assigned to lower ranks are considered to be more plausible than interpretations assigned to higher ranks. In particular, interpretations with the rank '0' are the most preferred ones and represent agent's current beliefs.

Among existing merging approaches, we can distinguish two important ones: utilitarist approaches (or majority approaches) and egalitarist approaches. Examples of majority approaches are those based on the "sum" operator, see [1], and examples of egalitarist approaches are those based on the maximum operator (and its extension Gmax), see [1].

Both utilitarist and egalitarist approaches, when applied to merge ranked belief bases, are based on the assumption that belief bases to merge are commensurable. Namely all sources share a same common scale to order their pieces of information. These commensurability assumptions may make sense in some applications, but can appear to be too strong for other applications.

This paper deals with egalitarist-based fusion modes, which are majority independent, of incommensurable ranked bases. We use the maximum-based merging operator as an example of egalitarist-based information fusion.

One way to deal with incommensurability problem is to use a so-called Pareto ordering [7]. Interpretations (or solutions) are compared with respect to their compatibility with each individual base. Namely, given a multi-set of consistent ranked bases E, a solution s is strictly preferred to another solution s', if $i)$ s' is not a model of each base in E, and $ii)$ for each base $B_i \in E$, either s is strictly preferred in s' with respect to B_i, or s and s' are both models of B_i.

In [8], another natural way to define merging operator is to consider all compatible scales. A compatible scale is simply a re-assignment of ranks associated with beliefs in bases, such that the original relative ordering between beliefs is preserved. Then a solution s is said to be strictly preferred to another solution s', if s is preferred to s' in each compatible scale using maximum-based merging operator.

We have shown in [8] that using all compatible scales is equivalent to considering a Pareto-like ordering which is informally described above. This paper goes one step further and shows that there is no need to define compatible rankings over the whole set of integers $\mathbb{N}$. In fact, we show that it is enough to use finite scales to recover the result of merging. In the last part of this paper, we show in fact that there is no need to use numbers, and one can only use total pre-orders induced by initial ranked bases.

The rest of this paper is organized as follows. First we present the concept of ranked belief bases, and maximum-based merging of commensurable bases. Then we present a first way to deal with commensurability assumption using finite compatible scales. Next we show how the result of merging can be obtained using a variant of Pareto ordering. Lastly another characterization of the result of fusion is obtained using total orders induced by initial belief bases.

II. Preliminaries

Let $\mathcal{L}$ be a $finite$ propositional language. We denote by Ω the set of interpretations of $\mathcal{L}$ and by ω an element of Ω. Greek letters ϕ, ψ denote propositional formulas. $Mod(\phi)$ represents the set of models of ϕ, namely $Mod(\phi) = \{\omega \in \Omega : \omega \models \phi\}$.

A. Ranked bases

Ranked belief bases are convenient frameworks for representing uncertain (or prioritized) pieces of information. Ranked belief bases are used in different frameworks, such as possibility theory [9], [10] or ordinal conditional functions (OCF) [11]–[14].

In this paper, the term beliefs is used when pieces of information provided by sources are uncertain. We will reserve the term constraints to completely sure and consistent pieces of information. Constraints should be present in the result of the fusion process, while beliefs can be accepted, weakened or if necessary ignored in the fusion process.

In this paper, ranked belief bases are assumed to be multi-sets of ranked propositional formulas. Namely:

Definition 1 (Ranked bases): A ranked base B_i is a multi-set of ranked propositional formulas,

$$B_i = \{(\phi_{ij}, R_{B_i}(\phi_{ij})),\ j \in \{1, ..., m_i\}\},$$

where $\phi_{ij} \in \mathcal{L}$, and $R_{B_i}(\phi_{ij}) \in \mathbb{N}^*$.

Intuitively, $(\phi_{ij}, R_{B_i}(\phi_{ij}))$ means that ϕ_{ij} has a priority rank of at least $R_{B_i}(\phi_{ij})$ (where a higher rank is better). Only strictly positive ranks are represented. Moreover, we reserve the infinity symbol $+\infty$ for integrity constraints.

