Resolving Inconsistencies by Variable Forgetting

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Abstract

In this paper, a fairly general framework for reasoning from inconsistent propositional belief bases is defined. Variable forgetting is used as a basic operation for weakening beliefs so as to restore consistency. The key notion is that of *recoveries*, which are sets of variables whose forgetting enables restoring consistency. Several criteria for defining preferred recoveries are proposed, depending on whether the focus is laid on the relative relevance of the variables or the relative entrenchment of the pieces of information (or both). Our framework encompasses several previous approaches as specific cases, including reasoning from preferred consistent subsets, and some forms of information merging and belief revision. Interestingly, the gain in flexibility and generality offered by our framework does not imply a complexity shift compared to these specific cases.

1 INTRODUCTION

Nontrivial processing of inconsistent sets of formulas is an important topic in artificial intelligence. Both the complexity of the problem and its significance are reflected by the number of approaches to paraconsistent reasoning that can be found in the literature under various names, like paraconsistent logics, belief revision, argumentative inference, information merging, model fitting, arbitration, knowledge integration, knowledge purification, etc.

Corresponding to these approaches are many different mechanisms to avoid trivialization. For instance, inconsistency can be removed by identifying wrong pieces of information through knowledge-gathering actions (see *e.g.* [Lang and Marquis, 2000] [Liberatore and Donini, 2000]). Inconsistency can also be dealt with. In this case, trivialization is avoided by weakening the set of consequences that can be derived from the belief base; this can be achieved either by considering an *approximation by below of classical entailment* (like in many paraconsistent logics, see [Besnard and Hunter, 1998] [Hunter, 1998] for a survey) or by *weakening the input belief base while keeping classical entailment*. This last technique is at work in so-called coherence-based approaches to paraconsistent inference (see *e.g.* [Rescher and Manor, 1970] [Fagin *et al.*, 1983] [Ginsberg, 1986] [Brewka, 1989], [Baral *et al.*, 1991] [Pinkas and Loui, 1992] [Benferhat *et al.*, 1993]), as well as in belief merging (see *e.g.* [Liberatore and Schaerf, 1998] [Revesz, 1997] [Konieczny and Pino Pérez, 1998] [Lin and Mendelzon, 1999]).

While some of these approaches take account for the relative importance of pieces of information, they do not handle the relative importance of *variables* in the problem at hand. This is problematic in many situations where some variables are less central than others, especially when some variables are meaningful only in the presence of others. Thus, it makes no sense to reason about whether John's car is grey if there is some strong conflict about whether John actually *has* a car.

In order to address this issue, we define in the following a new framework for reasoning from inconsistent propositional belief bases, using *variable forgetting* as a basic operation for weakening beliefs. Belief bases are viewed as (finite) vectors of formulas, conjunctively interpreted. W.l.o.g., each formula is assumed to be issued from a specific source of information. The key notion of our approach is that of *recoveries*, which are sets of variables whose forgetting enables restoring consistency. Several criteria for defining preferred recoveries are proposed, depending on whether the focus is laid on the relative relevance of the variables or the relative entrenchment of the pieces of information (or both). Our framework offers several advantages compared with many existing approaches to paraconsistent reasoning. First of all, it is quite general and flexible. Especially, it enables to model situations where some sources of information are considered more reliable than others in an absolute way, but also relatively to some topics of interest. Equity between some sources of information can also be achieved by imposing to forget the same variables in the sources. Accordingly, our framework encompasses several previous approaches as specific cases, including reasoning from preferred consistent subsets, and forms of information merging and belief revision. Interestingly, the gain in generality and flexibility offered by our framework does not imply a complexity shift compared with these specific cases.

The rest of this paper is organized as follows. Formal preliminaries including a presentation of forgetting are given in Section 2. Our framework is presented in Section 3. Computational issues are discussed in Section 4. Section 5 concludes the paper. Proof sketches can be found in an appendix.

2 FORMAL PRELIMINARIES

2.1 PROPOSITIONAL LOGIC

$PROP_{PS}$ denotes the propositional language built up from a finite set PS of symbols, the Boolean constants $\top$ (true) and $\bot$ (false), and the usual connectives. $Var(\phi)$ denote the set of propositional variables occurring in the formula ϕ. $\phi_{x \leftarrow 0}$ (resp. $\phi_{x \leftarrow 1}$) denotes the formula obtained by replacing in ϕ every occurrence of variable x by $\bot$ (resp. $\top$).

An interpretation ω is an assignment of a truth value to each variable of PS. $\Omega = 2^{PS}$ is the set of all interpretations. Formulas are interpreted in the classical way. Every finite set of formulas is interpreted conjunctively. $Mod(\phi)$ denotes the set of models of ϕ. Finally, ω and ω' being two interpretations, $Diff(\omega, \omega')$ is the set of propositional variables assigned different truth values by ω and ω'.

2.2 FORGETTING

Our approach to restore consistency is based on *variable forgetting*, also known as projection or marginalization. Variable forgetting can be defined as follows (see [Lin and Reiter, 1994] [Lang and Marquis, 1998] for more details):

Definition 1 (forgetting) *Let ϕ be a formula from $PROP_{PS}$ and $V \subseteq PS$. The* forgetting *of V in ϕ, noted $\exists V.\phi$, is a quantified boolean formula over PS, equivalent to a formula from $PROP_{PS}$ that can be inductively defined as follows:*

- $\exists \emptyset.\phi \equiv \phi$;
- $\exists \{x\}.\phi \equiv \phi_{x \leftarrow 0} \vee \phi_{x \leftarrow 1}$;
- $\exists (\{x\} \cup V).\phi \equiv \exists V.(\exists \{x\}.\phi)$.

For example, $\exists \{a\}.\neg a \wedge b \equiv b$ and $\exists \{a\}.(a \vee b) \equiv \top$.

$\exists V.\phi$ represents the logically strongest consequence ψ of ϕ that is independent of V, which means that there exists a formula ψ' from $PROP_{PS}$ s.t. $\psi \equiv \psi'$ and $Var(\psi') \cap V = \emptyset$. Accordingly, forgetting a set of variables within a formula leads to *weaken* it. To be more precise, if $V \subseteq W$ holds, then $\exists V.\phi \models \exists W.\phi$ holds. Moreover, ϕ is consistent if and only if $\exists Var(\phi).\phi$ is valid (see [Lang and Marquis, 1998]).

Many characterizations of variable forgetting, together with complexity results, are reported in [Lang *et al.*, 2001]. For instance, for every $V \subseteq PS$ and every formula ϕ, we clearly have $\exists V.\phi \equiv \exists V_\phi.\phi$, where $V_\phi = V \cap Var(\phi)$.

3 REASONING FROM PREFERRED RECOVERIES

3.1 BELIEF BASES AND RECOVERIES

Let us first make precise the belief bases we are interested in.

Definition 2 (belief base) *A* belief base *B is a vector $\langle \phi_1, \ldots, \phi_n \rangle$ of n formulas from $PROP_{PS}$, where n is a positive integer.*

$B = \langle \phi_1, \ldots, \phi_n \rangle$ is conjunctively interpreted, so that it is said to be *inconsistent* if and only if $\{\phi_1, \ldots, \phi_n\}$ is inconsistent. Each i ($1 \leq i \leq n$) identifies a source of information and ϕ_i denotes the piece of belief conveyed by source i. Note that it can be the case that a formula occurs more than once in B, which can be used to model the situation where two different sources (or more) give the same information.

