DA^2 Merging Operators ⋆

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Abstract

A new framework for propositional merging is presented. *DA^2 merging operators*, parameterized by a distance between interpretations and two aggregation functions, are introduced. Many distances and aggregation functions can be used and many merging operators already defined in the literature (including both model-based ones and syntax-based ones) can be encoded as specific DA^2 operators. Both logical and complexity properties of those operators are studied. An important result is that (under very weak assumptions) query entailment from merged bases is "only" at the first level of the polynomial hierarchy when any of the DA^2 operators is used. As a by-product, complexity results for several existing merging operators are derived as well.

Key words: Knowledge Representation, Belief Merging, Computational Complexity.

1 Introduction

Belief merging is an important issue of many AI fields (see (1) for a panorama of applications of data and belief fusion.) Although particular requirements can be asked for each application, several pieces of information are usually brought into play when propositional base merging is concerned. In the following:

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- A *belief profile* $E = \{K_1, \ldots, K_n\}$ is a finite multi-set of belief bases, where each *belief base* K_i represents the set of beliefs from source i. Each K_i is a finite set of consistent propositional formulas $\varphi_{i,j}$ encoding the explicit beliefs from source i.
- IC is a propositional formula encoding some *integrity constraints*. IC represents some information the result of the merging has to obey (e.g. some physical constraints, norms, etc..)

The purpose of merging E is to characterize a formula (or a set of formulas) $\triangle_{IC}(E)$, considered as the overall belief from the n sources given the integrity constraints IC. Recently, several families of such merging operators have been defined and characterized in a logical way (2; 3; 4; 5; 6). Among them are the so-called *model-based* merging operators (2; 3; 4; 5) where the models of $\triangle_{IC}(E)$ are defined as the models of IC which are preferred according to some criterion depending on E. Often, such preference information takes the form of a total pre-order over interpretations, induced by a notion of distance $d(\omega, E)$ between an interpretation ω and the belief profile E. The distance $d(\omega, E)$ is typically defined by aggregating the distances $d(\omega, K_i)$ for every K_i. Usually, model-based merging operators take only into account consistent belief bases K_i. Other merging operators are so-called *syntax-based* ones (7; 8; 9). They are based on the selection of some consistent subsets of the set-theoretic union $\bigcup_{i=1}^n K_i$ of the belief bases. This allows for taking inconsistent belief bases K_i into account and to incorporate some additional preference information into the merging process. Indeed, as in belief revision, relying on the syntax of K_i is a way to specify (implicitly but in a cheap way with respect to representation) that explicit beliefs are preferred to implicit beliefs (10; 11). But the price to be paid is the introduction of an additional connective ",", which is not truth functional. Moreover, since they are based on the set-theoretic union $\bigcup_{i=1}^n K_i$ of the bases, such operators usually do not take into account the frequency of each explicit piece of belief into the merging process (the fact that $\varphi_{i,j}$ is believed in one source only or in the n sources under consideration is not considered relevant, which is often counter-intuitive.)

In this paper, a new framework for defining propositional merging operators is provided. A family of merging operators parameterized by a distance d between interpretations and two aggregation functions $\oplus$ and $\odot$ is presented. Accordingly, DA^2 merging operator is a short for Distance-based merging operator, obtained through 2 Aggregation steps. The parameters d, $\oplus$, $\odot$ are used to define a notion of distance between an interpretation and a belief profile E in a two-step fashion. Like in existing model-based approaches to merging, the models of the merging $\triangle_{IC}^{d,\oplus,\odot}(E)$ of E given some integrity constraints IC are exactly the models of IC that are as close as possible to E with respect to the distance. Moreover, the first aggregation step allows to take into account the syntax of belief bases within the merging process (and to handle inconsistent ones in a satisfying way.)

The contribution of this work is many fold. First, our framework is general enough to encompass many model-based merging operators as specific cases, especially those given in (2; 3; 4; 5; 12; 13; 6). In addition, despite the model-theoretic ground of our approach, several syntax-based merging operators provided so far in the literature can be captured as well (7; 8; 9). We show that, by imposing few conditions on the parameters, several logical properties that are expected when merging operators are considered, are satisfied by DA^2 operators.

Another very strong feature offered by our framework is that query entailment from $\triangle_{IC}^{d,\oplus,\odot}(E)$ is guaranteed to lay at the first level of the polynomial hierarchy provided that d, $\oplus$ and $\odot$ can be computed in polynomial time. Accordingly, improving the generality of the model-based merging operators framework through an additional aggregation step does not result in a complexity shift.

We specifically focus on some simple families of distances and aggregation functions. By letting the parameters d, $\oplus$ and $\odot$ vary in these respective sets, several merging operators are obtained; some of them were already known and are thus encoded as specific cases in our framework, and others are new operators. In any case, we investigate the logical properties and identify the complexity of each operator under consideration. As a by-product, the complexity of several model-based merging operators already pointed out so far is also identified.

The remaining of the paper is as follows. In Section 2 we give some formal preliminaries, and we recall some notions of computational complexity and some axiomatic properties for belief merging. In Section 3 we give a glimpse at the two main families of merging methods: model-based merging operators and syntax-based ones. In Section 4 we introduce DA^2 merging operators and give some examples. In Section 5 we study the computational complexity of this class of operators. This section also gives complexity results for some specific operators from the class. In Section 6 we address the logical properties of the operators. Finally Section 7 concludes the paper and presents some directions for future work.

2 Formal Preliminaries

We consider a propositional language $PROP_{PS}$ built up from a finite set PS of propositional symbols in the usual way. $\top$ (resp. $\bot$) denotes the Boolean constant interpreted to 1 (true) (resp. 0 (false)). An interpretation is a total function from PS to $BOOL = \{0, 1\}$. It is denoted by a tuple of literals over PS (or a tuple of truth values 0, 1 when a total ordering over PS is given).

The set of all interpretations is denoted by $\mathcal{W}$. An interpretation ω is a model of a formula if it makes it true in the usual classical truth functional way.

Provided that φ is a formula from $PROP_{PS}$, $Mod(\varphi)$ denotes the set of models of φ, i.e., $Mod(\varphi) = \{\omega \in \mathcal{W} \mid \omega \models \varphi\}$. Conversely, let M be a set of interpretations, $form(M)$ denotes the logical formula (unique up to logical equivalence) whose models are M.

Two belief bases K_1 and K_2 are said to be logically equivalent ($K_1 \equiv K_2$) if $\bigwedge K_1 \equiv \bigwedge K_2$, and two belief profiles E_1 and E_2 are said to be equivalent ($E_1 \equiv E_2$) if and only if there is a bijection between E_1 and E_2 such that each belief base of E_1 is logically equivalent to its image in E_2. A belief base K_i is said to be consistent if and only if the conjunction $\bigwedge K_i$ of its formulas is consistent. Similarly, a belief profile E is said to be consistent if the conjunction of its belief bases $\bigwedge E = \bigwedge_{K_i \in E} \bigwedge_{\varphi_{i,j} \in K_i} \varphi_{i,j}$ is consistent. $\sqcup$ denotes the multi-set union. For every belief profile E and for every integer n, E^n denotes the multi-set containing E n times.

For any set A, let $\leq$ be any binary relation over $A \times A$. $\leq$ is said to be *total* if $\forall a, b \in A$, $a \leq b$ or $b \leq a$; *reflexive* if $\forall a \in A$, $a \leq a$; *transitive* if $\forall a, b, c \in A$, ($a \leq b$ and $b \leq c$) implies $a \leq c$. Let $\leq$ be any binary relation, $<$ is its strict counterpart, i.e., $a < b$ if and only if $a \leq b$ and $b \not\leq a$, and $\simeq$ is its indifference relation, i.e. $a \simeq b$ if and only if $a \leq b$ and $b \leq a$. We denote $\min(A, \leq)$ the set $\{a \in A \mid \nexists b \in A \; b < a\}$.

2.1 Computational Complexity

The complexity results we give in this paper refer to some complexity classes which we now briefly recall (see (14) for more details), especially the classes Δ_2^p and Θ_2^p (15; 16) from the polynomial hierarchy PH, as well as the class BH_2 from the Boolean hierarchy. We assume the reader familiar with the classes P, NP et coNP and we now introduce the following three classes located at the first level of the polynomial hierarchy:

- BH_2 (also known as DP) is the class of all languages L such that $L = L_1 \cap L_2$, where L_1 is in NP and L_2 in coNP. The canonical BH_2-complete problem is SAT−UNSAT: given two propositional formulas φ and ψ, $\langle \varphi, \psi \rangle$ is in SAT−UNSAT if and only if φ is consistent and ψ is inconsistent.
- $\Delta_2^p = \mathsf{P}^{\mathsf{NP}}$ is the class of all languages that can be recognized in polynomial time by a deterministic Turing machine equipped with an NP oracle, where an NP oracle solves whatever instance of a problem from NP in unit time.
- $\Theta_2^p = \Delta_2^p[\mathcal{O}(log\ n)]$ is the class of all languages that can be recognized in polynomial time by a deterministic Turing machine using a number of calls to an NP oracle bounded by a logarithmic function of the size of the input

data.

Note that the following inclusions hold:

$$\mathsf{NP} \cup \mathsf{coNP} \subseteq \mathsf{BH}_2 \subseteq \Theta_2^p \subseteq \Delta_2^p \subseteq \mathsf{PH}.$$

Finally, $\mathsf{F}\Delta_2^p$ is the class of function problems associated wih Δ_2^p, i.e. those that can be solved in deterministic polynomial time on a Turing machine equipped with an NP oracle.

2.2 Logical Properties for Belief Merging

Some work in belief merging aims at finding sets of axiomatic properties operators may exhibit the expected behaviour (17; 2; 4; 18; 5; 12). We focus here on the characterization of Integrity Constraints (IC) merging operators (5; 13).

Definition 1 (IC merging operators) *Let E, E_1, E_2 be belief profiles, K_1, K_2 be consistent belief bases, and IC, IC_1, IC_2 be formulas from $PROP_{PS}$. $\triangle$ is an* IC merging operator *if and only if it satisfies the following postulates:*

(IC0) $\triangle_{IC}(E) \models IC$.
(IC1) *If IC is consistent, then $\triangle_{IC}(E)$ is consistent.*
(IC2) *If $\bigwedge E$ is consistent with IC, then $\triangle_{IC}(E) \equiv \bigwedge E \wedge IC$.*
(IC3) *If $E_1 \equiv E_2$ and $IC_1 \equiv IC_2$, then $\triangle_{IC_1}(E_1) \equiv \triangle_{IC_2}(E_2)$.*
(IC4) *If $K_1 \models IC$ and $K_2 \models IC$, then $\triangle_{IC}(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle_{IC}(\{K_1, K_2\}) \wedge K_2$ is consistent.*
(IC5) $\triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2) \models \triangle_{IC}(E_1 \sqcup E_2)$.
(IC6) *If $\triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2)$ is consistent , then $\triangle_{IC}(E_1 \sqcup E_2) \models \triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2)$.*
(IC7) $\triangle_{IC_1}(E) \wedge IC_2 \models \triangle_{IC_1 \wedge IC_2}(E)$.
(IC8) *If $\triangle_{IC_1}(E) \wedge IC_2$ is consistent, then $\triangle_{IC_1 \wedge IC_2}(E) \models \triangle_{IC_1}(E)$.*

The intuitive meaning of the properties is the following: (IC0) ensures that the result of merging satisfies the integrity constraints. (IC1) states that, if the integrity constraints are consistent, then the result of merging will be consistent. (IC2) states that if possible, the result of merging is simply the conjunction of the belief bases with the integrity constraints. (IC3) is the principle of irrelevance of syntax : the result of merging has to depend only on the expressed opinions and not on their syntactical presentation. (IC4) is a fairness postulate meaning that the result of merging of *two* belief bases should not give preference to one of them (if it is consistent with one of both, it has to be consistent with the other one.) It is a symmetry condition, that aims to rule out operators that can give priority to one of the bases. Note that (IC4) is a strong impartiality requirement and may appear very strong in some cases, but nevertheless it is satisfied by many interesting merging operators. Note that stating this property makes sense only because the belief bases K_i are required to be consistent. (IC5) expresses the following idea: if belief profiles are viewed as expressing the beliefs of the members of a group, then if E_1 (corresponding to a first group) compromises on a set of alternatives which A belongs to, and E_2 (corresponding to a second group) compromises on another set of alternatives which contains A too, then A has to be in the chosen alternatives if we join the two groups. (IC5) and (IC6) together state that if one could find two subgroups which agree on at least one alternative, then the result of the global merging will be exactly those alternatives the two groups agree on. (IC7) and (IC8) state that the notion of closeness is well-behaved, i.e., that an alternative that is preferred among the possible alternatives (IC_1), will remain preferred if one restricts the possible choices ($IC_1 \wedge IC_2$).