There exist different ways to induce an ordering on possible interpretations from a given ranked belief base. In this paper, we use the so-called best out ordering [15] , which is defined as: an interpretation ω is preferred to another interpretation ω', if and only if the highest belief falsified by ω is less important than the highest belief falsified by ω'. Hence, each interpretation is associated with the ranks of highest formula that it falsifies. Interpretations which are models of B_i^* have a rank equal to 0 and are the preferred ones. More precisely:

Definition 2 (Ranking functions): A ranking function κ_{B_i} associated with a ranked belief base B_i is a function that maps each interpretation $\omega \in \Omega$ to an integer $\kappa_{B_i}(\omega)$ such that:

$$\kappa_{B_i}(\omega) = \begin{cases} 0 \text{ if } \forall (\phi_{ij},\ R_i(\phi_{ij})) \in B_i,\ \omega \models \phi_{ij} \\ max\{R_{B_i}(\phi_{ij}) : \omega \not\models \phi_{ij}, \\ \qquad\qquad (\phi_{ij}, R_{B_i}(\phi_{ij})) \in B_i\} \\ \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Best-out ordering is the basis of possibilistic logic semantics [9] and adjustment revision [12].

Example 1: Consider a ranked belief base:
$B = \{(\neg a \vee b, 8), (a \vee b, 5), (a, 2)\}$.

The following table gives the ranking function κ_B associated with B.

Table I
AN EXAMPLE OF RANKING FUNCTION

$\omega_i \in \Omega$	a	b	$\kappa_B(\omega_i)$
ω_0	0	0	5
ω_1	0	1	2
ω_2	1	0	8
ω_3	1	1	**0**

B. Maximum-based fusion

Let $E = \{B_1, ..., B_n\}$ be a multi-set of n ranked bases issued from n sources, and let μ be a propositional formula representing integrity constraints to be satisfied. We suppose in this section that all the source share the same meaning of ranks assigned with formulas. We also suppose that each ranked belief base is consistent (but of course, their union may be inconsistent).

The aim of merging is, given n commensurable ranked bases, to compute $\Delta(E)$, a propositional formula representing the result of the fusion of these bases. In the literature, different methods for merging E have been proposed.

This paper focuses on an egalitarist fusion, and uses the maximum operator to illustrate the fusion process.

But first, we need to introduce the notion of $profile$ associated with an interpretation ω, denoted by $\nu_E(\omega)$, and defined by

$$\nu_E(\omega) =< \kappa_{B_1}(\omega), ..., \kappa_{B_n}(\omega) > .$$

It represents the degree of surprise (or consistency) of an interpretation ω with respect to the multi-set of ranked bases. The computation of the result of merging $\Delta(E)$ is achieved in two step: first combine the surprise degrees $\kappa_{B_i}(\omega)$'s with a merging operator (here the maximum operator), and then select interpretations with lowest ranks. This leads to define a strict order, denoted by $\lhd_{Max}$, between interpretations as follows: an interpretation ω is preferred to another interpretation ω' if the maximum element of the profile of ω is smaller than the maximum element of the profile of ω'. More formally:

Definition 3 (definition of $\lhd^E_{Max}$): Let E be a multi-set of ranked bases. Let ω and ω' be two interpretations and $\nu_E(\omega)$, $\nu_E(\omega')$ be their associated profiles. Then:

$$\omega \lhd^E_{Max} \omega' \text{ iff } Max(\nu_E(\omega)) < Max(\nu_E(\omega')),$$

where

$$Max(\nu_E(\omega)) = Max\{\kappa_{B_i}(\omega) : i \in \{1, ..., n\}\}.$$

The result of the merging $\Delta^{max}_{\mu}(E)$ is a propositional formula whose models are interpretations which are models of μ and which are minimal with respect to $\lhd^E_{Max}$. More formally:

Definition 4 (Maximum-based merging operator): Let $E = \{B_1, ..., B_n\}$ be a multi-set of ranked belief bases and μ be an integrity constraint. The result of merging is a propositional formula, denoted by $\Delta^{max}_{\mu}(E)$, defined by:

$$Mod(\Delta^{max}_{\mu}(E)) = \{\omega \in Mod(\mu) : \nexists \omega' \in Mod(\mu), \omega' \lhd_{Max} \omega\}$$

Let us illustrate these definitions with the following example.

Example 2: Let $E = \{B_1,\ B_2\}$ such that:
$B_1 = \{(a, 8), (\neg b, 4)\}$ and $B_2 = \{(b, 2), (\neg a, 1)\}$.
The profile of each interpretation is given in Table II

Table II
PROFILES ASSOCIATED WITH INTERPRETATIONS

	a	b	$\kappa_{B_1}(\omega)$	$\kappa_{B_2}(\omega)$	$\nu_E(\omega)$	Max
ω_0	0	0	8	2	<8,2>	8
ω_1	0	1	8	0	<8,0>	8
ω_2	1	0	0	2	<0,2>	**2**
ω_3	1	1	4	1	<4,1>	4

The result of the Max-merging, considering $\mu \equiv a$ is such that:

$$Mod(\Delta_a^{max}(E)) = \{\omega_2\}$$

III. Fusion-based on compatible scalings

The merging operation defined above assume that the sources, who provide B_i's, are commensurable. In example 6, it is assumed that the weight associated with $\neg b$ in B_1 (namely 4) can be compared to the weight associated with $\neg a$ (namely 1) in B_2. Such assumption is not always true. In the following, we drop this commensurability assumption.