The key notions of our approach are *forgetting vectors* and *recoveries*. A forgetting vector consists of a collection of sets of variables to be forgotten in each formula from the belief base. These sets of variables need not be identical, but they should obey some constraints bearing on the forgetting process:

Definition 3 (forgetting context) *A* forgetting context *$\mathcal{C}$ for a belief base $B = \langle \phi_1, \ldots, \phi_n \rangle$ is a triple $\langle F, G, H \rangle$ where:*

- $F = \langle F_i \rangle_{i=1\dots n}$ *where, for each* i, $F_i \subseteq PS$ *is the set of variables that can be forgotten in* ϕ_i *(*i.e.*, variables in* $PS \setminus F_i$ *must not be forgotten in* ϕ_i*);*

- $G = \langle G_i \rangle_{i=1\dots n}$ *where, for each* i, G_i *is a directed graph on* F_i*; an arc* (v, v') *in* G_i *means that* v' *is meaningful only in the presence of* v, i.e., *that if* v *is forgotten in* ϕ_i *then* v' *must be forgotten as well in* ϕ_i*;*

- $H \subseteq PS \times \{1 \dots n\} \times \{1 \dots n\}$ *is a set of triples* (v, i, j) *where* $v \in F_i \cap F_j$ *and* $i \neq j$*, meaning that if* v *is forgotten in* ϕ_i*, then it must be forgotten as well in* ϕ_j*.*

F, G and H impose some constraints over the way the pieces of belief ϕ_i from B can be weakened. Thus, if $x \notin F_i$, forgetting x in ϕ_i is forbidden. This is helpful to model the situation where source i is fully reliable to what concerns x, *i.e.*, what i says about x must be taken for sure. G_i enables to express that variables can be significant only in presence of others so forgetting the latter imposes to forget the former. Finally, H can be used to force some sources of information to be considered on equal terms w.r.t. weakening. For instance, if ϕ_i and ϕ_j are together inconsistent and consistency can be recovered by forgetting x in ϕ_i, then it could be expected that x should also be forgotten in ϕ_j.

Example 1 *(inspired from [Konieczny and Pino Pérez, 1998]). As a matter of illustration, let us consider the following preference merging scenario. Suppose that a group of four co-owners of a residence tries to agree about building a new tennis court* t *and/or a new swimming pool* s*. If it is constructed, the swimming pool can be either red (*s_r*) or blue (*s_b*). If both a tennis court and a pool – respectively, one of them, none of them – are (is) constructed, the induced cost is 2 money units (*c_2*) – respectively, 1 unit (*c_1*), nothing* c_0*). The first co-owner would not like to spend more than 1 unit, and prefers a red swimming pool, should it be constructed. The second co-owner would like to improve the quality of the residence by the construction of at least a tennis court or a swimming pool and would prefer a blue swimming pool, should it be constructed. The third co-owner just prefers a swimming pool to be built, whatever its colour. The fourth co-owner would like both a swimming pool (of whatever colour) and a tennis court to be built.*

Of course, if there is no agreement about whether the swimming pool is to be constructed, any preference concerning its colour must be ignored. Furthermore, it is meaningless forgetting about whether the pool should be blue without forgetting whether it should be red: either we forget about the colour of the swimming pool or not. Similarly, either we forget about the expenses, i.e., we forget $\{c_0, c_1, c_2\}$ *or not (i.e., we forget none of them).*

Clearly enough, the preferences of the group are jointly inconsistent. This scenario can be encoded in our framework using the following belief base $B = \langle \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \rangle$ *and forgetting context* $\mathcal{C}$*:*

- $\begin{array}{ll} \phi_1 = & (s \Rightarrow (s_r \oplus s_b)) \wedge (\neg s \Rightarrow \neg s_r \wedge \neg s_b) \\ & \wedge (c_2 \Leftrightarrow (s \wedge t)) \wedge (c_1 \Leftrightarrow (s \oplus t)) \\ & \wedge (c_0 \Leftrightarrow (\neg s \wedge \neg t)) \end{array}$;
- $\phi_2 = (c_0 \vee c_1) \wedge (s \Rightarrow s_r)$;
- $\phi_3 = (s \vee t) \wedge (s \Rightarrow s_b)$;
- $\phi_4 = s$;
- $\phi_5 = s \wedge t$;
- $F_1 = \emptyset$, $F_2 = F_3 = F_4 = F_5 = \{s, t, s_r, s_b, c_0, c_1, c_2\}$;
- $G_1 = \emptyset$, $G_2 = G_3 = G_4 = G_5 = \{(s, s_r), (s, s_b), (s_r, s_b), (s_b, s_r), (c_0, c_1), (c_0, c_2), (c_1, c_0), (c_1, c_2), (c_2, c_0), (c_2, c_1)\}$;
- $H = \emptyset$.

Given that $F_1 = \emptyset$, ϕ_1 *expresses the integrity constraint according to which the swimming pool must be either red or blue if it is constructed, as well as the logical definition of the induced price from the number of equipments built.* ϕ_2 *(resp.* ϕ_3, ϕ_4, ϕ_5*) encodes the preferences of the co-owners.* G_2 $(= G_3 = G_4 = G_5)$ *ensures that* s_r *and* s_b *are irrelevant if there is no agreement on* s*, that* s_r *is irrelevant if and only if* s_b *is, and that for any* $i, j \in \{0, 1, 2\}$, c_i *is irrelevant if and only if* c_j *is. In the situation where all co-owners must be considered on equal terms w.r.t. the set of variables to be forgotten,* H *must be changed to* $(\{s_r, s_b\} \times \{2,3,4,5\} \times \{2,3,4,5\}) \cup (\{c_0, c_1, c_2\} \times \{2,3,4,5\} \times \{2,3,4,5\})$.

As expected, our definition is very general and many different forgetting contexts can be considered. In some situations, the variables forgotten in each of the pieces of information must be identical so that all sources of information are considered on equal terms. This can be captured by considering *homogeneous* contexts, where a forgetting context $\mathcal{C} = \langle F, G, H \rangle$ for $B = \langle \phi_1, \dots, \phi_n \rangle$ is said to be homogeneous if and only if for every i, $j \in 1 \dots n$, we have $F_i = F_j$ and $G_i = G_j$, and $H = PS \times \{1 \dots n\} \times \{1 \dots n\}$. A simple case when $\mathcal{C}$ should not be homogeneous is when some of the pieces of information must be left intact: for such a ϕ_i, we set $F_i = \emptyset$. This is useful to encode integrity constraints (or revision formulas), *i.e.*, formulas that are required to be true (just like ϕ_1 in the example above). Another non-homogeneous forgetting context for $B = \langle \phi_1, \ldots, \phi_n \rangle$ is its *standard* forgetting context, noted $\mathcal{C_S}$ and given by $F_i = PS$, $G_i = \emptyset$ for every $i \in 1 \ldots n$, and $H = \emptyset$. In this context, every variable can be forgotten in any piece of belief, and variables can always be forgotten in an independent way.

We can now define our notion of recovery:

Definition 4 (recovery) *Let $B = \langle \phi_1, \ldots, \phi_n \rangle$ be a belief base and $\mathcal{C}$ be a forgetting context for B. A* recovery vector *(or recovery for short) $\vec{V}$ for B given $\mathcal{C}$ is a vector $\vec{V} = \langle V_1, \ldots, V_n \rangle$ of subsets of $Var(B)$*[1] *s.t.:*

- *for every $i \in 1 \ldots n$, $V_i \subseteq F_i$, and*
- *for every $i \in 1 \ldots n$, for any $(v, v') \in G_i$, $v \in V_i$ implies $v' \in V_i$, and*
- *for every $(v, i, j) \in H$, $v \in V_i$ implies $v \in V_j$, and finally*
- *the set $B \mid \vec{V} = \{\exists V_i.\phi_i \mid i \in 1 \ldots n\}$ is consistent.*

$B \mid \vec{V}$ is called the projection *of recovery $\vec{V}$ on B given $\mathcal{C}$. $\mathcal{R_C}(B)$ denotes the set of all recoveries for B given $\mathcal{C}$. B is said to be* recoverable *w.r.t. $\mathcal{C}$ if and only if $\mathcal{R_C}(B) \neq \emptyset$.*