Two sub-classes of IC merging operators have been defined. *IC Majority operators* aim at resolving conflicts by adhering to the majority wishes, while *IC arbitration operators* have a more consensual behaviour:

Definition 2 (majority and arbitration) *An* IC majority operator *is an IC merging operator that satisfies the following majority postulate:*

(Maj) $\exists n \ \triangle_{IC}(E_1 \sqcup E_2{}^n) \models \triangle_{IC}(E_2)$.

An IC arbitration operator *is an IC merging operator that satisfies the following arbitration postulate:*

$$\textbf{\textit{(Arb)}} \quad \left.\begin{array}{l} \triangle_{IC_1}(K_1) \equiv \triangle_{IC_2}(K_2) \\ \triangle_{IC_1 \Leftrightarrow \neg IC_2}(\{K_1, K_2\}) \equiv (IC_1 \Leftrightarrow \neg IC_2) \\ IC_1 \not\models IC_2 \\ IC_2 \not\models IC_1 \end{array}\right\} \Rightarrow \triangle_{IC_1 \vee IC_2}(\{K_1, K_2\}) \equiv \triangle_{IC_1}(K_1).$$

See (5; 12) for explanations about those two postulates and the behaviour of the two corresponding classes of merging operators. For the sake of simplicity, we simply refer to such operators in the following as majority (resp. arbitration) ones, omitting IC.

3 Model-based Merging vs Syntax-based Merging

In this section, we recall the two main families of belief merging operators: the model-based ones and the syntax-based ones.

The idea here is that the result of the merging process is a belief base (up to logical equivalence) whose models are the *best* ones for the given belief profile E. Formally, provided that $\leq_E$ denotes an arbitrary binary relation (usually $\leq_E$ is required to be total, reflexive and transitive) on $\mathcal{W}$:

$$Mod(\triangle_{IC}(E)) = \min(Mod(IC), \leq_E)$$

Accordingly, in order to define a model-based merging operator, one just has to point out a function that maps each belief profile E to a binary relation $\leq_E$ (see (5) for conditions on this function.)

A compact way to characterize $\leq_E$ consists in deriving it from a notion of distance between an interpretation ω and a belief profile E (in this case $\leq_E$ is a total pre-order):

$$\omega \leq_E \omega' \text{ if and only if } d(\omega, E) \leq d(\omega', E).$$

$d(\omega, E)$ is usually defined by choosing a distance between interpretations aiming at building "individual" evaluations of each interpretation for each belief base, and then by aggregating those evaluations in a "social" evaluation of each interpretation. Indeed, assume that we have a distance d between interpretations (cf. Definition 5) that fits our particular application. Then one can define an (individual) belief base evaluation of each interpretation as the minimal distance between this interpretation and the models of the belief base:

$$d(\omega, K) = \min_{\omega' \models K} d(\omega, \omega').$$

Then it remains to compute a (social) belief profile evaluation of the interpretations using some aggregation function $*$:

$$d(\omega, E) = *_{K \in E}\ d(\omega, K).$$

In the first works on model-based merging, the distance used was Dalal's distance (19), namely, the Hamming distance between interpretations, and the aggregation function was the *sum* or the *max* (2; 3). In (5; 12) it has been shown that one can take any distance between interpretations without changing the logical properties of the operators and a *leximax* aggregation function was proposed as an example of arbitration operator.

Syntax-based merging (also called formula-based merging) operators work from preferred consistent subsets of formulas. The differences between the operators of this family lie in the definition of the preference relation (maximality with respect to set inclusion for instance.)

Let us briefly present the operators given in (7; 8).

Definition 3 *Let* $\textsc{maxcons}(K, IC)$ *be the set of the* maxcons *of* $K \cup \{IC\}$ *that contain* IC*, i.e., the maximal (with respect to set inclusion) consistent subsets of* $K \cup \{IC\}$ *that contain* IC*. Formally,* $\textsc{maxcons}(K, IC)$ *is the set of all* M *such that:*

- $M \subseteq K \cup \{IC\}$*, and*
- $IC \subseteq M$*, and*
- *if* $M \subset M' \subseteq K \cup \{IC\}$*, then* $M' \models \bot$.

Let $\textsc{maxcons}(E, IC) = \textsc{maxcons}(\bigcup_{K_i \in E} K_i, IC)$*. When the maximality of the sets is defined in terms of cardinality, we will use the subscript "card", i.e. we will note the set* $\textsc{maxcons}_{card}(E, IC)$*.*

Let us define the following operators:

Definition 4 *Let* E *be a belief profile and* IC *be a belief base:*

$\triangle^{C1}{}_{IC}(E) = \bigvee \textsc{maxcons}(E, IC)$.
$\triangle^{C3}{}_{IC}(E) = \bigvee\{M : M \in \textsc{maxcons}(E, \top)$ *and* $M \cup \{IC\}$ *consistent*$\}$.
$\triangle^{C4}{}_{IC}(E) = \bigvee \textsc{maxcons}_{card}(E, IC)$.
$\triangle^{C5}{}_{IC}(E) = \bigvee\{M \cup \{IC\} : M \in \textsc{maxcons}(E, \top)$ *and* $M \cup \{IC\}$ *consistent*$\}$
if this set is non empty and IC *otherwise.*

The $\triangle^{C1}$ operator takes as result of the combination the set of the maximal consistent subsets of $E \cup \{IC\}$ that contain the constraints IC. The $\triangle^{C3}$ operator computes first the set of the maximal consistent subsets of E, and then selects those that are consistent with the constraints. The $\triangle^{C4}$ operator selects the set of consistent subsets of $E \cup \{IC\}$ that contain the constraints IC and that are maximal with respect to cardinality.

$\triangle^{C1}{}_{IC}(E)$, $\triangle^{C3}{}_{IC}(E)$ and $\triangle^{C4}{}_{IC}(E)$ correspond respectively to $Comb1(E, IC)$, $Comb3(E, IC)$ and $Comb4(E, IC)$ as defined in (8) (there is no actual need to to consider the $Comb2$ operator since it is equivalent to $Comb1$ (8).) The $\triangle^{C5}$ operator is a slight modification of $\triangle^{C3}$ in order to get more logical properties (9).

Once the union of the belief bases is performed, the problem is to extract some coherent piece of information from it. Thus, such an approach is very close to Rescher and Manor's inference (20), Brewka's preferred subtheories (21), to the work by Benferhat *et al.* on entailment from inconsistent databases (22; 23; 24), as well as to several approaches to belief revision (25; 10; 26) and to reasoning with counterfactuals (27).

A drawback of this approach is that the distribution of information is not taken into account in the consistency restoration process. To deal with this drawback, it has been proposed in (9) to select only the maxcons that best fit a merging criterion. Those selection functions are related to those used in the AGM belief revision framework for *partial meet revision functions* (28). In both cases the selection functions aim at selecting only some of the maxcons (the "best" ones.) The idea for belief merging is to use the selection function to incorporate a "social" evaluation of maxcons.

In (9) three particular criteria have been proposed and studied. The first one selects the maxcons that are consistent with as many belief bases as possible. The second one takes the maxcons that have the smallest symmetrical difference (with respect to cardinality) with the belief bases and the last one takes the maxcons that have the largest intersection (with respect to cardinality) with the belief bases.

4 DA^2 Merging

4.1 The General Framework

Defining a merging operator in our framework simply consists in setting three parameters: a distance d and two aggregation functions $\oplus$ and $\odot$. Let us first make it precise what such notions mean in this paper:

Definition 5 (distances) *A distance between interpretations*[1] *is a total function d from $\mathcal{W} \times \mathcal{W}$ to $I\!N$ such that for every ω_1, $\omega_2 \in \mathcal{W}$*

- $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$, *and*
- $d(\omega_1, \omega_2) = 0$ *if and only if* $\omega_1 = \omega_2$.

Any distance between interpretations d induces a distance between an inter-

[1] We slightly abuse words here, since d is only a pseudo-distance (triangular inequality is not required.)

pretation ω and a formula φ given by

$$d(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega').$$

Definition 6 (aggregation functions) *An aggregation function is a total function $\oplus$ associating a nonnegative integer to every finite tuple of nonnegative integers and verifying (non-decreasingness), (minimality) and (identity).*

- *if $x \leq y$, then $\oplus(x_1, \ldots, x, \ldots, x_n) \leq \oplus(x_1, \ldots, y, \ldots, x_n)$.* *(non-decreasingness)*
- *$\oplus(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \ldots = x_n = 0$.* *(minimality)*
- *for every nonnegative integer x, $\oplus(x) = x$.* *(identity)*

We are now in position to define DA^2 merging operators. Basically the distance gives the closeness between an interpretation and each formula of a belief base. Then a first aggregation function $\oplus$ evaluates the plausibility (resp. desirability) of the interpretation for an agent (belief base) K_i from those closeness degrees when formulas are interpreted as information items (resp. preference items). And finally the second aggregation function $\odot$ evaluates the plausibility (resp. desirability) of the interpretation for the whole group (belief profile.)

Definition 7 (DA^2 merging operators) *Let d be a distance between interpretations and $\oplus$ and $\odot$ be two aggregation functions. For every belief profile $E = \{K_1, \ldots, K_n\}$ and every integrity constraint IC, $\triangle_{IC}^{d,\oplus,\odot}(E)$ is defined in a model-theoretical way by:*

$$Mod(\triangle_{IC}^{d,\oplus,\odot}(E)) = \min(IC, \leq_E^{d,\oplus,\odot}).$$

$\leq_E^{d,\oplus,\odot}$ is defined as $\omega \leq_E^{d,\oplus,\odot} \omega'$ if and only if $d(\omega, E) \leq d(\omega', E)$, where

$$d(\omega, E) = \odot(d(\omega, K_1), \ldots, d(\omega, K_n)),$$

and for every $K_i = \{\varphi_{i,1}, \ldots, \varphi_{i,n_i}\}$,

$$d(\omega, K_i) = \oplus(d(\omega, \varphi_{i,1}), \ldots, d(\omega, \varphi_{i,n_i})).$$

Defining two separate aggregation steps is not a theoretical fantasy that is only motivated by a struggle for generalization; rather, it formalizes the different nature of belief bases and belief profiles:

- a belief base is the set of elementary data reported by a given entity. The precise meaning of this rather vague formulation ("entity") depends on the context of the merging problem:

  - when merging several pieces of belief stemming from different "sources" (in practice, a source may be a sensor, an expert, a database...), the formulas inside a belief base K_i are the *pieces of information provided by source* i;
  - when evaluating alternatives with respect to different criteria, the formulas inside a belief base K_i are the *pieces of information pertaining to criterion* i;
  - when aggregating individual preferences in a group decision making context, the formulas inside a "belief base" K_i are the *elementary goals expressed by agent* i. In this case, the formulas $\varphi_{i,j}$ are no longer *beliefs* but *preferences* (which does not prevent one from using the same merging operators.) In this case, still calling these formulas "beliefs" is no longer appropriate, but, for the sake of simplicity, we nevertheless use the terminology "belief", rather than systematically writing "beliefs or preferences", which would be rather awkward.
- a belief profile E consists of the collection of all belief bases K_i corresponding to the different sources, criteria or agents involved in the problem.

Now, since the relationship between a belief base and its elementary pieces of information and the relationship between a belief profile and its belief bases are of different nature, there is no reason for not using *two* (generally distinct) aggregation functions $\oplus$ and $\odot$. In other words, both aggregation steps corresponds to different processes. The first step is an *intra-source* (more generally, intra-entity) aggregation: $\oplus$ aggregates scores with respect to the elementary (explicit) pieces of information contained in each K_i (it allows, in particular, to take inconsistent belief bases into account.) The second step is an *inter-source* (more generally, inter-entity) aggregation: $\odot$ aggregates the "$\oplus$-aggregated scores" pertaining to the different sources. Such a two-step approach is used in a group decision context by (29).