We present in this section a strategy for an egalitarist fusion of incommensurable ranked belief bases. In a technical paper [8], a natural way to make them commensurable is to apply a compatible scaling on existing ranks. A scaling is said to be compatible if it preserves original relative orders between beliefs of each ranked bases.

A scaling $\mathcal{S}$ affects new ranks to beliefs of each ranked base from the multi-set E. In [8] the new ranks assigned are defined on the set of integers $\mathbb{N}$. This paper shows that there is no need to consider the whole $\mathbb{N}$, and one can only consider a finite scale. More precisely, this finite scale, denoted by $\mathbb{L}$, is defined as:

$$\mathbb{L} = \{1, 2, ..., |B_1| + ... + |B_n|\}$$

where $|B_i|$ denotes the number of different degrees (or ranks) in B_i. Namely:

Definition 5 (compatible scaling): Let $E = \{B_1, ..., B_n\}$ where $B_i = \{(\phi_{ij}, R_{B_i}(\phi_{ij}))\}$. Then a scaling $\mathcal{S}$ is defined by:

$$\begin{array}{llll} \mathcal{S}: & B_1 \sqcup ... \sqcup B_n & \rightarrow & \mathbb{L} \\ & (\phi_{ij}, R_{B_i}(\phi_{ij})) & \mapsto & \mathcal{S}(\phi_{ij}) \end{array}$$

Where $\sqcup$ represents union of multi-sets.
A scaling $\mathcal{S}$ is said to be compatible with $R_{B_1}, ..., R_{B_n}$ if and only if:

$$\forall B_i \in E,\ \forall (\phi, R_{B_i}(\phi)), (\phi', R_{B_i}(\phi')) \in B_i,$$

$$R_{B_i}(\phi) \leq R_{B_i}(\phi') \ \text{ iff } \ \mathcal{S}(\phi) \leq \mathcal{S}(\phi').$$

Clearly, a compatible scaling is not unique, as it is illustrated by the following example.

Example 3 (continued): Let us consider again $B_1 = \{(a, 8), (\neg b, 4)\}$ and $B_2 = \{(b, 2), (\neg a, 1)\}$. We have $|B_1| = |B_2| = 2$ and $\mathbb{L} = \{1, 2, 3, 4\}$. Table III gives 2 scalings: $\mathcal{S}_1$ and $\mathcal{S}_2$.

Table III
EXAMPLES OF SCALING

	ϕ_{ij}	$R_{B_i}(\phi_{ij})$	$\mathcal{S}_1(\phi_{ij})$	$\mathcal{S}_2(\phi_{ij})$
B_1	a	8	3	3
	$\neg b$	4	2	4
B_2	b	2	4	2
	$\neg a$	1	1	1

The scaling $\mathcal{S}_1$ is a compatible one, because it preserves the order inside each ranked base. However, the scaling $\mathcal{S}_2$ is not a compatible one: it inverses the priorities inside B_1.

The set of compatible scalings with E is denoted by $\mathbb{S}_E$. Note that $\mathbb{S}_E$ is never empty.

Given a compatible scaling $\mathcal{S}$, we denote by $B_i^{\mathcal{S}}$ the ranked base obtained from B_i by replacing each pair $(\phi_{ij}, R_i(\phi_{ij}))$ by $(\phi_{ij}, \mathcal{S}(\phi_{ij}))$. Similarly, we denote by $E^{\mathcal{S}}$ the multi-set obtained from E by replacing each B_i in E by $B_i^{\mathcal{S}}$.