Example 1 (continued) *Here is a list of recoveries for B given $\mathcal{C}$. For each recovery $\vec{V}$ and every $i \in 1 \ldots 5$ we compute $\exists V_i.\phi_i$, and finally $B|\vec{V}$.*

All the vectors $\vec{V}^1$ to $\vec{V}^9$ considered on the table 1 are recoveries for B given $\mathcal{C}$, while the following ones are not *recoveries for B given $\mathcal{C}$:*

- *$\langle \emptyset, \{t\}, \{t\}, \{t\}, \{t\} \rangle$, because $\exists V_1.\phi_1 \land \exists V_2.\phi_2 \land \exists V_3.\phi_3 \land \exists V_4.\phi_4 \land \exists V_5.\phi_5$ is inconsistent;*
- *$\langle \emptyset, \{s\}, \{s\}, \{s\}, \{s\} \rangle$, because it does not satisfy the constraints of $\mathcal{C}$.*

By construction, replacing the beliefs of B by the projection on B of any of its recoveries is sufficient to restore consistency, provided that B is recoverable. In the general case, it may happen that $\mathcal{R_C}(B)$ is empty, because some variables must not be forgotten. Another important sufficient condition for a belief base not to be recoverable is when it incorporates some inconsistent ϕ_i. In some situations, this can reflect the fact that the i^{th} source of information is fully unreliable, so it should not be taken into account at all (hence the corresponding ϕ_i should be removed from B).

[1]We note $Var(\langle \phi_1, \ldots, \phi_n \rangle) = \bigcup_{i=1}^{n} Var(\phi_i)$.

3.2 PREFERRED RECOVERIES

Generally, many recoveries for a belief base given a forgetting context are possible, but most of the time, some of the corresponding projections are more expected than others. Discriminating them calls for a notion of *preferred recovery*. Many preference criteria can be considered so as to capture in formal terms several intuitions about the way the pieces of information should be merged.

For instance, we may prefer the recoveries $\vec{V} = \langle V_1, ..., V_n \rangle$ for $B = \langle \phi_1, \ldots, \phi_n \rangle$ in which the set V_i of variables forgotten in some of the ϕ_i's are as close to each other as possible[2], and eventually coincide (which always is the case whenever homogeneous contexts are considered). We may also prefer recoveries that lead to forget minimal sets of variables, where minimality may be defined using a preference criterion induced by some priorities or penalties on the set of variables. Finally, we may prefer recoveries that preserve as much as preferred beliefs as possible (like in the coherence-based approaches to paraconsistent reasoning).

In order to capture these various preferences in formal terms, $\mathcal{R_C}(B)$ can be equipped with a preordering (*i.e.*, a reflexive and transitive relation) $\sqsubseteq$. For every vector $\vec{V}$, $\vec{V'}$ of length n, $\vec{V} \subseteq_p \vec{V'}$ means that $V_i \subseteq V'_i$ for each $i \in 1 \ldots n$.

Definition 5 (preference relation) *Given a belief base $B = \langle \phi_1, \ldots, \phi_n \rangle$ and a forgetting context $\mathcal{C}$ for it, a* preference relation *is a reflexive and transitive relation $\sqsubseteq$ on $\mathcal{R_C}(B)$ satisfying the monotonicity property:*

$$\forall \vec{V}, \vec{V'} \in \mathcal{R_C}(B), \text{ if } \vec{V} \subseteq_p \vec{V'} \text{ then } \vec{V} \sqsubseteq \vec{V'}.$$

We note $\vec{V} \sqsubset \vec{V'}$ for $\vec{V} \sqsubseteq \vec{V'}$ and $\vec{V'} \not\sqsubseteq \vec{V}$, and $\vec{V} \sim \vec{V'}$ for $\vec{V} \sqsubseteq \vec{V'}$ and $\vec{V'} \sqsubseteq \vec{V}$.

It is easy to prove that for any $\vec{V}$, $\vec{V'} \in \mathcal{R_C}(B)$, if $\vec{V} \subseteq_p \vec{V'}$ then $B \mid \vec{V} \models B \mid \vec{V'}$ holds. Accordingly, among the recoveries from $\mathcal{R_C}(B)$, the minimal ones w.r.t. $\sqsubseteq = \subseteq_p$ lead to projections that preserve as much information as possible given $\mathcal{C}$. This is helpful to represent some inertia principle in our framework.

[2]For instance, if both $\vec{V} = \langle \{a\}, \{b, c\} \rangle$ and $\vec{V'} = \langle \{a\}, \{a, c\} \rangle$ are recoveries for B, $\vec{V'}$ should (in some contexts) be preferred because it is more uniform.

Table 1:

recovery	$\exists V_1.\phi_1$	$\exists V_2.\phi_2$	$\exists V_3.\phi_3$	$\exists V_4.\phi_4$	$\exists V_5.\phi_5$	$B \mid \vec{V}$
$\vec{V}^1 = \langle \emptyset, \{s, s_b, s_r\}, \{s, s_b, s_r\}, \{s, s_b, s_r\}, \{s, s_b, s_r\}\rangle$	ϕ_1	$c_0 \vee c_1$	$\top$	$\top$	t	$\phi_1 \wedge \neg s \wedge t$
$\vec{V}^2 = \langle \emptyset, \{t, s_b, s_r\}, \{t, s_b, s_r\}, \{t, s_b, s_r\}, \{t, s_b, s_r\}\rangle$	ϕ_1	$c_0 \vee c_1$	$\top$	s	s	$\phi_1 \wedge s \wedge \neg t$
$\vec{V}^3 = \langle \emptyset, \{c_0, c_1, c_2, s_b, s_r\}, \{c_0, c_1, c_2, s_b, s_r\}, \{c_0, c_1, c_2, s_b, s_r\}, \{c_0, c_1, c_2, s_b, s_r\}\rangle$	ϕ_1	$\top$	$s \vee t$	s	$s \wedge t$	$\phi_1 \wedge s \wedge t$
$\vec{V}^4 = \langle \emptyset, \{s_b, s_r\}, \{s_b, s_r\}, \emptyset, \{t\}\rangle$	ϕ_1	$c_0 \vee c_1$	$s \vee t$	s	s	$\phi_1 \wedge s \wedge \neg t$
$\vec{V}^5 = \langle \emptyset, \{c_0, c_1, c_2\}, \{s_b, s_r\}, \emptyset, \emptyset\rangle$	ϕ_1	$s \Rightarrow s_r$	$s \vee t$	s	$s \wedge t$	$\phi_1 \wedge s \wedge t \wedge s_r$
$\vec{V}^6 = \langle \emptyset, \{s_b, s_r\}, \emptyset, \emptyset, \{t\}\rangle$	ϕ_1	$c_0 \vee c_1$	$(s \vee t) \wedge (s \Rightarrow s_b)$	s	s	$\phi_1 \wedge s \wedge \neg t \wedge s_b$
$\vec{V}^7 = \langle \emptyset, \emptyset, \emptyset, \{s, s_b, s_r\}, \{s, s_b, s_r\}\rangle$	ϕ_1	$(c_0 \vee c_1) \wedge (s \Rightarrow s_r)$	$(s \vee t) \wedge (s \Rightarrow s_b)$	$\top$	t	$\phi_1 \wedge \neg s \wedge t$
$\vec{V}^8 = \langle \emptyset, \{c_0, c_1, c_2, s_b, s_r\}, \emptyset, \emptyset, \emptyset\rangle$	ϕ_1	$\top$	$(s \vee t) \wedge (s \Rightarrow s_b)$	s	$s \wedge t$	$\phi_1 \wedge s \wedge t \wedge s_b$
$\vec{V}^9 = \langle \emptyset, \emptyset, \{s_b, s_r\}, \emptyset, \{t\}\rangle$	ϕ_1	$(c_0 \vee c_1) \wedge (s \Rightarrow s_r)$	$s \vee t$	s	s	$\phi_1 \wedge s \wedge \neg t \wedge s_r$

Example 1 (cont'd) For instance, we have $\vec{V}^6 \subseteq_p \vec{V}^4$. Accordingly, $B \mid \vec{V}^6$ is at least as logically strong as $B \mid \vec{V}^4$.