Interestingly, few conditions are imposed on d, $\oplus$, and $\odot$. As we will see in the next section, many distances and aggregation functions can be used. Often, the aggregation functions $\oplus$ and $\odot$ are required to be symmetric (i.e., no priority is given to some explicit beliefs in a belief base, and no priority is given to some belief base in a belief profile.) However, this condition is not mandatory here and this is important when some preference information are available, especially when all sources i are not equally reliable. For instance, the *weighted sum* aggregation function gives rise to (non-symmetric) merging operators.

Let us stress that, contrarily to usual model-based operators, our definition allows for inconsistent belief bases to take (a non-trivial) part in the merging process.

Example 1 *Assume that we want to merge* $E = \{K_1, K_2, K_3, K_4\}$ *under the integrity constraints* $IC = \top$*, where*

- $K_1 = \{a, b, c, a \Rightarrow \neg b\}$,
- $K_2 = \{a, b\}$,
- $K_3 = \{\neg a, \neg b\}$,
- $K_4 = \{a, a \Rightarrow b\}$.

In this example, K_1 believes that c holds. Since this piece of information is not involved in any contradiction, it seems sensible to be confident in K_1 about the truth of c. Model-based merging operators can not handle this situation: inconsistent belief bases can not be taken into account. Thus, provided that the Hamming distance d_H between interpretations is considered, the operator $\triangle^{d_H,\Sigma}$ (2; 3; 5; 13) gives a merged base whose models (over $\{a, b, c\}$) are: $(a, b, \neg c)$ and (a, b, c); the operator $\triangle^{d_H,Gmax}$ (5; 13) gives a merged base whose models are: $(\neg a, b, \neg c)$, $(\neg a, b, c)$, $(a, \neg b, \neg c)$, and $(a, \neg b, c)$. In any of these two cases, nothing can be said about the truth of c in the merged base, which is often counter-intuitive since no argument against it can be found in the input data.

Syntax-based operators render possible the exploitation of inconsistent belief bases. Thus, on the previous example, c holds in the merged base, whatever the syntax-based operator at work (among those considered in the paper.) Obviously, this would not be the case, would the inconsistent base K_1 be replaced by an equivalent one, as $\{a, \neg a\}$. However, syntax-based operators are not affected by how the formulas are distributed among the belief bases. Consider the two standard syntax-based operators $\triangle^{C1}$ and $\triangle^{C4}$, selecting the maximal subsets of E with respect to set inclusion and to cardinality, respectively. On the previous example, $\triangle^{C1}$ returns a merged base equivalent to c and $\triangle^{C4}$ to $c \wedge \neg a$. So, a is in the result for none of these two operators, whereas a holds in three of four input bases.

Our DA^2 operators achieve a compromise between model-based operators and syntax-based operators, by taking into account the way information is distributed and by taking advantage of the information stemming from inconsistent belief bases. For instance, our operator $\triangle^{d_D,sum,sum}$ (cf. Section 4.2) gives a merged base whose single model is (a, b, c), and $\triangle^{d_D,sum,lex}$ returns a merged base whose models are $(\neg a, b, c)$ and $(a, \neg b, c)$. So, using any of these two operators, we can conclude that c holds from the merged base.

DA^2 merging operators can be viewed as a generalization of model-based merging operators, with an additional aggregation step. One can then ask why we restrict the approach to two aggregation steps instead of characterizing DA^3 merging operators and so on... Actually, it can be sensible to use those additional aggregation steps to characterize the common belief of an organization structured in a hierarchical way. For example if an organization is composed of several departments, which are divided in services, that group several teams, etc., we can figure out an aggregation step for the team level, a second one for the service level, etc. At each step it is possible to use a different aggregation scheme. A detailed study of such operators is left to further research. In the light of our results, we can nevertheless make some important remarks concerning DA^n operators. On the one hand, the first aggregation step has a specific role since it allows to take inconsistent belief bases into account in the merging process. This underlies a main difference between DA^n operators (with n $\geq$ 2) and DA^1 operators, the usual model-based merging operators. The latter are not suited to use inconsistent belief bases in a valuable way. The differences induced by the second and the third aggregation steps are in some sense less significant. On the other hand, it is easy to show that all our complexity results pertaining to DA^2 operators can be extended to DA^n operators (the complexity does not change provided that the number of aggregation steps is bounded a priori.) Finally, as a tool for modelizing corporation merging, we think that DA^n operators are not fully adequate. Indeed, they would suppose that the number of hierarchical divisions is the same in all the branches, and that all the groups at a given level use the same aggregation method; this is a strong, unrealistic assumption.

4.2 Instantiating our Framework

Let us now instantiate our framework and focus on some simple families of distances and aggregation functions.

Definition 8 (some distances) *Let ω_1, $\omega_2 \in \mathcal{W}$ be two interpretations.*

- *The* drastic distance d_D *is defined by*

$$d_D(\omega_1, \omega_2) = \begin{cases} 0 \textit{ if } \omega_1 = \omega_2, \\ 1 \textit{ otherwise.} \end{cases}$$

- *The* Hamming distance d_H *is defined by*

$$d_H(\omega_1, \omega_2) = |\{x \in PS \mid \omega_1(x) \neq \omega_2(x)\}|.$$

- *Let q be a total function from PS to $\mathbb{N}^*$. The* weighted Hamming distance d_{H_q} *induced by q is defined by*

$$d_{H_q}(\omega_1, \omega_2) = \textstyle\sum_{\{x \in PS \mid \omega_1(x) \neq \omega_2(x)\}} q(x).$$

These distances satisfy the requirements imposed in Definition 7. The Hamming distance is the distance most commonly considered in model-based merging. It is very simple to express, but it is very sensitive to the representation language of the problem (i.e., the choice of propositional symbols.) Interestingly, many other distances can be used. For instance, weighted Hamming distances are relevant when some propositional symbols are known as more important than others. [2]

As to aggregation functions, many choices are possible. We just give here two well-known classes of such functions.

Definition 9 (weighted sums) *Let q be a total function from $\{1, \ldots, n\}$ to $I\!N^*$ such that $q(1) = 1$ whenever $n = 1$. The weighted sum WS_q induced by q is defined by*

$$WS_q(e_1, \ldots, e_n) = \sum_{i=1}^{n} q(i)e_i.$$

q is a *weight function*, that gives to each formula (resp. belief base) φ_i (resp. K_i) of index i its weight $q(i)$ denoting the formula (resp. belief base) reliability. The requirement $q(1) = 1$ whenever $n = 1$ ensures that when we merge a singleton, the aggregation function has no impact.

Definition 10 (ordered weighed sums) *Let q be a total function from $\{1, \ldots, n\}$ to $I\!N$ such that $q(1) = 1$ whenever $n = 1$, and $q(1) \neq 0$ in any case. The ordered weighted sum OWS_q induced by q is defined by*

$$OWS_q(e_1, \ldots, e_n) = \sum_{i=1}^{n} q(i)e_{\sigma(i)}$$

where σ is a permutation of $\{1, \ldots, n\}$ such that $e_{\sigma(1)} \geq e_{\sigma(2)} \geq \ldots \geq e_{\sigma(n)}$.

The requirement $q(1) \neq 0$ is needed to meet the minimality condition (definition 6.) When using q with OWS_q, $q(i)$ reflects the importance given to the i^{th} largest value. With the slight difference that q is normalized (but without requiring that $q(1) = 1$ whenever $n = 1$), the latter family is well-known in multi-criteria decision making under the terminology "Ordered Weighted Averages" (OWAs) (30).

When $q(i) = 1$ for every $i \in 1, \ldots, n$, WS_q and OWS_q are the usual *sum*. When $q(1) = 1$ and $q(2) = \ldots = q(n) = 0$, we have $OWS_q(e_1, \ldots, e_n) = max(e_1, \ldots, e_n)$. Lastly, let M be a upper bound of the scores, i.e., for any possible $(e_1, \ldots, e_n)$ we have $e_i < M$, and let $q(i) = M^{n-i}$ for all i. Then the rank order on vectors of scores induced by OWS_q is exactly the *leximax* (abbreviated by *lex*) ordering $\leq_{lex}$. Namely, we have $(e_1, \ldots, e_n) <_{lex} (e'_1, \ldots, e'_n)$

[2] Consider this example where information items about a murder coming from different witnesses; let a stand for "the murderer is a male" and b stand for "the murderer had an umbrella". Attaching a larger weight to a than to b means that the interpretation (a, b) is closer to $(a, \neg b)$ than to $(\neg a, b)$, reflecting that a mistake about b is more plausible than a mistake about a.

if and only if there exists k in $1, \ldots, n$ such that for all $i < k$, $e_{\sigma(i)} = e'_{\sigma'(i)}$ and $e_{\sigma(k)} < e'_{\sigma'(k)}$ if and only if $OWS_q(e_1, \ldots, e_n) < OWS_q(e'_1, \ldots, e'_n)$.

All these functions satisfy the requirements imposed in Definition 7; all of them are symmetric but *weighted sum* when q is not uniform. [3]

Many other possible choices for $\oplus$ and $\odot$ can be found in the literature of multi-criteria decision making (31). Noticeable examples of such aggregation functions are the *Choquet integral*, which generalizes both the weighted sum and the ordered weighted sum, and its ordinal counterpart, the *Sugeno integral* (32). These aggregation functions are still polynomially computable, which makes the following complexity results applicable when instantiating $\oplus$ and $\odot$ with such functions.

Note that functions such as the purely utilitarian *sum* or *weighted sum* allow for compensation between scores (and lead to majority-like operators), while the egalitarian functions *max* and *lex* do not.

By letting the parameters d, $\oplus$ and $\odot$ vary, several merging operators are obtained; some of them were already known and are thus encoded as specific cases in our framework, while others are new operators. For example, $\triangle^{d_D,max,max}$ is the *basic* merging operator (5), giving $\bigwedge E \wedge IC$ if consistent and IC otherwise. $\triangle^{d_D,max,sum}$ is the *drastic* merging operator which amounts to select the models of IC satisfying the greatest number of belief bases from E. It is equivalent to the drastic majority operator as defined in (9) when working with deductively closed belief bases. $\triangle^{d_D,sum,sum}$ corresponds to the intersection operator of (9). $\triangle^{d_D,WS_q,max}$ corresponds to an operator used in (29) in a group decision context. When singleton belief bases are considered [4] (in this case $\oplus$ is irrelevant) every $\triangle^{d,\oplus,max}$ operator is a $\triangle^{Max}$ operator (2; 13), every $\triangle^{d,\oplus,sum}$ operator is a $\triangle^{\Sigma}$ operator (2; 3; 5), and every $\triangle^{d,\oplus,lex}$ operator is a $\triangle^{GMax}$ operator (5; 13). Still with singleton belief bases, $\triangle^{d_D,\oplus,WS_q}$ is a penalty-based merging operator (where one minimizes the sum of the penalties $q(i)$ attached to the K_i's) (33), and taking $d = d_D$ and $\oplus = WMAX_q$ (defined by $WMAX_q(x_1, \ldots, x_n) = \max_{i=1,\ldots,n} \min(q(i), x_i))$ we get a possibilistic merging operator (6) (the scales used for scores are different but it is easy to show that this difference has no impact, i.e., the induced orderings over interpretations coincide.) Finally, the operators $\triangle^{d_D,sum,\odot}$, with $\odot \in \{sum, WS_q, max, lex\}$ have been proposed in (34) as a compromise between model-based and syntax-based approaches and a way to take into account inconsistent belief bases in the merging process.

We will now illustrate the behaviour of these different operators on an example.

[3] q is uniform when $\forall i, j \in 1, \ldots, n,\ q(i) = q(j)$.