A natural question now is, given the set of all compatible scalings $\mathbb{S}_E$, how to define the result of merging. Different options exist, either we use some uncertainty measure to select one compatible scaling from $\mathbb{S}_E$, or we consider all compatible scalings. In this paper, we adopt for a skeptical option and consider all compatible scalings, in order to avoid arbitrary choices. An interpretation ω is then said to be preferred to ω', if for each compatible scaling $\mathcal{S}$, ω is preferred to ω' using Definition 3 (namely, $\omega \lhd_{Max}^{E^{\mathcal{S}}} \omega'$). More precisely,

Definition 6 (Ordering between interpretations): Let E be a multi-set of ranked belief bases, $\mathbb{S}_E$ be the set of all compatible scalings associated with E. Let ω, ω' be two interpretations. Then:

$$\omega <_{\forall}^{E} \omega' \ \text{ iff } \ \forall \mathcal{S} \in \mathbb{S}_E,\ \omega \lhd_{Max}^{E^{\mathcal{S}}} \omega'$$

where $\lhd_{Max}^{E^{\mathcal{S}}}$ is the result of applying Definition 3 on $E^{\mathcal{S}}$.

Models of $\Delta_{\mu}^{\forall}(E)$ are those which are models of μ and minimal for $<_{\forall}^{E}$, namely:

$$Mod(\Delta_{\mu}^{\forall}(E)) = \{\omega \in Mod(\mu) :\ \nexists \omega' \in Mod(\mu),\ \omega' <_{\forall}^{E} \omega\}.$$

Note that $<_{\forall}^{E}$ is only a partial order.

The following proposition shows that an interpretation ω is a model of $\Delta_{\mu}^{\forall}$ if and only if there exists a compatible scaling where this interpretation belongs to the result fusion, namely is a model of $\Delta_{Max}^{E^{\mathcal{S}}}$. More formally:

Proposition 1: Let E be a multi-set of ranked belief bases. Then $\omega \in Mod(\Delta_{\mu}^{\forall}(E))$, if and only if there exists a compatible scaling $\mathcal{S}$ such that $\omega \in Mod(\Delta_{\mu}^{max}(E^{\mathcal{S}}))$.

Let us illustrate the fusion based on all compatible scalings with the following example.

Example 4 (continued): Assume that $\mu \equiv \top$. Let us consider again $B_1 = \{(a, 8), (\neg b, 4)\}$ and $B_2 = \{(b, 2), (\neg a, 1)\}$.

We have $\mathbb{L} = \{1, 2, 3, 4\}$.

Let us consider again $\mathcal{S}_1$ where $B_1^{\mathcal{S}_1} = \{(a, 4), (\neg b, 3)\}$ and $B_2^{\mathcal{S}_1} = \{(b, 2), (\neg a, 1)\}$ and a scaling $\mathcal{S}_2$, where $B_1^{\mathcal{S}_2} = \{(a, 3), (\neg b, 2)\}$ and $B_2^{\mathcal{S}_2} = \{(b, 3), (\neg a, 2)\}$. Both of them are compatible. Table IV presents the profile of each interpretation for each scaling.

Table IV
TWO EQUIVALENT COMPATIBLE SCALINGS

	a	b	$\nu_{E^{\mathcal{S}_1}}(\omega)$	Max	$\nu_{E^{\mathcal{S}_2}}(\omega)$	Max
ω_0	0	0	$< 4, 2 >$	4	$< 3, 3 >$	3
ω_1	0	1	$< 4, 0 >$	4	$< 3, 0 >$	3
ω_2	1	0	$< 0, 2 >$	**2**	$< 0, 3 >$	3
ω_3	1	1	$< 3, 1 >$	3	$< 2, 2 >$	**2**

Note that in the compatible scaling $\mathcal{S}_1$, ω_2 is the preferred interpretation and in the compatible scaling $\mathcal{S}_2$, ω_3 is the preferred one. Table V shows six additional compatible scalings, Table VI and Table VII provide their associated profiles.

Table V
REPRESENTATIVE COMPATIBLE SCALINGS

	ϕ_{ij}	R_{B_i}	$\mathcal{S}_3$	$\mathcal{S}_4$	$\mathcal{S}_5$	$\mathcal{S}_6$	$\mathcal{S}_7$	$\mathcal{S}_8$
B_1	a	8	4	3	2	2	2	2
	$\neg b$	4	3	2	1	1	1	1
B_2	b	2	2	2	2	3	3	4
	$\neg a$	1	1	1	1	1	2	3

Table VI
PROFILES OF COMPATIBLE SCALINGS

	$\nu_{E^{\mathcal{S}_3}}(\omega)$	$\nu_{E^{\mathcal{S}_4}}(\omega)$	$\nu_{E^{\mathcal{S}_5}}(\omega)$
ω_0	$< 4, 2 >$	<3,2>	<2,2>
ω_1	$< 4, 0 >$	<3,0>	<2,0>
ω_2	$< \mathbf{0}, \mathbf{2} >$	**<0,2>**	<0,2>
ω_3	$< 3, 1 >$	**<2,1>**	**<1,1>**