Depending on the problem at hand, many other properties on $\sqsubseteq$ can be imposed so as to capture various intuitions about the result of the merging process.

Thus, given $B = \langle \phi_1, \ldots, \phi_n \rangle$, a preference relation $\sqsubseteq$ is said to satisfy

- the *binaricity* property if and only if $\forall \vec{V}, \vec{V'} \in \mathcal{R}_{\mathcal{C}}(B)$, if (for every $i \in 1 \ldots n$, $(V_i \neq \emptyset \Leftrightarrow V'_i \neq \emptyset)$), then $\vec{V} \sim \vec{V'}$.
- the *gathering* property if and only if $\forall \vec{V} \in \mathcal{R}_{\mathcal{C}}(B)$, $\vec{V} \sim \langle \bigcup \vec{V}, \ldots, \bigcup \vec{V} \rangle$, where $\bigcup \langle V_1, \ldots, V_n \rangle = V_1 \cup \ldots \cup V_n$ (obviously, it is required that $\langle \bigcup \vec{V}, \ldots, \bigcup \vec{V} \rangle \in \mathcal{R}_{\mathcal{C}}(B)$ for this notion to make sense).

In many cases, it is desirable to assume that $\sqsubseteq$ is a *complete* preordering. In this situation, we can equivalently represent $\sqsubseteq$ by a *ranking function* μ from $\mathcal{R}_{\mathcal{C}}(B)$ to $I\!N$ such that $\mu(\langle \emptyset, \emptyset, ..., \emptyset \rangle) = 0$, and $\forall \vec{V}, \vec{V'} \in \mathcal{R}_{\mathcal{C}}(B)$, if $\vec{V} \subseteq_p \vec{V'}$, then $\mu(\vec{V}) \leq \mu(\vec{V'})$. The preference relation $\sqsubseteq_\mu$ induced by μ is the total preordering defined by $\forall \vec{V} \in \mathcal{R}_{\mathcal{C}}(B)$, $\vec{V} \sqsubseteq_\mu \vec{V'}$ if and only if $\mu(\vec{V}) \leq \mu(\vec{V'})$. We say that a ranking function μ is *anonymously decomposable* if and only if there exists a total function $\mu_0 : 2^{PS} \rightarrow I\!N$ and a non-decreasing, symmetric, total function $H : I\!N^n \rightarrow I\!N$ s.t. $\mu_0(\emptyset) = 0$ and $\forall X, Y \subseteq PS$, if $X \subseteq Y$, then $\mu_0(X) \leq \mu_0(Y)$, and $H(0, ..., 0) = 0$, and $\mu(\vec{V}) = H(\mu_0(V_1), ..., \mu_0(V_n))$.

We are now in position to define the family of inference relations $\approx^{\mathcal{C}}_{\sqsubseteq}$ that can be defined in our framework by letting $\mathcal{C}$ and $\sqsubseteq$ to vary. We call them *forget* inference relations.

Definition 6 (skeptical forget inference) *Given a belief base B and a forgetting context $\mathcal{C}$ for it, let $\sqsubseteq$ be a preference relation on $\mathcal{R}_{\mathcal{C}}(B)$ (possibly induced by a ranking function μ).*

- *A recovery $\vec{V} \in \mathcal{R}_{\mathcal{C}}(B)$ is said to be* preferred *(w.r.t. $\sqsubseteq$) if and only if it is minimal in its set w.r.t. $\sqsubseteq$, i.e., there is no $\vec{V'} \in \mathcal{R}_{\mathcal{C}}(B)$ s.t. $\vec{V'} \sqsubset \vec{V}$.*
- *Let ψ be any formula from $PROP_{PS}$. ψ is said to be (skeptically) inferred from B w.r.t. $\sqsubseteq$, denoted by $B \approx^{\mathcal{C}}_{\sqsubseteq} \psi$, if and only if for any preferred recovery $\vec{V}$ (w.r.t. $\sqsubseteq$), we have $B \mid \vec{V} \models \psi$.*

Since skeptical inference is considered, among the preferred recoveries for B, only the maximal ones w.r.t. $\subseteq_p$ are relevant for inference, in the sense that the other ones can be ignored without modifying the inference relation $\approx^{\mathcal{C}}_{\sqsubseteq}$. Indeed, such maximal elements correspond to the logically weakest projections.

Definition 7 (prudent forget inference) *Given* *a belief base B and a forgetting context $\mathcal{C}$ for it, let $\sqsubseteq$ be a preference relation on $\mathcal{R}_{\mathcal{C}}(B)$ (possibly induced by a ranking function μ). Let ψ be any formula from $PROP_{PS}$. ψ is a* prudent *consequence of B w.r.t. $\sqsubseteq$, denoted $B \mid\!\approx_{\sqsubseteq}^{\mathcal{C}_{prudent}} \psi$, if and only if $B \mid \vec{V}_{prudent}^{\sqsubseteq} \models \psi$, where $\vec{V}_{prudent}^{\sqsubseteq}$ is a recovery for B, defined as the the pointwise union $\langle \bigcup_{\vec{V} \in min(\mathcal{R}_{\mathcal{C}}(B), \sqsubseteq)} V_1, \ldots, \bigcup_{\vec{V} \in min(\mathcal{R}_{\mathcal{C}}(B), \sqsubseteq)} V_n \rangle$ of all preferred recoveries for B w.r.t. $\sqsubseteq$ (especially, the relation does not hold whenever $\vec{V}_{prudent}^{\sqsubseteq}$ does not satisfy the constraints imposed by $\mathcal{C}$).*

Example 1 (cont'd) *If $\sqsubseteq \; = \; \subseteq_p$ is chosen as a preference relation, we have $min(\mathcal{R}_{\mathcal{C}}(B), \sqsubseteq) = \{\vec{V}^5, \vec{V}^6, \vec{V}^7, \vec{V}^8, \vec{V}^9\}$. Accordingly, $\vec{V}_{prudent}^{\sqsubseteq} = \langle \emptyset, \{c_0, c_1, c_2, s_b, s_r\}, \{s_b, s_r\}, \{s, s_b, s_r\}, \{s, s_b, s_r, t\} \rangle$. We obtain $B \mid\!\approx_{\sqsubseteq}^{\mathcal{C}} \phi_1 \wedge (s \vee t)$ and $B \mid\!\approx_{\sqsubseteq}^{\mathcal{C}_{prudent}} \phi_1 \wedge (s \vee t)$.*

3.3 ON THE GENERALITY OF OUR FRAMEWORK

We now show how several well-known paraconsistent inference relations belong to the family of forget relations.

3.3.1 Reasoning from Preferred Consistent Subsets

Let us first recall the definition of inference drawn from preferred consistent subsets, also called *maxcons* inference relation (see *e.g.* [Rescher and Manor, 1970] [Fagin *et al.*, 1983] [Ginsberg, 1986] [Brewka, 1989] [Baral *et al.*, 1991] [Pinkas and Loui, 1992] [Benferhat *et al.*, 1993]). Let $B = \{\phi_1, \ldots, \phi_n\}$ be a finite set of formulas s.t. each ϕ_i $(1 \leq i \leq n)$ is consistent, and $\succeq$ a preference pre-ordering on subsets of B s.t. $\forall X$, $Y \subseteq B$, if $X \supseteq Y$, then $X \succeq Y$[3]. Then $B \mid\!\approx_{\succeq}^{\forall} \psi$ if and only if for any preferred (w.r.t. $\succeq$) consistent subset X of B, we have $X \models \psi$.