[4] Or when each K_i is replaced by $\{\bigwedge K_i\}$ before merging.

$$
\begin{aligned}
\triangle_{IC}^{d_D,max,max}(E) &\equiv \top. \\
\triangle_{IC}^{d_D,max,sum}(E),\ \triangle_{IC}^{d_D,max,lex}(E),\ \triangle_{IC}^{d_H,max,sum}(E) &\equiv a \wedge b\ . \\
\triangle_{IC}^{d_D,sum,max}(E) &\equiv \neg b\ . \\
\triangle_{IC}^{d_D,sum,sum}(E) &\equiv (\neg a \wedge \neg b) \vee (a \wedge b \wedge c)\ . \\
\triangle_{IC}^{d_D,sum,lex}(E) &\equiv \neg a \wedge \neg b\ . \\
\triangle_{IC}^{d_H,sum,max}(E),\ \triangle_{IC}^{d_H,sum,lex}(E) &\equiv a \wedge \neg b \wedge c. \\
\triangle_{IC}^{d_H,max,max}(E), \triangle_{IC}^{d_H,max,lex}(E) &= (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c)\ . \\
\triangle_{IC}^{d_H,sum,sum} &\equiv a \wedge c\ .
\end{aligned}
$$

Fig. 1. Result of merging for the operators of Example 2

Example 2 *Consider the following belief profile $E = \{K_1, K_2, K_3, K_4\}$ that we want to merge under the integrity constraints $IC = \top$.*

- $K_1 = \{a \wedge b \wedge c, a \Rightarrow \neg b\}$,
- $K_2 = \{a \wedge b\}$,
- $K_3 = \{\neg a \wedge \neg b, \neg b\}$,
- $K_4 = \{a, a \Rightarrow b\}$.

The result of merging E according to the different operators with $d \in \{d_D, d_H\}$, $\oplus \in \{max, sum\}$ and $\odot \in \{max, sum, lex\}$ under no constraints (i.e., $IC = \top$) is given on Figure 1.

Table 1 gives an example of computation with the $\triangle^{d_H,sum,lex}$ operator. In the leftmost column of this table, every interpretation (x, y, z) with $x, y, z \in \{0, 1\}$ is the one mapping a to x, b to y and c to z. Each cell except those of the extreme columns gives the (Hamming) distance between the interpretation indexing its row and the formula or the belief base indexing its column. Each cell of the rightmost column contains the vector (ordered in a decreasing way) of distances from the interpretation ω indexing the corresponding row and each belief base K_i $(i = 1, \ldots, 4)$. As explained before, each such vector can be encoded as an integer using an OWS_q function, and such a score can be interpreted as the distance between ω and E. The main point is that the natural ordering over such scores representing vectors coincides with the leximax one over the corresponding vectors.

For the example, the result of merging process with $d = d_H, \oplus = sum, \odot = lex$ is $Mod(\triangle_{\top}^{d_H,sum,lex}(E)) = \{(1, 0, 1)\}$ since $I = (1, 0, 1)$ is the unique interpretation leading to the minimal vector 1111 (corresponding to a minimal distance to E.)

	$a \wedge b \wedge c$	$a \Rightarrow \neg b$	$a \wedge b$	$\neg a \wedge \neg b$	$\neg b$	a	$a \Rightarrow b$	$\mathbf{K_1}$	$\mathbf{K_2}$	$\mathbf{K_3}$	$\mathbf{K_4}$	$\mathbf{E}$
$(0,0,0)$	3	0	2	0	0	1	0	3	2	0	1	3210
$(0,0,1)$	2	0	2	0	0	1	0	2	2	0	1	2210
$(0,1,0)$	2	0	1	1	1	1	0	2	1	2	1	2211
$(0,1,1)$	1	0	1	1	1	1	0	1	1	2	1	2111
$(1,0,0)$	2	0	1	1	0	0	1	2	1	1	1	2111
$(1,0,1)$	1	0	1	1	0	0	1	1	1	1	1	1111
$(1,1,0)$	1	1	0	2	1	0	0	2	0	3	0	3200
$(1,1,1)$	0	1	0	2	1	0	0	1	0	3	0	3100

Table 1
$\triangle^{d_H,sum,lex}$ operator

The wide variety of the results we obtained shows the degree of flexibility achieved by our framework. The example illustrates several aspects of merging operators: the belief base K_1 is not consistent, but it is the only base that gives an information about c, so it can be sensible to take c as true in the result of merging. DA^2 operators can encode merging operators that are syntax-dependent: for example, K_3 is logically equivalent to $\neg a \wedge \neg b$, but replacing K_3 by this formula would lead to different results of the merging operation. Syntax is relevant for DA^2 merging operators since one has to consider that different formulas of a same base are distinct reasons to believe in the same information. Taking syntax into account is important from the point of view of representation of beliefs (or goals). In our framework, unlike with the classical model-based merging operators, the symbol "," can be taken to be a connective that is interpreted differently from "$\wedge$".

We do not consider the case $\oplus = lex$, since this choice induces some specific difficulties as to the second aggregation step. The first one is definitional: what does it mean to aggregate vectors (instead of atomic values) using an OWS_q function, especially when the vectors have different sizes? Several conflicting intuitions may exist. But would the induced operators exhibit the expected behavioural properties of merging? One may argue that, as shown above, it is possible to find out an OWS_q function the total pre-order induced by it coincides with leximax; however, this leads to another problem, namely, a representational problem: how to encode in a faithful way an aggregation function over vectors using some aggregation function over (atomic) values, so that the induced pre-orders coincide? A solution to both problems may come from a systematic study of more general aggregation functions than those used in this paper. This idea is of interest, but has not been considered here. It can be considered as an open question of this paper (however see (35) for a related issue.)

5 Computational Complexity

Let us now turn to the complexity issue. First of all, we can get a general hardness result that holds for *any* merging operator satisfying (IC1) and (IC2); this result, extremely close to a similar BH_2-hardness result for belief revision in (36), gives us a general lower bound of the complexity of inference from merging.

Proposition 1 *For any merging operator Δ satisfying properties (IC1) and (IC2), the complexity of inference from a merged base is BH_2-hard.*

Proof : Let $\langle \varphi, \psi \rangle$ be a pair of propositional formulas; without loss of generality, assume that φ and ψ do not share any propositional symbols. Then with $\langle \varphi, \psi \rangle$ we associate the following instance of INFERENCE-FROM-MERGING: $IC = \top$, $E = \{\{\varphi \vee x\}, \{\varphi \vee \neg x\}\}$, where x is a new symbol (appearing neither in φ nor in ψ), and $\alpha = \varphi \wedge \neg\psi$. Then we have $\Delta(E) \models \alpha$ if and only if φ is satisfiable and ψ is unsatisfiable, that is, if and only if $\langle \varphi, \psi \rangle$ is a positive instance of SAT-UNSAT. Indeed: consider first the case φ is satisfiable; in this case, (IC2) implies that $\Delta(E) \equiv (\varphi \vee x) \wedge (\varphi \vee \neg x) \equiv \varphi$; now, $\Delta(E) \models \varphi \wedge \neg\psi$ if and only if $\varphi \models \varphi \wedge \neg\psi$, which, since φ and ψ do not share any symbol, holds if and only if ψ is unsatisfiable. Consider now the case φ is unsatisfiable. Then $\alpha = \varphi \wedge \neg\psi$ is unsatisfiable, and property (IC1) tells that it cannot be the case that $\Delta(E) \models \alpha$. Therefore, we have $\Delta(E) \models \alpha$ if and only if $\langle \varphi, \psi \rangle$ is a positive instance of SAT-UNSAT.

$\square$

Now, as to finding an upper bound, we obtain a fairly general membership result which states that provided that d, $\oplus$ and $\odot$ can be computed in polynomial time, determining whether a given formula is entailed by the merging of a belief profile is in Δ_2^p; in addition to this, if d, $\oplus$ and $\odot$ are bounded by polynomial functions, then the above problem falls in Θ_2^p. Let us now state this more formally:

Proposition 2 *Let $\triangle^{d,\oplus,\odot}$ be a DA^2 merging operator. Given a belief profile E and two formulas IC and α:*

- *If d, $\oplus$ and $\odot$ are computable in polynomial time, then determining whether $\triangle_{IC}^{d,\oplus,\odot}(E) \models \alpha$ holds is in Δ_2^p.*
- *If d, $\oplus$ and $\odot$ are computable in polynomial time and are polynomially bounded,[5] then determining whether $\triangle_{IC}^{d,\oplus,\odot}(E) \models \alpha$ holds is in Θ_2^p.*

---

[5] A function $f : \mathbb{N}^n \to \mathbb{N}$ is polynomially bounded if and only if it is bounded by a polynomial function; more formally, when f is a function with a variable number of arguments, such as our aggregation functions, f is polynomially bounded if and

Proof : These results are consequences of the two following lemmata:

Lemma 1 *Let k be an integer; if d, $\oplus$ and $\odot$ are computable in polynomial time, then the problem of determining whether* $\min_{\omega \models IC} d(\omega, E) \leq k$ *given IC, E and k is in* NP.

Proof : It is sufficient to consider the following nondeterministic algorithm:

i) guess an interpretation ω and N interpretations $\omega_{i,j}$ ($i = 1, \dots, n$, $j = 1, \dots, n_i$) over $Var(E \cup \{IC\})$, where $N = \Sigma_{i=1,\dots,n} n_i$ is the total number of formulas $\varphi_{i,j}$ in E;
ii) check that $\omega \models IC$ and that $\omega_{i,j} \models \varphi_{i,j}$ for all $i = 1, \dots, n$ and all $j = 1, \dots, n_i$;
iii) compute $d(\omega, \omega_{i,j})$ for all $i = 1, \dots, n$ and all $j = 1, \dots, n_i$;
iv) compute $d(\omega, K_i)$ for all $i = 1, \dots, n$;
v) compute $d(\omega, E)$ and check that $d(\omega, E) \leq k$.

This algorithm runs in polynomial time in the size of the input (E, IC, and k represented in binary notation) since d, $\oplus$, $\odot$ are computable in polynomial time.

□

Lemma 2 *If for any $\omega \in \mathcal{W}$ the value of $d(\omega, E)$ is bounded by the value $h(|E| + |IC|)$ (where h is a function with values in $I\!N$), then* $\min_{\omega \models IC} d(\omega, E)$ *can be computed using* $\lceil \log_2 h(|E| + |IC|) \rceil$ *calls to an* NP *oracle.*

Proof : $min = \min_{\omega \models IC} d(\omega, E)$ can be computed using binary search on $\{0, \dots, h(|E| + |IC|)\}$ with at each step a call to an NP oracle to check whether $\min_{\omega \models IC} d(\omega, E) \leq k$ (that is in NP from Lemma 1.) Since a binary search on $\{0, \dots, h(|E| + |IC|)\}$ needs at most $\lceil \log_2 h(|E| + |IC|) \rceil$ steps, the result follows.

□

- *Point 1. of Proposition 2*

If d, $\oplus$ and $\odot$ are computable in polynomial time, then for every belief profile E and every $\omega \in \mathcal{W}$, the binary representation of $d(\omega, E)$ is bounded by $p(|E| + |IC|)$, where p is a polynomial. Hence, the value of $d(\omega, E)$ is bounded by $2^{p(|E|+|IC|)}$. From Lemma 2, we can conclude that $min = \min_{\omega \models IC} d(\omega, E)$ can be computed using a polynomial number of calls to an NP oracle. Now, let E be a belief profile, IC be a formula, k be an integer and α be a formula, the problem of determining whether there exists a model ω of IC such that $d(\omega, E) = k$ and such that $\omega \not\models \alpha$ is in NP (note the similarity between this proof and the one of Lemma 1):

---

only if there exists a collection of polynomial functions $\{pol_i \mid i \geq 1\}$ such that $f(x_1, ..., x_n) \leq pol_n(x_1, ..., x_n)$ for every n and for all $x_1, ..., x_n$.

i) guess an interpretation ω and N interpretations $\omega_{i,j}$ ($i = 1, \ldots, n$, $j = 1, \ldots, n_i$) over $Var(E \cup \{IC, \alpha\})$, where $N = \Sigma_{i=1,\ldots,n} n_i$ is the total number of formulas $\varphi_{i,j}$ in E;
ii) check that $\omega \models IC \wedge \neg\alpha$ and that $\omega_{i,j} \models \varphi_{i,j}$ for all $i = 1, \ldots, n$ and all $j = 1, \ldots, n_i$;
iii) compute $d(\omega, \omega_{i,j})$ for all $i = 1, \ldots, n$ and all $j = 1, \ldots, n_i$;
iv) compute $d(\omega, K_i)$ for all $i = 1, \ldots, n$;
v) compute $d(\omega, E)$ and check that $d(\omega, E) = k$.