Table VII
PROFILES OF COMPATIBLE SCALINGS (CONTINUED)

	$\nu_{E^{\mathcal{S}_6}}(\omega)$	$\nu_{E^{\mathcal{S}_7}}(\omega)$	$\nu_{E^{\mathcal{S}_8}}(\omega)$
ω_0	<2,3>	<2,3>	<2,4>
ω_1	<2,0>	**<2,0>**	**<2,0>**
ω_2	<0,3>	<0,3>	<0,4>
ω_3	**<1,1>**	**<1,2>**	<1,3>

In fact, it can be shown that these six compatible scalings given in Table V are enough to characterize the result of fusion. Namely, for each compatible scaling $\mathcal{S}$, there exists a scaling $\mathcal{S}_i \in \{3, ..., 8\}$ given in Table V, such that $\omega \lhd_{Max}^{\mathcal{S}} \omega'$ iff $\omega \lhd_{Max}^{\mathcal{S}_i} \omega'$, for all ω and ω'.

Bold elements in Table VI and VII represent models of Δ_μ^{max} for a given scaling. For instance, the interpretations ω_1 and ω_3 are models of Δ_μ^{max} for the compatible scaling $\mathcal{S}_5$ from Tables 5.

Finally, the strict partial order between interpretations is only defined by $\omega_3 <_{\forall}^{E} \omega_1$.

Hence, models of $\Delta_\mu^{\forall}(E)$ are $\{\omega_1, \omega_2, \omega_3\}$, and $\Delta_\mu^{\forall}(E) \equiv a \vee b$.

As a skeptical approach, conclusions obtained using all compatible scalings are safe. However, considering all possible compatible scales does not means that the approach is too cautious and for instance, only tautologies can be derived from the result of merging. In particular, if the union of bases is consistent, then the result of merging is simply the conjunct of the bases. More formally:

Proposition 2: Let $E = \{B_1, ..., B_n\}$. Then if $\bigwedge_{B_i \in E}(B_i^*) \wedge \mu$ is consistent, then

$$\Delta_\mu^{\forall}(E) \equiv \bigwedge_{B_i \in E}(B_i^*) \wedge \mu$$

Proof: (sketch) The proof is immediate. Let ω be a model of $\bigwedge_{B_i \in E}(B_i^*)$. For each compatible scaling $\mathcal{S}$, it can be checked that its associated profile is $\nu_{E^{\mathcal{S}}}(\omega) =< 0, ..., 0 >$, namely $Max(\nu_{E^{\mathcal{S}}}(\omega)) = 0$. Hence, ω is minimal. Now let ω' be such that it falsifies at least one belief of some base in E, then for each compatible scaling $Max(\nu_{E^{\mathcal{S}}}(\omega')) > 0$. Therefore, for each compatible scale, ω is minimal iff ω is a model of $\bigwedge_{B_i \in E}(B_i^*)$.

■

IV. CHARACTERIZATIONS OF COMPATIBLE-BASED FUSION

A. A Pareto-like ordering

This section describes how identifying preferred interpretations from Ω (or from Mod(μ)) for merging E without computing $\mathbb{S}_E$, using a variant strict Pareto ordering.

Intuitively, an interpretation ω is strictly preferred to ω' if:

1) ω' is not a model of each base B_i^*, and
2) for each B_j, either ω and ω' are models of B_j^*, or $\kappa_{B_j}(\omega) < \kappa_{B_j}(\omega')$ (namely, ω is preferred to ω' with respect to B_j).

More formally:

Definition 7 (Pareto-like ordering): Let $E = \{B_1, ..., B_n\}$. ω is pareto-preferred to ω', denoted by $\omega \lhd_{Pareto} \omega'$, iff the two following conditions are satisfied:

- $\exists i \in \{1, ..., n\},\ \kappa_{B_i}(\omega') \neq 0$,
- $\forall i \in \{1, .., n\},\ \kappa_{B_i}(\omega) = \kappa_{B_i}(\omega') = 0,\ or\ \kappa_{B_i}(\omega) < \kappa_{B_i}(\omega')$.