Proposition 1 *$\mid\!\approx_{\succeq}^{\forall}$ is a maxcons inference relation if and only if there exists a preference relation $\sqsubseteq$ on $\mathcal{R}_{\mathcal{C}_S}(B)$ satisfying binaricity and s.t. for any ψ, $\{\phi_1, \ldots, \phi_n\} \mid\!\approx_{\succeq}^{\forall} \psi$ if and only if $\langle \phi_1, \ldots, \phi_n \rangle \mid\!\approx_{\sqsubseteq}^{\mathcal{C}_S} \psi$.*

The first direction of Proposition 1 expresses that under the assumption of binaricity and standard forgetting context, inference from preferred recoveries comes down to inference from preferred maximal consistent subsets. The other direction of Proposition 1 states that the set of all maxcons inference relations is contained in the set of forget inference relations, and therefore that the latter family is at least as general as the former. In particular, if preference over recoveries is defined by $\forall \vec{V}$, $\vec{V'} \in \mathcal{R}_{\mathcal{C}_S}(B)$, $\vec{V} \sqsubseteq \vec{V'}$ if and only if $\{i \mid V_i \neq \emptyset\} \subseteq \{i \mid V'_i \neq \emptyset\}$ then $B \mid\!\approx_{\sqsubseteq}^{\mathcal{C}_S} \psi$ if and only if $S \models \psi$ holds for any maximal (w.r.t. set inclusion) consistent subset S of B. Other criteria such that maximum cardinality, "discrimin" or "leximin", or minimum penalty, can be recovered as well. The assumption that each ϕ_i $(1 \leq i \leq n)$ in B is consistent is necessary (and sufficient) to ensure that B is recoverable given its standard forgetting context. It can be made w.l.o.g., just because B and $B \setminus \{\phi_i \mid 1 \leq i \leq n$ and ϕ_i is inconsistent$\}$ have the same consistent subsets. As an immediate consequence of Proposition 1, our framework encompasses other important frameworks, like supernormal default theories with priorities [Brewka, 1989] and syntax-based belief revision [Nebel, 1992], as specific cases.

3.3.2 Belief Merging

A *merging operator* (see *e.g.* [Liberatore and Schaerf, 1998] [Revesz, 1997] [Konieczny and Pino Pérez, 1998] [Lin and Mendelzon, 1999]) maps any finite collection (multiset) of consistent propositional formulas $B = \{\phi_1, \ldots, \phi_n\}$, with n being unfixed, into a new consistent propositional formula $Merge(\phi_1, \ldots, \phi_n)$. When n is fixed, we define a n-merging operator $Merge_n(\phi_1, \ldots, \phi_n)$ as the restriction of $Merge$ to n-uples of formulas. In order to simplify notations, we will write $Merge(\phi_1, \ldots, \phi_n)$ instead of $Merge_n(\phi_1, \ldots, \phi_n)$ in the following.

A special class of n-merging operators is the class of *decomposable merging operators induced by (pseudo-)distances*. Let $d : \Omega \times \Omega \rightarrow I\!N$ be a total function satisfying $\forall \omega$, $\omega' \in \Omega$, $d(\omega, \omega') = 0$ if and only if $\omega = \omega'$, and $d(\omega, \omega') = d(\omega', \omega)$. Let $*$ be a total function from $I\!N^n$ to $I\!N$, symmetric and monotonic in each of its arguments. For every $B = \{\phi_1, \ldots, \phi_n\}$ s.t. each ϕ_i $(1 \leq i \leq n)$ is consistent, the *decomposable n-merging operator* $Merge_d^*$ induced by d and $*$ is defined by $Merge_d^*(B) = \{\omega \in \Omega \mid d(\omega, B)$ is minimal$\}$, where $d(\omega, B) = *(d(\omega, \phi_1), \ldots, d(\omega, \phi_n))$ when $B = \{\phi_1, \ldots, \phi_n\}$ and $d(\omega, \phi_i) = \min_{\omega' \models \phi_i} d(\omega, \omega')$. The n-merging inference relation $\mid\!\approx_{d,*}^{M}$ induced by d and $*$ is defined by $B \mid\!\approx_{d,*}^{M} \psi$ if and only if $Merge_d^*(\{\phi_1, \ldots, \phi_n\}) \subseteq Mod(\psi)$.

In order to characterize the decomposable n-merging inference relations as forget relations, we need to focus on the *differential* ones, *i.e.*, those based on a *differential* distance, where a distance d is said to be

[3] Among usual preference relations, we find set inclusion: $X \succeq Y$ if and only if $X \supseteq Y$, and cardinality: $X \succeq Y$ if and only if $|X| \geq |Y|$.

differential if and only if there exists a total function $f : 2^{PS} \rightarrow I\!N$ satisfying (i) $f(\emptyset) = 0$ and (ii) $\forall A, B \subseteq PS$, if $A \subseteq B$, then $f(A) \leq f(B)$, and s.t. $\forall \omega, \omega' \in \Omega$, $d(\omega, \omega') = f(Diff(\omega, \omega'))$.

Proposition 2 *$\{\phi_1, \ldots, \phi_n\} \approx^M_{d,*}$ is a decomposable and differential n-merging inference relation if and only if there exists a complete preference relation $\sqsubseteq_\mu$ on $\mathcal{R}_{\mathcal{C}_S}(B)$, induced by a ranking measure μ satisfying anonymous decomposability, such that for any ψ, $\{\phi_1, \ldots, \phi_n\} \approx^M_{d,*} \psi$ if and only if $\langle \phi_1, \ldots, \phi_n \rangle \approx^{\mathcal{C}_S}_{\sqsubseteq_\mu} \psi$*

The first direction of Proposition 2 expresses that under the assumptions of completeness of $\sqsubseteq_\mu$, anonymous decomposability, and standard forgetting context, forget inference comes down to inference from a merging operator. The other direction expresses that a particularly interesting subclass of inferences from merging operators is contained in the set of forget inference relations. In particular, usual arbitration and majority merging operators (see [Konieczny and Pino Pérez, 1998]) are recovered by letting (1) $\mu_i(A) = |A|$ for each $i \in 1 \ldots n$, which implies that the induced distance d is the Hamming distance between interpretations, and (2) $* = max$ or $* = +$, respectively.

3.3.3 Belief Revision

Our framework also includes some approaches to *model-based*[4] belief revision, including Satoh's $\circ_S$ [Satoh, 1988] (or equivalently, the operator given in [Delgrande and Schaub, 2000]), Dalal's $\circ_D$ [Dalal, 1988] and Weber's $\circ_W$ [Weber, 1986], provided that the belief set under consideration is not $PROP_{PS}$ (*i.e.*, it can be encoded by a consistent formula ϕ).

Given two formulas ϕ and ψ, we have:
- $Diff^{min}(\psi, \phi) = min(\{Diff(\omega, \omega') \mid \omega \in Mod(\psi), \omega' \in Mod(\phi)\}, \subseteq)$;
- $||Diff^{min}(\psi, \phi)|| = min(\{|Diff(\omega, \omega')| \mid \omega \in Mod(\psi), \omega' \in Mod(\phi)\}, \leq)$;
- $Mod(\phi \circ_S \psi) = \{\omega \in Mod(\psi) \mid \exists \omega' \in Mod(\phi)$ s.t. $Diff(\omega, \omega') \in Diff^{min}(\psi, \phi)\}$;
- $Mod(\phi \circ_D \psi) = \{\omega \in Mod(\psi) \mid \exists \omega' \in Mod(\phi)$ s.t. $|Diff(\omega, \omega')| = ||Diff^{min}(\psi, \phi)||\}$.
- $Mod(\phi \circ_W \psi) = \{\omega \in Mod(\psi) \mid \exists \omega' \in Mod(\phi)$ s.t. ω and ω' coincides except possibly on variables from $\bigcup Diff^{min}(\psi, \phi)\}$.