So we can show that $\triangle_{IC}^{d,\oplus,\odot}(E) \not\models \alpha$ using first a polynomial number of calls to an NP oracle in order to compute min, and then using an additional call to an NP oracle in order to determine whether there exists a model ω of IC such that $d(\omega, E) = min$ and $\omega \not\models \alpha$. Hence the membership to Δ_2^p for this problem. The fact that Δ_2^p is closed for the complement concludes the proof.

- *Point 2. of Proposition 2*

When d, $\oplus$ and $\odot$ are polynomially bounded, the proof is similar to the one of point 1., but the computation of $\min_{\omega \models IC} d(\omega, E)$ needs only a logarithmic number of steps since h is polynomially bounded, hence the membership to Θ_2^p.

$\square$

As shown by the previous proposition, improving the generality of the model-based merging operators framework through an additional aggregation step does not result in a complexity shift : the decision problem for query entailment is still at the first level of PH.

Importantly, our results rely on the assumption that distances and aggregation functions can be computed in polynomial time. First, it should be remarked that all Δ_2^p membership results would still hold provided that distances and aggregation functions are in $\mathsf{F}\Delta_2^p$. Second, let us discuss the reasonableness of this assumption. On the one hand, all "usual" distance and aggregation functions used in the Knowledge Representation and in the Multicriteria Decision Making communities are consistent with it. On the other hand, there do exist interesting non polynomially-computable distances (and maybe also aggregation functions, although this is less clear). [6]

[6] Here is an example. Consider a set of deterministic events (or actions) E, where the dynamics of each event is described by a STRIPS list, and let us define the distance d_E by $d_E(\omega, \omega') = \min(L_E(\omega, \omega'), L_E(\omega', \omega))$ where $L_E(\omega, \omega')$ is the length of the shortest event sequence (or the shortest plan, if E is a set of actions) leading from ω to ω'. Then, using well-known results about the complexity of propositional STRIPS planning (37) imply that unless $\mathsf{P} = \mathsf{PSPACE}$, d_E is not polynomially computable.

We have also identified the complexity of query entailment from a merged base for the following DA^2 merging operators. Due to some similarity in the proofs of the three following Propositions, their proofs are written in one block. For these three Propositions, when X is a complexity class, X-c means X-complete.

Proposition 3 (complexity results for $d = d_D$)
Given a belief profile E and two formulas IC and α from $PROP_{PS}$, the complexity of $\triangle_{IC}^{d_D,\oplus,\odot}(E) \models^? \alpha$ is reported in the following table.

$\oplus/\odot$	max	sum	lex	WS_q	OWS_q
max	BH_2-c	Θ_2^p-c	Θ_2^p-c	Δ_2^p-c	Θ_2^p-c
sum	Θ_2^p-c	Θ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
WS_q	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
OWS_q	Θ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c

Proposition 4 (complexity results for $d = d_H$)
Given a belief profile E and two formulas IC and α from $PROP_{PS}$, the complexity of $\triangle_{IC}^{d_H,\oplus,\odot}(E) \models^? \alpha$ is reported in the following table

$\oplus/\odot$	max	sum	lex	WS_q	OWS_q
max	Θ_2^p-c	Θ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
sum	Θ_2^p-c	Θ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
WS_q	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
OWS_q	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c

Proposition 5 (complexity results for $d = d_{H_q}$)
Given a belief profile E and two formulas IC and α from $PROP_{PS}$, the complexity of $\triangle_{IC}^{d_{H_q},\oplus,\odot}(E) \models^? \alpha$ is reported in the following table.

$\oplus/\odot$	max	sum	lex	WS_q	OWS_q
max	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
sum	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
WS_q	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c
OWS_q	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c	Δ_2^p-c

Proof :

- Membership: All the membership results (for Propositions 3, 4 and 5) are direct consequences of Proposition 2, except to what concerns the basic merging operator ($d = d_D, \oplus = \odot = max$) and the membership-to-$\Theta_2^p$ results reported in Proposition 3 in the situation one of the two aggregation functions is a OWS_q function while the other one is max. Indeed, in all the remaining cases, all the distances and aggregation functions considered in the three tables can be computed in polynomial time. In addition, the distances d_D and d_H and the aggregation functions sum and max are polynomially bounded. As a consequence, we obtain immediately the membership to Θ_2^p of the inference problem with $\triangle^{d,\oplus,\odot}$ with $d \in \{d_D, d_H\}$, $\oplus \in \{sum, max\}$, $\odot \in \{sum, max\}$.

Now, focusing on the situation $d = d_D$, let us consider the case $\oplus$ is max and $\odot$ is a OWS_q function. For every interpretation $\omega \in \mathcal{W}$, let us note $k_E(\omega)$ the number of belief bases K_i ($i \in 1, \ldots, n$) from E such that $\omega \models K_i$ holds. Then we have $d(\omega, E) = \sum_{i=1}^{n-k_E(\omega)} q_i$. For any $j \in 0, \ldots, n$, it is easy to determine in nondeterministic polynomial time whether there exists a model ω of IC such that $d(\omega, E) \leq \sum_{i=1}^{n-j} q_i$. Now, since $d(\omega, E)$ can only take at most $n+1$ different values, its minimal value min over $Mod(IC)$ can be computed through binary search using at most $\lceil \log_2 n \rceil$ calls to an NP oracle which implements the nondeterministic algorithm above (starting with $j = 0$.) Once min has been computed, a final call to an NP oracle can be used to determine whether there exists a model ω of IC such that $d(\omega, E) = min$ and $\omega \not\models \alpha$. The fact that Θ_2^p is closed for the complement concludes the proof. The case $\oplus$ is a OWS_q function and $\odot$ is max can be handled in a similar way. The main difference is that $d(\omega, E)$ can only take at most $max_{i=1,\ldots,n} card(K_i)$ different values.
Finally, as to the basic merging operator, determining whether a formula α is a logical consequence of the merged base E given IC can be achieved using the following algorithm:

if $sat(E \cup \{IC\})$
then return($unsat(E \cup \{IC, \neg\alpha\})$)
else return($unsat(\{IC, \neg\alpha\})$).

Since only one satisfiability test (sat) and one unsatisfiability test ($unsat$) are required, the decision problem is in BH_2.

- Hardness:
  - Proposition 3: The Θ_2^p-hardness results are direct consequences of hardness results for cardinality-maximizing base revision $\circ_C$ (Theorem 5.14 from (36)) since we have $\Delta_{IC}^{d_D,sum,max}(\{\{\varphi_1, \ldots, \varphi_n\}\}) \equiv \Delta_{IC}^{d_D,sum,sum}(\{\{\varphi_1, \ldots, \varphi_n\}\}) \equiv \Delta_{IC}^{d_D,max,sum}(\{\{\varphi_1\}, \ldots, \{\varphi_n\}\}) \equiv \Delta_{IC}^{d_D,max,lex}(\{\{\varphi_1\}, \ldots, \{\varphi_n\}\}) \equiv \{\varphi_1, \ldots, \varphi_n\} \circ_C IC$.

    Indeed, whenever a single aggregation step is done and the drastic distance d_D is considered, lex gives the same ordering as sum. Since lex is a

specific OWS_q function, the corresponding Θ_2^p-hardness results still hold in the case $\oplus$ is a OWS_q function and $\odot = max$, as well as in the case $\oplus = max$ and $\odot$ is a OWS_q function.
As to the case where $\oplus$ is a OWS_q function and $\odot = sum$, the Δ_2^p-hardness result can be established by considering the following polynomial reduction from the Δ_2^p-complete problem MAX-SAT-ASG$_{odd}$ (16). MAX-SAT-ASG$_{odd}$ is the following decision problem:

Input: Σ, a propositional formula such that $Var(\Sigma) = \{x_1, \ldots, x_n\}$.
Question: Is the greatest model ω of Σ (over $Var(\Sigma)$) with respect to the lexicographic ordering $\preccurlyeq$ induced by $x_1 < x_2 < \ldots < x_n$ such that $\omega(x_n) = 1$?

To every formula Σ such that $Var(\Sigma) = \{x_1, \ldots, x_n\}$, we associate in polynomial time the tuple $M(\Sigma) = \langle E, IC, \alpha\rangle$, where $E = \{K_i \mid i \in 1, \ldots, n\}$, $IC = \Sigma$, $\alpha = x_n$ and for each $i \in 1 \ldots n$, $K_i = \{\bigwedge_{k=1}^{n+2-j} x_i \mid j \in 1 \ldots n+2-i\}$. Accordingly, each K_i contains $n+2-i$ formulas that are syntactically distinct but all equivalent to x_i. We consider now the OWS_q function $\oplus$ induced by q such that $q(1) = 1$ and for every $j > 1$, $q(j) = 2^{i-2}$. By construction, for any $\omega \in \mathcal{W}$ and any $i \in 1, \ldots, n$, we have $d_D(\omega, K_i) = 0$ if $\omega \models x_i$ and $d_D(\omega, K_i) = 2^{n-i+1}$ if $\omega \not\models x_i$. Accordingly, $d_D(\omega, E) = \sum_{i=1}^{n} d_D(\omega, K_i) = \sum_{i=1,\ldots,n \mid \omega \not\models x_i} 2^{n-i+1}$. We immediately get that ω is a model of IC that minimizes $d_D(\omega, E)$ if and only if ω is the (unique) greatest model of Σ w.r.t. $\preccurlyeq$, which leads easily to the result.
The Δ_2^p-hardness result in the case $\oplus = sum$ and $\odot = lex$ can be easily derived by taking advantage of the Δ_2^p-hardness result in the case each K_i is a singleton reduced to a conjunction of atoms (hence $\oplus$ is irrelevant), $\odot$ is lex and the Hamming distance d_H is considered (the proof is given in the following.) Indeed, to each $K_i = \bigwedge_{j=1}^{n_i} x_{i,j}$, we can associate the *set* of formulas $K_i = \{x_{i,j} \mid j \in 1,\ldots,n_i\}$ and for every interpretation $\omega \in \mathcal{W}$, we have $d_H(\omega, K_i) = \sum_{j=1}^{n_i} d_D(\omega, x_{i,j})$. Roughly, the Hamming distance is encoded here through a first aggregation step (using $\oplus = sum$) based on the drastic distance. Since sum is a specific WS_q function and lex is a specific OWS_q function, this hardness result can be extended to the rest of the table, except for the case ($\oplus$ is a WS_q function and $\odot = max$ or $\odot = sum$) or $\odot$ is a WS_q function.
As to these cases, the Δ_2^p-hardness of linear base revision $\circ_L$ (Theorem 5.9 from (36)) can be used to obtain the desired result. Indeed, it is sufficient to consider belief bases K_i reduced to singletons (hence the first aggregation step using $\oplus$ is irrelevant) or similarly a belief profile E consisting of a singleton (so that $\odot$ is irrelevant) since we have $\triangle_{IC}^{d_D,\oplus,\odot}(\{K_1, \ldots, K_n\}) \equiv \{K_1, \ldots, K_n\} \circ_L IC$, where $\odot$ is the weighted sum induced by q such that $q(i) = 2^{n-i}$, and each K_i is viewed as the unique formula it contains. Here, the preference ordering over $\{K_1, \ldots, K_n\}$ is such that $K_1 < K_2 < \ldots < K_n$.
Finally, as to the basic merging operator, the BH_2-hardness result is a direct consequence of Propositions 1 and 6.

- Proposition 4: The Θ_2^p-hardness results still hold in the situation E contains only one belief base K, and K itself contains only one formula that is a conjunction of atoms. This merely shows that our hardness result is independent from the aggregation functions $\oplus$ and $\odot$ under consideration (since they are irrelevant whenever E and K are singletons) but is a consequence of the distance that is used (Hamming). Indeed, in this restricted case, $\triangle_{IC}^{d_H,\oplus,\odot}(\{K\})$ is equivalent to $K \circ_D IC$ where $\circ_D$ is Dalal's revision operator (19). The fact that the inference problem from $K \circ_D IC$ is Θ_2^p-hard (even in the restricted case where K is a conjunction of atoms) concludes the proof (see Theorem 6.9 from (15)).