The first condition simply means that ω cannot be strictly preferred to ω' if ω' satisfies all bases to merge. The second condition is a variant of pareto-ordering. Recall that the usual definition of pareto-ordering is: ω strictly Pareto-dominates ω' if for all i, $\kappa_{B_i}(\omega) \leq \kappa_{B_i}(\omega')$ and $\exists j$ such that $\kappa_{B_j}(\omega) < \kappa_{B_j}(\omega')$. This is different from the one given in our definition 7, as it is illustrate in the following example:

Example 5: Let us consider two ranked bases: $B_1 = \{(a,3), (\neg a \wedge b, 2)\}$ and $B_2 = \{(b,5), (\neg a \wedge \neg b, 1)\}$. Table VIII presents the profile of each interpretation of Ω.

Table VIII
PROFILES ASSOCIATED WITH INTERPRETATIONS

	a	b	$\kappa_{B_1}(\omega)$	$\kappa_{B_2}(\omega)$	$\nu_E(\omega)$
ω_0	0	0	3	5	<3,5>
ω_1	0	1	3	1	<3,1>
ω_2	1	0	2	5	**<2,5>**
ω_3	1	1	2	1	**<2,1>**

From Table IX, ω_3 is strictly preferred to ω_2 using classic Pareto ordering, but not with our pareto-like ordering defined above.

Let us illustrate Definition 7 by the following example:

Example 6 (continued): Let us consider again $B_1 = \{(a,8), (\neg b, 4)\}$ and $B_2 = \{(b,2), (\neg a, 1)\}$. Table IX presents the profile of each interpretation of Ω.

Table IX
PROFILES ASSOCIATED WITH INTERPRETATIONS

	a	b	$\kappa_{B_1}(\omega)$	$\kappa_{B_2}(\omega)$	$\nu_E(\omega)$
ω_0	0	0	8	2	<8,2>
ω_1	0	1	8	0	**<8,0>**
ω_2	1	0	0	2	**<0,2>**
ω_3	1	1	4	1	**<4,1>**

From Table IX we have ω_3 strictly preferred to ω_0 using $\lhd_{Pareto}$, since for each belief base, the rank associated with ω_3 is strictly lower than the rank associated with ω_0.

Given definition 7, we define the result of merging as following:

$$Mod(\Delta_\mu^{Pareto}(E)) = \{\omega \in Mod(\mu) : \nexists \omega' \in Mod(\mu),\ \omega' \lhd_{Pareto} \omega\}.$$

Example 7 (continued): Let us consider again $B_1 = \{(a,8), (\neg b, 4)\}$ and $B_2 = \{(b,2), (\neg a, 1)\}$. From Table IX, ω_1, ω_2 and ω_3 are models of Δ_μ^{Pareto} because they are minimal with respect to $\lhd_{Pareto}$ ordering defined in Definition 7.

Hence, $\Delta_\mu^{Pareto}(E) = a \vee b$ for $\mu = \top$. This is exactly the same result as the one given in Example 4.

As we have shown for the case of compatible scales with Proposition 2 , when the bases are consistent then the result is simply the conjunction of bases using Pareto-like ordering. Namely,

Proposition 3: Let μ be an integrity constraint. Let E be a multi-set of ranked belief bases. If $\bigwedge_{B_i^* \in E} B_i^* \wedge \mu$ is consistent, then

$$Mod(\Delta_\mu^{Pareto}(E)) = Mod(\mu \wedge \bigwedge_{B_i \in E} B_i^*)$$

Proof: Assume that ω is a model of $\bigwedge_{B_i^*}$ (which is consistent). Then, $\forall i \in 1,...,n$, $\kappa_{B_i}(\omega) = 0$. Hence, ω is minimal with respect to $\lhd_{Pareto}$

Now, assume that ω' is not a model of $\bigwedge_{B_i^*}$. Then $\exists j \in \{1,...,n\}$ such that $\kappa_{B_i}(\omega') > 0$. Using Definition 7, ω' is not minimal with respect to $\lhd_{Pareto}$.

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The following proposition generalizes the above results and shows the equivalence between Pareto-like ordering and compatible scaling ordering:

Proposition 4 (Equivalence between $<_\forall^E$ and $\lhd_{Pareto}$): Let E be a multi-set of ranked belief bases, $<_\forall^E$ and $\lhd_{Pareto}$ be the two partial orders defined respectively by Definition 6 and Definition 7. Then:

$$\forall \omega,\ \omega' \in \Omega,\ \omega <_\forall^E \omega' \ \text{ iff } \ \omega \lhd_{Pareto} \omega'$$

The proof is given in the appendix.
Thus, this Pareto-like ordering allow us to merge incommensurable ranked bases, without computing all compatible scalings.