The inference relations induced by $\circ_S$, $\circ_D$ and $\circ_W$ are forget ones since:

[4] As evoked previously, it incorporates as well the usual syntax-based ones.

Proposition 3 *Let $\mathcal{C}$ be the forgetting context for $\langle \psi, \phi \rangle$ s.t. $F_1 = G_1 = G_2 = H = \emptyset$ and $F_2 = PS$. Let $\sqsubseteq_S$ (resp. $\sqsubseteq_D$) be a preference relation on $\mathcal{R}_\mathcal{C}(B)$ s.t. $\forall \vec{V}, \vec{V'} \in \mathcal{R}_\mathcal{C}(B)$, $\vec{V} \sqsubseteq_S \vec{V'}$ if and only if $V_2 \subseteq V'_2$ (resp. $\vec{V} \sqsubseteq_D \vec{V'}$ if and only if $|V_2| \leq |V'_2|$). Provided that ϕ is consistent, we have:*

- *for any α, $\phi \circ_S \psi \models \alpha$ if and only if $\langle \psi, \phi \rangle \approx^{\mathcal{C}}_{\sqsubseteq_S} \alpha$;*
- *for any α, $\phi \circ_D \psi \models \alpha$ if and only if $\langle \psi, \phi \rangle \approx^{\mathcal{C}}_{\sqsubseteq_D} \alpha$;*
- *for any α, $\phi \circ_W \psi \models \alpha$ if and only if $\langle \psi, \phi \rangle \approx^{\mathcal{C}_{prudent}}_{\sqsubseteq_S} \alpha$.*

3.4 And Many Others!

Up to now, we showed in this section that reasoning from preferred consistent subsets and some forms of belief merging and of belief revision are particular cases of forget inference relations. Now, there are many other forget inference relations which do not belong to any of the previous families (provided that no new propositional symbols are added).

A first example is the family of *homogeneous forget inference relations*. Such relations can be obtained by requiring the forgetting context to be homogenous, or requiring the preference relation $\sqsubseteq$ to satisfy the gathering property (which is less demanding). When homogeneous inference relations are considered, it is enough to define a preference relation on subsets of PS (instead of vectors of subsets of PS). Many such relations can be obtained by letting the preference relation on 2^{PS} vary, in quite the same manner as maxcons inference relations are obtained by letting the preference relation on subsets vary. We may for instance minimize the *set* of forgotten variables (*i.e.*, $\vec{V} \sqsubseteq \vec{V'}$ if and only if $\bigcup \vec{V} \subseteq \bigcup \vec{V'}$), or minimize the *number* of forgotten variables, or, more generally, make use of a predefined penalty function or priority preordering on variables.

Here is a variety of conclusions that can be achieved through a judicious choice of $\sqsubseteq$ on Example 1.

Example 1 (continued) *Consider the following preference relations and the induced inference relations $\approx^{\mathcal{C}}_{\sqsubseteq}$. We define the mapping $k : 2^{Var(B)} \rightarrow I\!N$ by $k = k_c + k_s + k_t$ where $k_c(V) = 3$ if $\{c_0, c_1, c_2\} \subseteq V$, $k_c(V) = 0$ otherwise ; $k_s(V) = 3$ if $\{s, s_b, s_r\} \subseteq V$, $k_s(V) = 1$ if $V \cap \{s, s_b, s_r\} = \{s_b, s_r\}$, and $k_s(V) = 0$ otherwise ; $k_t(V) = 2$ if $t \in V$, $k_t(V) = 0$ otherwise.*

- *$\sqsubseteq_1 = \subseteq_p$;*

- $\sqsubseteq_2$: $\vec{V} \sqsubseteq_2 \vec{V}'$ *if and only if* $\sum_{i=1}^{5} k(V_i) \leq \sum_{i=1}^{5} k(V_i')$
- $\sqsubseteq_3$: $\vec{V} \sqsubseteq_3 \vec{V}'$ *if and only if* $\bigcup \vec{V} \subseteq \bigcup \vec{V}'$;
- $\sqsubseteq_4$: $\vec{V} \sqsubseteq_4 \vec{V}'$ *if and only if* $k(\bigcup \vec{V}) \leq k(\bigcup \vec{V}')$;
- $\sqsubseteq_5$: $\vec{V} \sqsubseteq_5 \vec{V}'$ *if and only if* $\{i \mid V_i \neq \emptyset\} \subseteq \{i \mid V_i' \neq \emptyset\}$.

The results are synthesized in table 2. $\sqsubseteq_1$ is the pointwise inclusion relation. $\sqsubseteq_2$ is a merging inference relation. $\sqsubseteq_5$ is a maxcons inference relation. $\sqsubseteq_3$ and $\sqsubseteq_4$ correspond to homogeneous inference relations: $\sqsubseteq_3$ minimizes the set of variables forgotten everywhere while $\sqsubseteq_4$ minimizes their cumulated cost.

A second family of forget inference relations which does not degenerate into one of the previous types of inference is obtained in the situation where reliability between sources is a matter of topic[5], as explained in the following example:

Example 2 *Let be $B = \langle \phi_1, \phi_2 \rangle$, with $\phi_1 = a \wedge b$, $\phi_2 = \neg a \wedge \neg b$. Let us consider the standard forgetting context for B. Assume that source 1. is more reliable to what concerns a than to what concerns b and conversely, source 2. is more reliable to what concerns b than to what concerns a. To capture it in our framework, the following ranking function can be considered (among other possible choices): $\mu(\langle V_1, V_2 \rangle) = \mu_0(V_1) + \mu_0(V_2)$ where $\mu_0(V_i) = \sum_{v \in V_i} k_i(v)$ and $k_1(a) = 2$, $k_1(b) = 1$, $k_2(a) = 1$, $k_2(b) = 2$. Then the (unique) preferred recovery for B is $\langle \{b\}, \{a\} \rangle$ and therefore we have $B \mid\!\approx^{\mathcal{C}}_{\sqsubseteq_\mu} a \wedge \neg b$.*

Finally, in order to illustrate the flexibility offered by our framework, let us consider the two following bases:

Example 3 *Let $B = \langle \phi_1, \phi_2 \rangle$, and let $B' = \langle \phi_1', \phi_2' \rangle$, with:*

	$\phi_1 = (a \vee b) \wedge c$ $\phi_2 = \neg a \wedge \neg b$	$\phi_1' = a \wedge b$ $\phi_2' = \neg a \wedge \neg b$
$\sqsubseteq_1$	$\phi_1 \vee \phi_2$	$a \Leftrightarrow b$
$\sqsubseteq_2$	$(\neg a \vee \neg b) \wedge c$	$\top$
$\sqsubseteq_3$	$(a \vee b \vee c) \wedge (\neg a \vee b)$	$a \not\Leftrightarrow b$
$\sqsubseteq_4$	$c \wedge (\neg a \vee \neg b)$	$\top$
$\sqsubseteq_5$	$c \wedge (\neg a \vee \neg b)$	$\top$

[5]Here, a topic simply is a propositional variable x and a source i conveys some information about it as soon as ϕ_i is not independent of x, *i.e.*, $\phi_i \not\equiv \exists\{x\}.\phi_i$. For a very different approach where topics are taken into account, see [Cholvy, 1995].

What should be concluded from such inconsistent bases, especially when no information about the (absolute or relative to topics) reliability of the two sources are available? There is no absolute answer to this question! The result depends on various assumptions on the merging process, and the variety of conclusions obtained through many paraconsistent inference relations, as reported in the table, exemplifies it.