  We now show that the Δ_2^p-hardness results hold in the restricted case each K_i is a singleton, reduced to a conjunction of literals (which means that the $\oplus$-aggregation step is irrelevant), whenever $\odot = lex$. Since lex can be viewed as a specific OWS_q, the hardness result holds for OWS_q functions as well. We consider the following polynomial reduction M from MAX-SAT-ASG$_{odd}$ to the inference problem from a merged base. Let Σ be a propositional formula such that $Var(\Sigma) = \{x_1, \ldots, x_n\}$. Let $M(\Sigma) =$

$$\langle E = \{K_i = \{x_i \wedge \bigwedge_{j=i+1}^{2n-i+1} new_j\} \mid i \in 1, \ldots, n\}, IC = \Sigma \wedge \bigwedge_{j=2}^{2n} \neg new_j, \alpha = x_n\rangle$$

  where each new_j ($j \in 2,\ldots,2n$) is a new variable (not occurring in Σ.) Now, for every model ω of IC and for every $i \in 1,\ldots,n-1$, we have

$$d_H(\omega, K_i = \{x_i \wedge \bigwedge_{j=i+1}^{2n-i+1} new_j\}) > d_H(\omega, K_{i+1} = \{x_{i+1} \wedge \bigwedge_{j=i+2}^{2n-i+2} new_j\}).$$

  This shows that the vectors L_ω^E obtained by sorting the set $\{d_H(\omega, K_i) \mid i \in 1, \ldots, n\}$ in decreasing lexicographic order are always sorted in the same way (independently of ω): the first element is $d_H(\omega, K_1)$, the second one is $d_H(\omega, K_2)$, etc. Furthermore, whenever a model ω_1 of IC is strictly smaller than a model ω_2 of IC with respect to the lexicographic ordering $\preccurlyeq$ induced by $x_1 < x_2 < \ldots < x_n$, then $L_{\omega_1}^E$ is strictly greater than $L_{\omega_2}^E$ (with respect to the lexicographic ordering over vectors of integers.) Since the models of IC are totally ordered with respect to $\preccurlyeq$, exactly one model of IC is minimal with respect to the preference ordering induced by E: this is the model of IC that is maximal with respect to $\preccurlyeq$. Accordingly, x_n is true in this model if and only if $\triangle_{IC}^{d_H,\oplus,lex}(E) \models \alpha$ holds. This concludes the proof.

  Finally, we show that the remaining Δ_2^p-hardness results hold in the case one of the aggregation function is a WS_q function, i.e., whenever each K_i is a singleton (even reduced to an atom) or E is a singleton. In the first case, this merely shows that our hardness result is independent from the aggregation function $\oplus$ under consideration but holds in the case $\odot$ is a WS_q function and $d = d_H$ is the Hamming distance. Let us consider the following polynomial reduction M from MAX-SAT-ASG$_{odd}$ to the inference problem from a merged base. Let Σ be a propositional formula such that $Var(\Sigma) = \{x_1, \ldots, x_n\}$. Let

$$M(\Sigma) = \langle E = \{K_i = \{x_i\} \mid i \in 1, \ldots, n\}, IC = \Sigma, \alpha = x_n\rangle$$

and the aggregation function $\odot$ is the weighted sum operator induced by $q(i) = 2^{n-i}$. Accordingly, for any interpretation $\omega \in \mathcal{W}$, we have $d(\omega, E) = \sum_{i=1}^{n} q(i) d_H(\omega, K_i)$. By construction, for any interpretations ω_1, $\omega_2 \in \mathcal{W}$, we have $d_H(\omega_1, E) \leq d_H(\omega_2, E)$ if and only if $\omega_2 \preccurlyeq \omega_1$ where $\preccurlyeq$ is the lexicographic ordering induced by $x_1 < x_2 < \ldots < x_n$. Accordingly, the greatest model ω of Σ with respect to $\preccurlyeq$ is the unique model of $\triangle_{IC}^{d_H,\oplus,\odot}(E)$. As a consequence, the greatest model ω of Σ with respect to $\preccurlyeq$ is such that $\omega(x_n) = 1$ if and only if $\triangle_{IC}^{d_H,\oplus,\odot}(E) \models \alpha$. This concludes the proof.

– Proposition 5: We show that Δ_2^p-hardness holds in the very restricted case E contains only one belief base K, and K itself contains only one formula that is a conjunction of atoms. This merely shows that our hardness result is independent from the aggregation functions $\oplus$ and $\odot$ under consideration (since they are irrelevant whenever E and K are singletons) but is a consequence of the family of distances that is used (weighted Hamming). Let us consider the following polynomial reduction M from MAX-SAT-ASG$_{odd}$ to the inference problem from a merged base. Let Σ be a propositional formula such that $Var(\Sigma) = \{x_1, \ldots, x_n\}$. Let

$$M(\Sigma) = \langle E = \{\{\bigwedge_{i=1}^{n} x_i\}\}, IC = \Sigma, \alpha = x_n\rangle$$

and the weighted Hamming distance d_{H_q} induced by q such that $\forall i \in 1, \ldots, n$, $q(x_i) = 2^{n-i}$. By construction, for any interpretations ω_1, $\omega_2 \in \mathcal{W}$, we have $d_{H_q}(\omega_1, \bigwedge_{i=1}^{n} x_i) \leq d_{H_q}(\omega_2, \bigwedge_{i=1}^{n} x_i)$ if and only if $\omega_2 \preccurlyeq \omega_1$ where $\preccurlyeq$ is the lexicographic ordering induced by $x_1 < x_2 < \ldots < x_n$. Accordingly, the greatest model ω of Σ with respect to $\preccurlyeq$ is the unique model of $\triangle_{IC}^{d_{H_q},\oplus,\odot}(E)$. As a consequence, the greatest model ω of Σ with respect to $\preccurlyeq$ is such that $\omega(x_n) = 1$ if and only if $\triangle_{IC}^{d_{H_q},\oplus,\odot}(E) \models \alpha$. This concludes the proof.

□

Looking at the tables above, we can observe that the choice of the distance d has a great influence on the complexity results. Thus, whenever $d = d_H$ or $d = d_{H_q}$, the complexity results for inference from a merged base coincide whenever $\oplus$ (or $\odot$) is a WS_q function or a OWS_q function. This is no longer the case when $d = d_D$ is considered.

Together with Proposition 2, the complexity of many model-based merging operators already pointed out in the literature are derived as a by-product of the previous complexity results. To the best of our knowledge, the complexity of such operators has not been identified up to now[7], hence this is an additional contribution of this work. We can also note that, while the complexity of our DA^2 operators is not very high (first level of PH, at most), finding out significant tractable restrictions seems a hard task since intractability is still the case in many restricted situations (see the proofs.) Finally, our results show that some syntax-based merging operators (the ones based on set inclusion instead of cardinality and "located" at the second level of PH) cannot be encoded in polynomial time as DA^2 operators (unless PH collapses.)

6 Logical properties

Let first see what are the logical properties of DA^2 merging operators in the general case.

Proposition 6 *Let d be any distance, and let f and g be two aggregation functions. $\triangle^{d,\oplus,\odot}$ satisfies (IC0), (IC1), (IC2), (IC7), (IC8). The other postulates are not satisfied in the general case.*

Proof :

(IC0) By definition $Mod(\triangle_{IC}(E)) \subseteq Mod(IC)$.

(IC1) $\oplus$ and $\odot$ are functions with values in $I\!N$, so if $Mod(IC) \neq \emptyset$, there is always a minimal model ω of IC such that for every model ω' of IC $d(\omega, E) \leq d(\omega', E)$. So $\omega \models \triangle_{IC}(E)$ and $\triangle_{IC}(E) \not\models \perp$.

(IC2) By assumption, $\bigwedge E$ is consistent, i.e., there exists ω such that $\omega \models (\varphi_{11} \wedge \ldots \wedge \varphi_{1n_1}) \wedge \ldots \wedge (\varphi_{n1} \wedge \ldots \wedge \varphi_{nn_n})$. By definition of the distance, $d(\omega, \varphi) = 0$ if $\omega \models \varphi$, so by (minimality) of $\oplus$ we get $\oplus(d(\omega, \varphi_{i1}), \ldots, d(\omega, \varphi_{in_i})) = d(\omega, K_i) = 0$ if and only if $\omega \models \varphi_{i1} \wedge \ldots \wedge \varphi_{in_i}$. By (minimality) of $\odot$ we have that $\odot(d(\omega, K_1), \ldots, d(\omega, K_n)) = d(\omega, E) = 0$ if and only if $\omega \models K_1 \wedge \ldots \wedge K_n$. So $\omega \models \triangle_{IC}(E)$ if and only if $\omega \models \bigwedge E \wedge IC$.

(IC7) Suppose $\omega \models \triangle_{IC_1}(E) \wedge IC_2$. For any $\omega' \models IC_1$, we have $d(\omega, E) \leq d(\omega', E)$. Hence $\omega' \models IC_1 \wedge IC_2$, $d(\omega, E) \leq d(\omega', E)$. Subsequently $\omega \models \triangle_{IC_1 \wedge IC_2}(E)$.

(IC8) Suppose that $\triangle_{IC_1}(E) \wedge IC_2$ is consistent. Then there exists a model ω' of $\triangle_{IC_1}(E) \wedge IC_2$. Consider a model ω of $\triangle_{IC_1 \wedge IC_2}(E)$ and suppose that $\omega \not\models \triangle_{IC_1}(E)$. We have $d(\omega', E) < d(\omega, E)$, and since $\omega' \models IC_1 \wedge IC_2$, we have $\omega \notin \min(Mod(IC_1 \wedge IC_2), \leq_E^{d,\oplus,\odot})$, hence $\omega \not\models \triangle_{IC_1 \wedge IC_2}(E)$. Contradiction.

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[7] However, $(\triangle_{IC}^{d_H,sum,sum}(E) \overset{?}{\models} \alpha) \in \Delta_2^p$ can be recovered from a complexity results given in (38), page 151.

$\square$

Clearly enough, it is not the case that every DA^2 merging operator is an IC merging operator (not satisfying some postulates is motivated by the need to give some importance to the syntax in order to take inconsistent belief bases into account.)

Concerning the operators examined in the previous section, we have identified the following properties:

Proposition 7 $\triangle^{d,\oplus,\odot}$ *satisfies the logical properties stated in Tables 2 and 3. Since all these operators are already known to satisfy (IC0), (IC1), (IC2), (IC7) and (IC8) (cf. Proposition 6), we refrain from repeating such postulates here. For the sake of readability, postulate* (ICi) *is noted* i *and* M *(resp.* A*) stands for (Maj) (resp. (Arb)).*

<table>
<tr><th>$\oplus/\odot$</th><th>max</th><th>sum</th><th>lex</th><th>WS_q</th><th>OWS_q</th></tr>
<tr><td>max</td><td>3,4,5,A</td><td colspan="2">3,4,5,6,M,A</td><td>5,6,M</td><td>3,4</td></tr>
<tr><td>sum</td><td>5,A</td><td>5,6,M</td><td>5,6,A</td><td>5,6,M</td><td></td></tr>
<tr><td>$WS_q - OWS_q$</td><td>5,A</td><td>5,6,M</td><td>5,6,A</td><td>5,6,M</td><td></td></tr>
</table>

Table 2
Logical properties ($d = d_D$)

$\oplus/\odot$	max	sum	lex	WS_q	OWS_q
max	5,A	5,6,M	5,6,A	5,6,M	
sum	5,A	5,6,M	5,6,A	5,6,M	
$WS_q - OWS_q$	5,A	5,6,M	5,6,A	5,6,M	

Table 3
Logical properties ($d = d_H$ or $d = d_{H_q}$)

Proof :

(IC3) Most operators of the table do not satisfy (IC3). For the operators with $\oplus = WS_q$, this is because (IC3) refers to the equivalence of belief profiles, and the definition of this equivalence does not take weights into account. A counter-example for operators with $d = d_D$ is $K_1 = \{a, b\}$, $K_2 = \{a \wedge b\}$, $K_3 = \{\neg b\}$ giving $\triangle_\top(\{K_1, K_2\}) \not\equiv \triangle_\top(\{K_1, K_3\})$. A counter-example for operators with $d = d_H$ is $K_1 = \{a, b\}$, $K_2 = \{a, b, b\}$, $K_3 = \{\neg b\}$. Nevertheless (IC3) holds for $d = d_D$, $\oplus = max$, and $\odot \in \{max, sum, lex, OWS_q\}$. It is because each $d(\omega, E)$ is a vector of 0 and 1 (0 is set whenever $\omega \models K_i$ and 1 otherwise.) It is not the case for $d = d_D$, $\oplus = max$, $\odot = WS_q$, since in this case the result is sensible to permutations (because of the weights.)