B. Focusing on pre-orders

This section briefly shows that total pre-orders induced by initial ranked bases to merge are sufficent to recover the result of fusion (either based on compatible finite scales or Pareto-like ordering). Let $B_i = \{(\phi_{ij}, R_i(\phi_{ij})), i \in \{1,...,m_i\}\}$ be a ranked belief base. We denote by $\leq_i$ a total pre-order defined on formulas of B_i^* such that: $\forall \phi_{ij} \in B_i^*, \forall \phi_{ik} \in B_i^*$,

$$\phi_{ij} \leq_i \phi_{ik} \text{ iff } R_{B_i}(\phi_{ij}) \leq R_{B_i}(\phi_{ik})$$

Then give $(\leq_i, ..., \leq_n)$ be the set of total pre-orders associated respectively with $E = (B_1, ..., B_n)$. We define a compatible ordering as a total pre-order on formulas of $B_1 \sqcup ... \sqcup B_n$, denoted by $<_c$, such that: $\forall \phi_{ij}, \forall \phi_{ik}$,

$$\phi_{ij} <_c \phi_{ik} \text{ iff } \phi_{ij} <_i \phi_{ik}$$

A compatible total pre-order $<_c$ on formulas induces a total order on interpretations, denoted by $\lhd_c$, as follows: $\forall \omega \in \Omega, \forall \omega' \in \Omega, \omega \lhd_c \omega'$ iff

$$\forall \phi \in B_1^* \sqcup ... \sqcup B_n^*\ , \omega \nvDash \phi$$

$$\exists \psi \in B_1^* \sqcup ... \sqcup B_n^*\ , \omega' \nvDash \psi$$

$$and\ \phi <_c \psi$$

Namely, for each formula falsified by ω, there exists a more important formula ψ (i.e $\psi <_c \phi$), which is falsified by

ω. Interpretations associated with the lowest ranks are the preferred ones.

Finally, we define the result of fusion as follows:

Definition 8: Let E be a multi-set of ranked bases. We say that ω is strictly preferred to ω', denoted by $\omega <_t^{\forall} \omega'$, iff for all compatible total pre-orders $\lhd_c$ we have $\omega \lhd_c \omega'$. The result of fusion, denoted $\lhd_t^{\forall}(E)$, is defined as usual:

$$Mod(\Delta_t^{\forall}(E)) = \{\omega : \ \omega \in Mod(\mu),$$

$$\nexists\omega' \in Mod(\mu) \text{ s.t. } \omega <_t^{\forall} \omega'\}.$$

Example 8 (continued): Let us consider again $B_1 = \{(a, 8), (\neg b, 4)\}$ and $B_2 = \{(b, 2), (\neg a, 1)\}$. Table X presents some compatible total pre-orders possibles and interpretations selected in the result of merging.

Table X
REPRESENTATIVE COMPATIBLE PRE-ORDERS

	$\leq_i$	$\lhd_i$
$\leq_1$	$\neg a <_1 b <_1 \neg b <_1 a$	$\omega_2 \lhd_1 \omega_3 \lhd_1 \omega_0, \omega_1$
$\leq_2$	$\neg a <_2 b, \neg b <_2 a$	$\omega_2, \omega_3 \lhd_2 \omega_0, \omega_1$
$\leq_3$	$\neg a, \neg b <_3 b, a$	$\omega_3 \lhd_3 \omega_0, \omega_1, \omega_2$
$\leq_4$	$\neg a, \neg b <_4 a <_4 b$	$\omega_3 \lhd_4 \omega_1 \lhd_4 \omega_0, \omega_2$
$\leq_5$	$\neg b <_5 \neg a, a <_5 b$	$\omega_1, \omega_3 \lhd_5 \omega_0, \omega_2$
$\leq_6$	$\neg b <_6 a <_6 \neg a <_6 b$	$\omega_1 \lhd_6 \omega_3 \lhd_6 \omega_0, \omega_2$

From Table X we can check that the result of merging is $Mod(\Delta_t^{\forall}) = \{\omega_1, \ \omega_2, \ \omega_3\}$ which is exactly the same as the one obtained using the compatible finite scales or Pareto-like ordering.

The following result shows that considering compatible scales, or a variant of Pareto-ordering, or total pre-orders induced by ranked bases leads to the same result, namely:

Proposition 5: Let E be a multi-set of ranked bases. Then:

$$Mod(\Delta_{\forall}^{Max}(E)) = Mod(\Delta_{\mu}^{Pareto}(E)) = Mod(\Delta_t^{\forall}(E))$$

V. CONCLUSION

This paper has addressed an issue which is not widely considered in belief fusion. It concerns the problem of merging incommensurable ordered belief bases. We have proposed a natural definition of bases on the idea of compatible rankings using finite scales. We have also proposed two equivalent characterizations of the result of merging. the first one defines equivalently the result of merging by ordering interpretations with respect to a variant of Pareto criteria. The second one defines the result of merging by only using total pre-orders induced by ranked belief bases.