The very important point here is that our framework enables to capture each of them:

- *$\sqsubseteq_1$ corresponds to inference from maximal (w.r.t. $\subseteq$ or cardinality) consistent subsets.*
- *$\sqsubseteq_2$ (resp. $\sqsubseteq_3$) corresponds to distance-based merging where d is the Hamming distance and $* = +$ (resp. $* = \max$).*
- *$\sqsubseteq_4$ = the closure of $\subseteq_p$ under gathering.*
- *$\sqsubseteq_5$ = $\subseteq_p$ under the standard forgetting context.*

Compared with the weakening by inhibition mechanism at work in the coherence-based approaches, weakening by variable forgetting is typically less drastic; indeed, instead of inhibiting a whole formula – or equivalently, replacing it by $\top$ – it is possible to keep all its consequences that are not involved in any contradiction (see *e.g.* the piece of information c in Example 3). Subsequently, more information can be preserved.

4 COMPUTATIONAL ISSUES

We now investigate the computational complexity of forget inference relations. We suppose the reader familiar with computational complexity (see *e.g.* [Papadimitriou, 1994]), and especially the complexity classes NP, Δ_2^p and Π_2^p of the polynomial hierarchy.

We have obtained the following results:

Proposition 4

1. *Given a belief base B and a forgetting context $\mathcal{C}$ for it, determining whether B is recoverable is* NP*-complete.*
2. *Provided that the preference relation $\sqsubseteq$ on $\mathcal{R}_{\mathcal{C}}(B)$ can be tested in polynomial time, the inference problem associated to $\mid\!\approx^{\mathcal{C}}_{\sqsubseteq}$ is in Π_2^p.*
3. *If moreover $\sqsubseteq$ is induced by a ranking function μ computable in polynomial time, the inference problem associated to $\mid\!\approx^{\mathcal{C}}_{\sqsubseteq}$ is in Δ_2^p.*

These results show that the gain in generality and flexibility offered by our framework does not induce

Table 2:

preference relation	preferred recoveries that are maximal w.r.t. $\subseteq_p$	$\approx^{\mathcal{C}}_{\sqsubseteq}$
$\sqsubseteq_1$	$\vec{V}^5$, $\vec{V}^6$, $\vec{V}^7$, $\vec{V}^8$, $\vec{V}^9$	$\phi_1 \wedge (s \vee t)$
$\sqsubseteq_2$	$\vec{V}^6$, $\vec{V}^9$	$\phi_1 \wedge (s \wedge \neg t)$
$\sqsubseteq_3$	$\vec{V}^1$, $\vec{V}^2$, $\vec{V}^3$	$s \vee t$
$\sqsubseteq_4$	$\vec{V}^1$, $\vec{V}^2$, $\vec{V}^6$	$\phi_1 \wedge (s \oplus t)$
$\sqsubseteq_5$	$\langle \emptyset, \emptyset, \emptyset, F_4, F_5 \rangle$, $\langle \emptyset, \emptyset, F_3, \emptyset, F_5 \rangle$, $\langle \emptyset, F_2, \emptyset, \emptyset, \emptyset \rangle$	$\phi_1 \wedge (s \vee t) \wedge (s \wedge t \Rightarrow s_b) \wedge (s \wedge \neg t \Rightarrow s_r)$

a complexity shift compared with many more specific approaches (such as maxcons inferences and forms of belief revision), that are already Π^p_2-complete. This holds as well for distance-based belief merging which is Δ^p_2-complete in the general case (see [Liberatore and Schaerf, 2000] [Konieczny *et al.*, 2002]), while the problem of forget inference with a complete preference relation induced by a polytime ranking function is not harder. Hardness results for our forget inference relations (in the general case) can be easily obtained from known results for maxcons inference relations, belief revision or merging (see *e.g.* [Eiter and Gottlob, 1992] [Cayrol *et al.*, 1998] [Nebel, 1998] [Konieczny *et al.*, 2002]), thanks to the forget-based characterizations of such approaches, as reported in Section 3.3.

In the general case (especially, when $\sqsubseteq$ is not induced by a ranking function), our complexity result show that two independent sources of complexity must be dealt with. One of them lies in the possibly exponential number of preferred recoveries which must be taken into account. The other one originates from the complexity of classical entailment in propositional logic.

One way to circumvent this intractability consists in *compiling* the belief base [Cadoli and Donini, 1997] [Coste-Marquis and Marquis, 2000]. Such a compilation may consist in computing off-line the set $min(\mathcal{R}_{\mathcal{C}}(B), \sqsubseteq)$ - or only the maximal elements of it w.r.t. $\subseteq_p$. In this situation, the source of complexity due to the number of preferred recoveries disappears (in practice, this is significant only when the number is small enough, which is always the case when prudent inference is considered); accordingly, the complexity of (skeptical or prudent) inference goes down to coNP. The other source of complexity can also be removed by imposing some restrictions on the belief base B. In this situation, on-line forget inference is tractable.

Proposition 5

1. *Provided that $min(\mathcal{R}_{\mathcal{C}}(B), \sqsubseteq)$ - or only the maximal elements of it w.r.t. $\subseteq_p$ - is part of the input, the inference problem associated to $\approx^{\mathcal{C}}_{\sqsubseteq}$ (or $\approx^{\mathcal{C}_{prudent}}_{\sqsubseteq}$) is in* coNP.
2. *If moreover the set $\{\phi_i \mid 1 \leq i \leq n\}$ of formulas from B belongs to a class of formulas that are tractable for clausal entailment and stable for new variable renaming[6], the inference problem associated to $\approx^{\mathcal{C}}_{\sqsubseteq}$ (or $\approx^{\mathcal{C}_{prudent}}_{\sqsubseteq}$) is in* P *when CNF queries are considered.*

Interestingly, several well-known classes that are tractable for the satisfiability problem are also stable for new variable renaming; For example, this is the case for the class of Krom formulas (CNF formulas consisting of binary clauses) and for the class of CNF Horn renamable formulas.

5 CONCLUSION

We have proposed a new framework for reasoning from inconsistent belief bases, which proceeds by selecting subsets of variables to be forgotten in the pieces of information, respecting some fixed constraints. We have shown that this framework encompasses many existing frameworks as specific cases, namely reasoning from preferred consistent subsets, forms of belief merging and of belief revision.

As evoked in the introduction, our approach is based on classical entailment so that the logical contents of the belief base is weakened instead of the inference relation itself. This gives to our family of forget relations some pros and some cons. Thus, if the belief base consists of a single inconsistent formula, our approach is not adapted (as well as the coherence-based approaches and belief merging, which are also based on classical entailment). Paraconsistent logics based on subclassical entailment relations can then be used with profit. Conversely, since weakening bears on the

[6]This means that for every formula ϕ from $PROP_{PS}$ and for every substitution σ that renames some occurrences of variables from ϕ into new variables (not occurring in ϕ), if ϕ belongs to the class under consideration, this is also the case for $\sigma(\phi)$.

inference relation in paraconsistent logics, it is uniform and the fact that information comes from several sources is not exploited. For instance, it is not possible to derive b from $\{a \wedge (\neg a \vee b), \neg a\}$ using the paraconsistent logic LP_m [Priest, 1991] (this is because $\neg a \vee b$ is a consequence of $\neg a$).

The idea of weakening beliefs instead of throwing them out is not new. For instance, in [Bessant *et al.*, 2001], each formula belonging to a conflict is weakened through the introduction of a new propositional symbol. This way, inconsistency is removed. Then, non-monotonic inference is conducted over the resulting base so as to prefer the introduced symbols as false. This enables a non-trivial treatment of iterated revision. Unfortunately, this approach comes down to the well-known WIDTIO approach ("When In Doubt Throw It Out") [Winslett, 1990] when only one revision is performed. Indeed, in this situation, weakening by variable introduction does not enable more information to be preserved than if beliefs were removed. Our approach does not suffer from this drawback because weakening is achieved through forgetting instead of variable introduction. This also enables classical entailment to be performed from the resulting base (i.e., after forgetting) instead of nonmonotonic inference.