(IC4) For most operators of the table, (IC4) is not satisfied, since those operators are sensible to the syntax of the base (in particular to the number of formulas.) Let us take as counter-example $K_1 = \{a, b, a \wedge b\}$ and $K_2 = \{\neg a\}$. Nevertheless (IC4) holds for $d = d_D$, $\oplus = max$, $\odot \in \{max, sum, lex, OWS_q\}$. Since if $K_1 \wedge K_2 \not\models \perp$, (IC4) holds trivially by (IC2), and if $K_1 \wedge K_2 \models \perp$, then if $\omega \models K_1$, then $d(\omega, \{K_1, K_2\}) = \odot(0, 1)$ and if $\omega \models K_2$, then $d(\omega, \{K_1, K_2\}) = \odot(1, 0)$. It is then sufficient to remark that every $\odot \in \{max, sum, lex,\ OWS_q\}$ is a symmetrical operator, so $\odot(0, 1) = \odot(1, 0)$.

(IC5) To show that the operators satisfy (IC5), it is enough to show that the following property holds: if $d(\omega, E_1) \leq d(\omega', E_1)$ and $d(\omega, E_2) \leq d(\omega', E_2)$, then $d(\omega, E_1 \sqcup E_2) \leq d(\omega', E_1 \sqcup E_2)$. This property depends only on $\odot$ and it is satisfied for $\odot \in \{max, sum, lex,\ WS_q\}$.

(IC6) To show that the operators satisfy (IC6), it is enough to show that the following property holds: if $d(\omega, E_1) < d(\omega', E_1)$ and $d(\omega, E_2) \leq d(\omega', E_2)$, then $d(\omega, E_1 \sqcup E_2) < d(\omega', E_1 \sqcup E_2)$. This property depends only on $\odot$ and it is satisfied for $\odot \in \{sum, lex,\ WS_q\}$.

(Maj) Showing that all operators with $\odot \in \{sum,\ WS_q\}$ satisfy (Maj) is easy from the properties of sum. It is also easy to show that operators with $\odot \in \{max, lex\}$ do not satisfy (Maj) since one can find a counter-example where the repetition of one base does not change the result [8]. Consider the following counter-examples: ($E_1 = \{K_1\} = \{\{a, b\}\}$ and $E_2 = \{K_2\} = \{\{\neg a, \neg b\}\}$), or ($E_1 = \{K_1\} = \{\{a \wedge b\}\}$ and $E_2 = \{K_2\} = \{\{\neg a \wedge \neg b\}\}$).

(Arb) It is easy to show that (Arb) holds for all operators with $\odot = max$ since the stronger following property holds: if $\triangle_{IC_1}(K_1) \equiv \triangle_{IC_2}(K_2)$, then $\triangle_{IC_1 \vee IC_2}(\{K_1, K_2\}) \equiv \triangle_{IC_1}(K_1)$.

To show that (Arb) holds for $\odot = lex$ operators, assume that $\triangle_{IC_1}(K_1) \equiv \triangle_{IC_2}(K_2)$, that is there exists a model ω of $IC_1 \wedge IC_2$ such that for every model ω' of IC_1, $d(\omega, K_1) \leq d(\omega', K_1)$ and for every model ω'' of IC_2, $d(\omega, K_2) \leq d(\omega'', K_2)$. W.l.o.g let us suppose that $d(\omega, K_1) \leq d(\omega, K_2)$. To show that $\triangle_{IC_1 \vee IC_2}(\{K_1, K_2\}) \equiv \triangle_{IC_1}(K_1)$, we show that if $\omega' \models IC_1 \vee IC_2$ and $\omega' \not\models \triangle_{IC_1}(K_1)$, then $\omega' \not\models \triangle_{IC_1 \vee IC_2}(\{K_1, K_2\})$. Consider the following three cases :

1. $\omega' \models IC_1 \wedge IC_2$. Then we have $d(\omega', K_1) > d(\omega, K_1)$ and $d(\omega', K_2) > d(\omega, K_2)$. As a consequence $d(\omega', \{K_1, K_2\}) > d(\omega, \{K_1, K_2\})$, hence $\omega' \not\models \triangle_{IC_1 \vee IC_2}(\{K_1, K_2\})$.
2. $\omega' \models IC_2 \wedge \neg IC_1$. Since $\omega' \models IC_2$ we know that $d(\omega', K_2) > d(\omega, K_2)$, by transitivity $d(\omega', K_2) > d(\omega, K_1)$. Then we have $d(\omega', \{K_1, K_2\}) > d(\omega, \{K_1, K_2\})$, hence $\omega' \not\models \triangle_{IC_1 \vee IC_2}(\{K_1, K_2\})$.
3. $\omega' \models IC_1 \wedge \neg IC_2$. Suppose that $\omega' \models \triangle_{IC_1 \vee IC_2}(\{K_1, K_2\})$. This implies that $d(\omega', \{K_1, K_2\}) \leq d(\omega, \{K_1, K_2\})$. This requires one of the following cases to hold (recall that we assume $d(\omega, K_1) \leq d(\omega, K_2)$):
    i. $d(\omega', K_1) < d(\omega, K_1)$ and $d(\omega', K_2) < d(\omega, K_1)$.

[8] Except for $d = d_D$, $\oplus = max$, $\odot = lex$, since in this case the lex operator induces the same ordering as the one induced by sum.

ii. $d(\omega', K_1) = d(\omega, K_1)$ and $d(\omega', K_2) \leq d(\omega, K_2)$.
iii. $d(\omega', K_2) = d(\omega, K_1)$ and $d(\omega', K_1) \leq d(\omega, K_2)$.

The first two cases are not possible since, as $\omega' \models IC_1$ and $\omega' \not\models \triangle_{IC_1}(K_1)$, we have $d(\omega', K_1) > d(\omega, K_1)$. So let us consider the last case and note that we have $d(\omega', K_1) \leq d(\omega, K_2)$ and $d(\omega', K_2) = d(\omega, K_1) \leq d(\omega, K_2)$. (Arb) requires that for every model ω'' of $IC_2 \wedge \neg IC_1$, $d(\omega'', \{K_1, K_2\}) = d(\omega', \{K_1, K_2\})$. So for any $\omega'' \models IC_2 \wedge \neg IC_1$ $d(\omega'', K_2) \leq d(\omega, K_2)$, hence $\omega'' \models \triangle_{IC_2}(K_2)$. But, by hypothesis, $\triangle_{IC_1}(K_1) \equiv \triangle_{IC_2}(K_2)$, hence $\omega'' \models IC_2$. Contradiction.

To show that operators with $\odot \in \{sum, WS_q\}$ do not satisfy (Arb), consider the following counter-example: $K_1 = \{a \wedge b\}$, $K_2 = \{\neg a \wedge \neg b\}$, $IC_1 = \neg(a \wedge b)$ and $IC_2 = a \wedge b$.

$\square$

The tables above show that our DA^2 operators exhibit different properties. We remark that only $\triangle^{d_D,max,sum}$ satisfies all listed properties. Failing to satisfy (IC3) (*irrelevance to the syntax*) in many cases is not surprising, since we want to allow our operators to take syntax into account. (IC4) imposes that, when merging two belief bases, if the result is consistent with one belief base, it has to be consistent with the other one – such fairness postulate is not expected when working with nonsymmetric operators (so, unsurprisingly, it is not satisfied for $\odot = WS_q$.) This postulate is not satisfied by any operator for which d is Hamming distance since cardinalities of the belief bases have an influence on $\oplus$, and more generally, it is hardly satisfiable when working with syntax-dependent operators. (IC5) and (IC6) correspond to Pareto dominance in social choice theory and are really important; so it is worth noting that almost all operators satisfy them (only operators for which $\odot = max$ or OWS_q do not satisfy (IC6).) As shown before, OWS_q gathers many aggregation functions; not surprisingly, the price to be paid is the lack of many logical properties in the general case.

We saw through the previous results that DA^2 merging operators do not (and aim not at) satisfy all IC merging operators properties. Hence this is natural to look for additional requirements under which all those properties would be satisfied.

Let us first define some natural additional properties on aggregation functions:

1) If $\varphi_1 \wedge \ldots \wedge \varphi_n$ is consistent,
then $\oplus(d(\omega, \varphi_1), \ldots, d(\omega, \varphi_n)) = \oplus(d(\omega, \varphi_1 \wedge \ldots \wedge \varphi_n))$.[9] *(and)*

[9] Since $\oplus$ is an aggregation function, we have $\oplus(d(\omega, \varphi_1 \wedge \ldots \wedge \varphi_n)) = d(\omega, \varphi_1 \wedge$

2) For any permutation σ, $\oplus(x_1, \ldots, x_n) = \oplus(\sigma(x_1, \ldots, x_n))$ *(symmetry)*
3) If $\oplus(x_1, \ldots, x_n) \leq \oplus(y_1, \ldots, y_n)$, then $\oplus(x_1, \ldots, x_n, z) \leq \oplus(y_1, \ldots, y_n, z)$. *(composition)*
4) If $\oplus(x_1, \ldots, x_n, z) \leq \oplus(y_1, \ldots, y_n, z)$, then $\oplus(x_1, \ldots, x_n) \leq \oplus(y_1, \ldots, y_n)$. *(decomposition)*

We have obtained the following representation theorem for DA^2 merging operators:

Proposition 8 *A DA^2 merging operator $\triangle^{d,\oplus,\odot}$ satisfies (IC0)-(IC8) if and only if the function $\oplus$ satisfies (and), and the function $\odot$ satisfies (symmetry), (composition) and (decomposition).*

Proof :

(If) We know that (IC0), (IC1), (IC2), (IC7) and (IC8) are directly satisfied (*cf.* Proposition 6). Let us consider the other properties. Let $E_1 = \{K_1, \ldots, K_n\}$ and $E_2 = \{K'_1, \ldots K'_n\}$.

(IC3) Assume that $E_1 \equiv E_2$. Hence we can find a permutation σ such that for every $i \in 1, \ldots, n$, $K_{\sigma(i)} \equiv K'_i$. Now, since $\oplus$ satisfies (and) and is non-decreasing in each argument, we have $d(\omega, K_{\sigma(i)}) = d(\omega, K'_i)$, so, as $\odot$ satisfies (symmetry) one gets $d(\omega, E_1) = \odot(d(\omega, K'_1), \ldots\ d(\omega, K'_n)) = d(\omega, E_2)$. Consequently $\triangle_{IC}(E_1) \equiv \triangle_{IC}(E_2)$. The result for $IC_1 \equiv IC_2$ is obvious from the definition of the operators.