APPENDIX

Proof: (proposition 4 : Equivalence between $<_{\forall}^{E}$ and $\lhd_{Pareto}$) Let ω and ω' be two interpretations. There are three cases to considere:

- ω and ω' are both models of $\bigwedge_{B_i \in E}(B_i^*)$. Then ω and ω' are both minimal with respect to $\lhd_{Pareto}$ using proposition 3, and minimal with repect to $<_{\forall}^{E}$ using proposition 2.
- assume that ω is a model of $\bigwedge_{B_i \in E}(B_i^*)$ and ω' is not. Hence, ω is minimal with respect to $\lhd_{Pareto}$ and ω' is not, using proposition 3, and ω is minimal with repect to $<_{\forall}^{E}$ and ω' is not, using proposition 2. In both case, ω is prefered to ω'.
- finally, assume that ω and ω' are not models of $\bigwedge_{B_i \in E}(B_i^*)$.
  First, let us show that $\omega \lhd_{Pareto} \omega' \Rightarrow \omega <_{\forall}^{E} \omega'$.
  By contraposition, suppose that $\omega \not<_{\forall}^{E} \omega'$; then by definition of $<_{\forall}^{E}$, there exists $\mathcal{S} \in \mathbb{S}_E$ such as $\omega \lhd_{Max}^{E^{\mathcal{S}}} \omega'$. Namely, there exists $\mathcal{S} \in \mathbb{S}_E$ such as $Max(\kappa_{B_i^{\mathcal{S}}}(\omega), \ i = 1, ..., n) \geq Max(\kappa_{B_i^{\mathcal{S}}}(\omega'), \ i = 1, ..., n)$.
  Hence, there exists B_i in E such as there exists $\mathcal{S}$ in $\mathbb{S}_E$ where $\kappa_{B_i^{\mathcal{S}}}(\omega) \geq Max(\kappa_{B_i^{\mathcal{S}}}(\omega'), \ i = 1, ..., n)$. Namely, there exists $B_i \in E$ and $\mathcal{S} \in \mathbb{S}_E$ such as $\kappa_{B_i^{\mathcal{S}}}(\omega) \geq \kappa_{B_i^{\mathcal{S}}}(\omega')$, where $\kappa_{B_i^{\mathcal{S}}}(\omega) \neq 0$ (since $Max(\kappa_{B_i^{\mathcal{S}}}(\omega) > 0$ due to the fact that ω is not a model) and $\kappa_{B_i^{\mathcal{S}}}(\omega') \neq 0$ (due to the same reasons). Because $\mathcal{S}$ is assumed to be compatible, there exists B_i in E such as $\kappa_{B_i}(\omega) \geq \kappa_{B_i}(\omega')$. Using definition of $\lhd_{Pareto}$, we obtain that $\nu_E(\omega) \not\lhd_{Pareto} \nu_E(\omega')$.

Let us show now that $\omega \lhd_{Pareto} \omega' \Leftarrow \omega <_{\forall}^{E} \omega'$.

By contraposition, suppose that $\omega \ /\!\lhd_{Pareto} \omega'$. B_i in E such as $\kappa_{B_i}(\omega) \geq \kappa_{B_i}(\omega')$. Hence, it is possible to build a compatible such as $\kappa_{B_i^{\mathcal{S}}}(\omega) = Max(\nu_{B_j^{\mathcal{S}}}(\omega), \ j = 1, ..., n)$ and $\kappa_{B_i^{\mathcal{S}}}(\omega') = Max(\nu_{B_j^{\mathcal{S}}}(\omega'), \ j = 1, ..., n)$. It is enought to have a compatible scale where new ranks of $\phi_j \in B_j \neq B_i$ are in 1,...,$max_{k \neq i}(|B_k|)$ and ranks of B_i are in $max_{k \neq i}(|B_k|)$,...,$max_{k \neq i}(|B_k|)$ + $|B_i|$

Using definition of $\lhd_{Max}^{E^{\mathcal{S}}}$, this compatible is such as $\omega \not\lhd_{Max}^{E^{\mathcal{S}}} \omega'$.

Using definition of $<_{\forall}^{E}$, we finally get $\omega \not<_{\forall}^{E} \omega'$;

■

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