Finally, weakening formulas by variable forgetting can be useful in many practical situations. For instance, if a series of tests has to be performed so as to "solve" inconsistency [Lang and Marquis, 2000; Liberatore and Donini, 2000], preferred recoveries give some hints about the variables that should be tested first, namely, the most important ones (w.r.t. the graph G in $\mathcal{C}$) that have to be forgotten. Similarly, in a decision-theoretic context where the ϕ_i's are no longer *beliefs* but *preferences* of several agents, variables that do not need to be forgotten are these the agents agree on, and preferred recoveries help finding the variables about which negociation should start.

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Appendix: proof sketches

Sketch of proof of Proposition 1:
($\Leftarrow$): Let $\sqsubseteq$ be a preference relation over $\mathcal{R}_{\mathcal{C}_S}(B)$ satisfying binaricity. Let $B = \langle\phi_1, \ldots, \phi_n\rangle$. For any subset S of $\{\phi_1, ..., \phi_n\}$, we define $\vec{V}_S = \langle V_1, \ldots, V_n\rangle$ where for each $i \in 1 \ldots n$, $V_i = \emptyset$ if $\phi_i \in S$ and $V_i = F_i$ otherwise. Now, we define a preference relation $\succeq$ on consistent subsets of $\{\phi_1, ..., \phi_n\}$ by $X \succeq Y$ if and only if $\vec{V}_X \sqsubseteq \vec{V}_Y$. It can then be shown that $\{\phi_1, ..., \phi_n\} \approx^{\forall}_{\succeq} \psi$ if and only if $B \approx^{\mathcal{C}_S}_{\sqsubseteq} \psi$.
($\Rightarrow$): Let $\succeq$ be a preference relation over consistent subsets of $\{\phi_1, ..., \phi_n\}$. For any vector $\vec{V} = \langle V_1, \ldots, V_n\rangle$ of subsets of PS, let $S_{\vec{V}}$ be the subset of $\{\phi_1, ..., \phi_n\}$ defined by $S_{\vec{V}} = \{\phi_i \mid V_i = \emptyset\}$. Let us now define the preference relation $\sqsubseteq$ over $\mathcal{R}_{\mathcal{C}_S}(B)$ by $\vec{V} \sqsubseteq \vec{V\prime}$ if and only if $S_{\vec{V}} \succeq S_{\vec{V'}}$. Note that $\sqsubseteq$ satisfies binaricity. Now, it can be shown that $\{\phi_1, ..., \phi_n\} \approx^{\forall}_{\succeq} \psi$ if and only if $\langle\phi_1, ..., \phi_n\rangle \approx^{\mathcal{C}_S}_{\sqsubseteq} \psi$.

Sketch of proof of Proposition 2:
($\Leftarrow$): A ssume that $\sqsubseteq_\mu$ is a complete preference relation on $\mathcal{R}_{\mathcal{C}_S}(B)$ induced by a ranking function μ satisfying decomposability, *i.e.*, there exists μ_0 and H s.t. $\mu(\vec{V}) = H(\mu_0(V_1), \ldots, \mu_0(V_n))$. Let us define the merging operator $\approx^M_{d,*}$ induced from the pseudo-distance defined by $d(\omega, \omega') = \mu_0(Diff(\omega, \omega'))$ and the aggregation function $* = H$. It can then be proven that $B \approx^M_{d,*} \psi$ if and only if $B \approx^{\mathcal{C}_S}_{\sqsubseteq_\mu}$.
($\Rightarrow$) Let $B \approx^M_{d,*}$ be a decomposable and differential n-merging inference relation induced by pseudo-distance d and aggregation function $*$, *i.e.*, $d(\omega, B) = *_{i=1}^n(d(\omega, \phi_i))$ and $d(\omega, \omega') = f(Diff(\omega, \omega')$. Let us consider the forget inference induced from the standard forgetting context and the function μ defined by $\mu(\vec{V}) = H(\mu_0(V_1), \ldots, \mu_0(V_n))$ where $\mu_0 = f$ and $H = *$. Then we prove that we get $\mu(\omega) = d(\omega, B)$, from which it follows that $B \approx^M_{d,*} = B \approx^{\mathcal{C}_S}_{\sqsubseteq_\mu}$.

Sketch of proof of Proposition 3:
We give the proof in the case of $\circ_S$. As a direct consequence of Corollary 3 from [Lang *et al.*, 2001], for every $\omega \in Mod(\psi)$, there exists $\omega' \in Mod(\phi)$ s.t. $Diff(\omega, \omega') = S$ if and only if $\omega \models \psi \wedge \exists S.\phi$ holds. Subsequently, we have $\{\omega \in Mod(\psi) \mid \exists\omega' \in Mod(\phi)$ s.t. $Diff(\omega, \omega') \in Diff^{min}(\psi, \phi)\} = \{\omega \in Mod(\psi \wedge \exists S.\phi) \mid S \in Diff^{min}(\psi, \phi)\}$. Now, $S \in Diff^{min}(\psi, \phi)$ if and only if S is a minimal (w.r.t. $\subseteq$) set of variables s.t. $\psi \wedge \exists S.\phi$ is consistent. This is equivalent to state that $\langle\emptyset, S\rangle$ is a preferred recovery for $\langle\psi, \phi\rangle$ w.r.t. $\sqsubseteq_S$ given $\mathcal{C}$, and the conclusion follows. The proof for the case $\circ_D$ is similar, with cardinality playing the role that set inclusion plays for $\circ_S$. We omit the case of $\circ_W$, the proof of which is not hard to establish.

Sketch of proof of Proposition 4: The membership proof for Point 1 goes as follows. Let $B = \langle\phi_1, \ldots, \phi_n\rangle$ be a belief base and let $\mathcal{C} = \langle F, G, H\rangle$ be a forgetting context for B. The following nondeterministic algorithm can be used to determine whether B is recoverable: 1. Guess n subsets $V_1, \ldots, V_n$ of $Var(B)$ and $n+1$ interpretations $\omega, \omega_1, \ldots, \omega_n$ over $Var(B)$; 2. For each $i \in 1 \ldots n$, check that: (a) $V_i \subseteq F_i$; (b) for every $(v, v') \in G_i$, if $v \in V_i$, then $v' \in V_i$; (c) $\omega_i \models \phi_i$; (d) ω coincides with ω_i on $Var(\phi_i) \setminus V_i$.
Due to lack of space we omit hardness (easy to establish) and points 2 and 3.

Proof of Proposition 5: Let $\vec{V} = \langle V_1, \ldots, V_n\rangle$ be any preferred recovery from $min(\mathcal{R}_\mathcal{C}(B), \sqsubseteq)$. Forget inference from B comes down to classical inference from $\bigvee_{\vec{V} \in min(\mathcal{R}_\mathcal{C}(B), \sqsubseteq)} B \mid \vec{V}$. The key point is the equivalence stating that $\{\exists V_1.\phi_1, \ldots, \exists V_n.\phi_n\} \models \psi$ if and only if $\bigwedge_{i=1}^n \phi_i[new(V_i)] \models \psi$, where $\phi_i[new(V_i)]$ is the formula obtained by renaming in an uniform way in ϕ_i every variable $v \in V_i$ by the new variable v^i, provided that ψ does not contain any new variable v^i, $i \in 1 \ldots n$. While the formula corresponding to $B \mid \vec{V}$ after such a renaming process is not logically equivalent to it[7], it is *query-equivalent* to it (*i.e.*, equivalence in preserved over the original language, only), but this is sufficient for our query answering purpose.

[7] In [Lang *et al.*, 2001], we show that in the general case and unless $\mathsf{NP} \cap \mathsf{coNP} \subseteq \mathsf{P/poly}$, there exists no propositional formula α equivalent to $\exists V.\phi$, s.t. the size of α is polynomially bounded in the size of ϕ plus the size of V.