(IC4) Suppose that $\triangle_{IC}(\{K_1, K_2\}) \wedge K_1 \not\models \bot$ and that $\triangle_{IC}(\{K_1, K_2\}) \wedge K_2 \models \bot$. As a consequence, we have $\min_{\omega\models K_1} \odot(d(\omega, K_1), d(\omega, K_2)) < \min_{\omega\models K_2} \odot(d(\omega,\ K_1), d(\omega, K_2))$. Since $\oplus$ satisfies (and), this is equivalent to $\min_{\omega\models K_1} \odot(0, d(\omega,\ K_2)) < \min_{\omega\models K_2} \odot(d(\omega, K_1), 0)$. Then by (symmetry), this is equivalent to $\min_{\omega\models K_1} \odot(d(\omega, K_2), 0) < \min_{\omega\models K_2} \odot(d(\omega, K_1), 0)$. Hence, since $\odot$ is non-decreasing in each argument, we get $\min_{\omega\models K_1} d(\omega, K_2) < \min_{\omega\models K_2} d(\omega, K_1)$. Now, let us take $K'_{j=1,2} = \bigwedge_{\varphi_i\in K_j} \varphi_i$. Since $\oplus$ satisfies (and), we have $d(\omega, K_j) = d(\omega, K'_j)$ for every interpretation ω. So we get $\min_{\omega\models K'_1} d(\omega, K'_2) < \min_{\omega\models K'_2} d(\omega, K'_1)$. Now, by definition of the distance $d(\omega, \omega') = d(\omega', \omega)$ for every pair of interpretations ω, ω'; from the definition of $d(\omega, \varphi)$, we have $\min_{\omega\models\varphi} d(\omega, \varphi') = \min_{\omega\models\varphi'} d(\omega, \varphi)$ for every pair of formulas φ, φ'. Since $\oplus$ is non-decreasing in each argument, we obtain $\min_{\omega\models\varphi} \oplus(d(\omega, \varphi')) = \min_{\omega\models\varphi'} \oplus(d(\omega, \varphi))$. But, taking $\varphi = K'_1$ and $\varphi' = K'_2$, this contradicts $\min_{\omega\models K'_1} d(\omega, K'_2) < \min_{\omega\models K'_2} d(\omega, K'_1)$.

(IC5) Consider $E_1 = \{K_1, \ldots K_n\}$ and $E_2 = \{K'_1, \ldots K'_n\}$. Suppose that ω is a model of $\triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2)$. The, for every model ω' of IC we have both:

$$\odot(d(\omega, K_1), \ldots, d(\omega, K_n)) \leq \odot(d(\omega', K_1), \ldots, d(\omega', K_n)), \text{ and}$$

$\ldots \wedge \varphi_n)$.

$$\odot(d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'})) \leq \odot(d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'})).$$

Since we have $\odot(d(\omega, K_1), \ldots, d(\omega, K_n)) \leq \odot(d(\omega', K_1), \ldots, d(\omega', K_n))$, using (composition) several times we obtain that:

$$\begin{aligned}\odot\,&(d(\omega, K_1), \ldots, d(\omega, K_n), d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'})) \\ &\leq \odot(d(\omega', K_1), \ldots, d(\omega', K_n), d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'})). \quad (1)\end{aligned}$$

Similarly, since $\odot(d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'})) \leq \odot(d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'}))$, using (composition) several times gives:

$$\begin{aligned}\odot\,&(d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'}), d(\omega', K_1), \ldots, d(\omega', K_n)) \\ &\leq \odot(d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'}), d(\omega', K_1), \ldots, d(\omega', K_n)). \quad (2)\end{aligned}$$

By (symmetry), we have that:

$$\begin{aligned}\odot\,&(d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'}), d(\omega', K_1), \ldots, d(\omega', K_n)) \\ &= \odot(d(\omega', K_1), \ldots, d(\omega', K_n), d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'})). \quad (3)\end{aligned}$$

By transitivity, using (1), (2) and (3), we have for every model ω' of IC:

$$\begin{aligned}\odot\,&(d(\omega, K_1), \ldots, d(\omega, K_n), d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'})) \\ &\leq \odot(d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'}), d(\omega', K_1), \ldots, d(\omega', K_n)).\end{aligned}$$

This exactly means that $\omega \models \triangle_{IC}(E_1 \sqcup E_2)$.

(IC6) Suppose that $\triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2) \not\models \bot$ and that $\triangle_{IC}(E_1 \sqcup E_2) \not\models \triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2)$. There exists ω such that $\omega \models \triangle_{IC}(E_1 \sqcup E_2)$ and $\omega \not\models \triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2)$. Let us assume w.l.o.g. that $\omega \not\models \triangle_{IC}(E_1)$. Since $\triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2) \not\models \bot$, let us consider any $\omega' \models \triangle_{IC}(E_1) \wedge \triangle_{IC}(E_2)$. Since $\omega' \models \triangle_{IC}(E_1)$ and $\omega \not\models \triangle_{IC}(E_1)$, we obtain:

$$\odot(d(\omega', K_1), \ldots, d(\omega', K_n)) < \odot(d(\omega, K_1), \ldots, d(\omega, K_n)).$$

Since $\omega' \models \triangle_{IC}(E_2)$, we have:

$$\odot(d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'})) \leq \odot(d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'})).$$

Using (decomposition) several times, we get:

$$\begin{aligned}\odot\,&(d(\omega', K_1), \ldots, d(\omega', K_n), d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'})) \\ &< \odot(d(\omega, K_1), \ldots, d(\omega, K_n), d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'})).\end{aligned}$$

Using (composition) several times, we get:

$$\begin{aligned}\odot\,&(d(\omega', {K'}_1), \ldots, d(\omega', {K'}_{n'}), d(\omega, K_1), \ldots, d(\omega, K_n)) \\ &\leq \odot(d(\omega, {K'}_1), \ldots, d(\omega, {K'}_{n'}), d(\omega, K_1), \ldots, d(\omega, K_n)).\end{aligned}$$

By (symmetry), we have:

$$\odot\,(d(\omega,K_1),\dots,d(\omega,K_n),d(\omega',K'_1),\dots,d(\omega',K'_{n'}))$$
$$= \odot(d(\omega',K'_1),\dots,d(\omega',K'_{n'}),d(\omega,K_1),\dots,d(\omega,K_n)).$$

Now by transitivity:

$$\odot\,(d(\omega',K_1),\dots,d(\omega',K_n),d(\omega',K'_1),\dots,d(\omega',K'_{n'}))$$
$$< \odot(d(\omega,K_1),\dots,d(\omega,K_n),d(\omega,K'_1),\dots,d(\omega,K'_{n'})).$$

That is $d(\omega',E_1\sqcup E_2) < d(\omega,E_1\sqcup E_2)$. This means $\omega \not\models \triangle_{IC}(E_1\sqcup E_2)$. Contradiction.

(Only if)

(symmetry) $\odot(x_1,\dots,x_n) = \odot(\sigma(x_1,\dots,x_n))$. Direct from (IC3).

(composition) If $\odot(x_1,\dots,x_n) \leq \odot(y_1,\dots,y_n)$, then let us consider two interpretations ω,ω' such that for every $i \in 1,\dots,n$, $d(\omega,K_i) = x_i$ and $d(\omega',K_i) = y_i$. From the definition of the DA^2 operators, we have $\omega \models \triangle_{form(\{\omega,\omega'\})}(\{K_1,\dots,K_n\})$. Now let us take a belief base K', such that $d(\omega,K') = d(\omega',K') = z$, we have both $\omega \models \triangle_{form(\{\omega,\omega'\})}(K')$ and $\omega' \models \triangle_{form(\{\omega,\omega'\})}(K')$. Now, from (IC5), we conclude that $\omega \models \triangle_{form(\{\omega,\omega'\})}(\{K_1,\dots,K_n,K\})$, or equivalently (from the definition of the operators) $\odot(x_1,\dots,x_n,z) \leq \odot(y_1,\dots,y_n,z)$.

(decomposition) We will show the equivalent condition:

if $\odot(x_1,\dots,x_n) < \odot(y_1,\dots,y_n)$, then $\odot(x_1,\dots,x_n,w) < \odot(y_1,\dots,y_n,w)$.

Suppose $\odot(x_1,\dots,x_n) < \odot(y_1,\dots,y_n)$. Let us consider two interpretations ω,ω' s. t. for every $i \in 1,\dots,n$, we have $d(\omega,K_i) = x_i$ and $d(\omega',K_i) = y_i$. From the definition of DA^2 operators, we get $\omega \models \triangle_{form(\{\omega,\omega'\})}(\{K_1,\dots,K_n\})$ and $\omega' \not\models \triangle_{form(\{\omega,\omega'\})}(\{K_1,\dots,K_n\})$. Now let us consider a base K', such that $d(\omega,K') = d(\omega',K') = z$; we have $\omega \models \triangle_{form(\{\omega,\omega'\})}(K')$ and $\omega' \models \triangle_{form(\{\omega,\omega'\})}(K')$. Since $\omega \models \triangle_{form(\{\omega,\omega'\})}(\{K_1,\dots,K_n\}) \wedge \triangle_{form(\{\omega,\omega'\})}(K')$, the conjunction is consistent and from (IC6) we obtain $Mod(\triangle_{form(\{\omega,\omega'\})}(\{K_1,\dots,K_n,K'\})) \subseteq Mod(\triangle_{form(\{\omega,\omega'\})}(\{K_1,\dots,K_n\})) = \{\omega\}$. So, by definition of the operator, we have $\odot(x_1,\dots,x_n,z) < \odot(y_1,\dots,y_n,z)$.

(and) Suppose that $\varphi = \varphi_1 \wedge \dots \wedge \varphi_n$ is consistent. We want to show that for every interpretation ω, $\oplus(d(\omega,\varphi_1),\dots,d(\omega,\varphi_n)) = \oplus(d(\omega,\varphi))$. There are 2 cases:

case 1: $\omega \models \varphi$.

By definition of the distances, we have $d(\omega,\varphi) = d(\omega,\varphi_i) = 0$; by (minimality) of $\oplus$, $\oplus(d(\omega,\varphi)) = \oplus(d(\omega,\varphi_1),\dots,d(\omega,\varphi_n)) = 0$.

case 2: $\omega \not\models \varphi$.

Consider the result of $\triangle_{form(\{\omega\})\vee\varphi}(\{\{form(\{\omega\})\},\{\varphi_1,\dots,\varphi_n\}\})$, by (IC0) and (IC1) this base has to be consistent, so it has to pick some models in

$\{\omega\} \cup Mod(\varphi)$. Furthermore (IC4) states that ω and some models of φ have to be in the result. Let us consider one such model ω' of φ. Then we have $d(\omega, \{\{form(\{\omega\})\}, \{\varphi_1, \ldots, \varphi_n\}\}) = d(\omega', \{\{form(\{\omega\})\}, \{\varphi_1, \ldots, \varphi_n\}\})$. Now, by definition of DA^2 merging operators:

$$d(\omega, E) = \odot(\oplus(d(\omega, form(\{\omega\}))), \oplus(d(\omega, \varphi_1), \ldots, d(\omega, \varphi_n))),$$

i.e., $\odot(0, \oplus(d(\omega, \varphi_1), \ldots, d(\omega, \varphi_n)))$. We have also:

$$d(\omega', E) = \odot(\oplus(d(\omega', form(\{\omega\}))), \oplus(d(\omega', \varphi_1), \ldots, d(\omega', \varphi_n))),$$

or equivalently $\odot(\oplus(d(\omega', \omega)), 0)$. Now by (symmetry) and (non-decreasingness) of $\odot$, we get that $\oplus(d(\omega, \varphi_1), \ldots, d(\omega, \varphi_n)) = \oplus(d(\omega', \omega))$. By the definition of the distance, this is equivalent to $\oplus(d(\omega, \varphi_1), \ldots, d(\omega, \varphi_n)) = \oplus(d(\omega, \varphi))$.

$\square$

7 Conclusion

The major contribution of this paper is a new framework for propositional merging. It is general enough to encompass many existing operators (both model-based ones and syntax-based ones) and to allow the definition of many new operators (symmetric or not.) Both the logical properties and the computational properties of the merging operators pertaining to our framework have been investigated. Some of our results are large-scope ones in the sense that they make sense under very weak conditions on the three parameters that must be set to define an operator in our framework. By instantiating our framework and considering several distances and aggregation functions, more refined results have also been obtained. Finally, a representation theorem for characterizing the "fully rational" DA^2 merging operators has been given.

This work calls for the investigation of several other perspectives. One of them consists in analyzing the properties of the DA^2 operators that are achieved when some other aggregation functions or some other distances are considered. For instance, suppose that a collection of formulas of interest (topics) is available. In this situation, the distance between ω_1 and ω_2 can be defined as the number of relevant formulas on which ω_1 and ω_2 differs (i.e., such that one of them satisfies the formula and the other one violates it.) Several additional distances could also be defined and investigated (see e.g., (39) for distances based on Choquet integrals.)

Finally, it would be interesting to extend our study to non-uniform DA^2 operators, i.e., those obtained by associating a specific aggregation function to each belief base K_i (instead of considering the same one for each K_i.)

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