A diff-Based Merging Operator

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Abstract

Merging operators aim at defining the beliefs (resp. the goal) of a group of agents from a profile of bases, gathering the beliefs (resp. the goals) of each member of the group. In the propositional setting, a well-studied family of merging operators are distance-based ones: the models of the merged base are the closest interpretations to the given profile. Closeness is, in this context, measured as a number resulting from the aggregation of the distances to each base of the profile. In this work we define a new kind of propositional merging operators, close to such distance-based merging operators, but relying on a set-theoretic definition of closeness, already at work in several revision/update operators from the literature. We study a specific merging operator of this family, obtained by considering set-product as the aggregation function.

Introduction

Information merging is a very important task in artificial intelligence: the issue is to determine the beliefs, or the goals, of a group of agents from their individual points of view. Much work has been devoted to the definition of merging operators in the propositional case (Revesz 1997; Liberatore & Schaerf 1998; Baral *et al.* 1992; Konieczny & Pino Pérez 2002a; Meyer, Pozos Parra, & Perrussel 2005), and to the study of their properties with respect to different criteria, mainly logical properties, strategy-proofness, complexity. See for instance (Konieczny & Pino Pérez 2002a; Revesz 1997; Liberatore & Schaerf 1998; Konieczny, Lang, & Marquis 2004) for logic-based characterizations, (Everaere, Konieczny, & Marquis 2007) for an investigation of strategy-proofness issues, and (Konieczny, Lang, & Marquis 2004; Everaere, Konieczny, & Marquis 2007) for computational complexity results. There exist also works on merging in richer logical settings than propositional logic, see for instance (Meyer 2001; Benferhat *et al.* 2002; Chopra, Ghose, & Meyer 2006; Benferhat, Lagrue, & Rossit 2007).

In (Konieczny & Pino Pérez 2002a) a set of postulates is proposed to characterize different families of merging operators, and several families of operators satisfying these postulates are defined. Such operators are called model-based merging operators because basically they select the models of a given integrity constraint (i.e., a formula encoding laws, norms, etc., used for constraining the result of the merging) that are the closest ones to the given profile of belief/goal bases of the group. Often, these operators are defined from a distance between interpretations.This distance between interpretations induce a distance between an interpretation and a base, which indicates how plausible/satisfactory the interpretation is with respect to the base. Once such distances are computed, an aggregation function is used to define the overall distance of each model (of the integrity constraints) to the profile. The models of the result of the merging are the closest models of the integrity constraints to the profile.

A commonly-used distance between interpretations is the Hamming distance (also called Dalal distance (Dalal 1988)). The Hamming distance between two interpretations is the number of propositional variables the two interpretations disagree on. The closeness between two interpretations is thus assessed as the number of atoms whose truth values must be flipped in one interpretation in order to make it identical to the second one. Such a distance is meaningful when no extra-information on the epistemic states of the agents are available.

The major problem with distance-based merging operators is that evaluating the closeness between two interpretations as a number may lead to lose too much information. Thus, the conflicting variables themselves (and not only how many they are) can prove significant. Especially, when variables express real-world properties, it can be the case that some variables are more important than others, or that some variables are logically connected. In these cases, distances are not mandatory.

As an alternative to distance, an interesting measure used to evaluate the closeness of two interpretations is diff, the symetrical difference between them. Instead of evaluating the degree of conflict between two interpretations as the number of variables on which they differ (as it is the case with the Hamming distance), the diff measure assesses it as the set of such variables.

In this paper, we consider the family of propositional merging operators based on the diff measure. We specifically focus on the operator $\Delta^{\mathsf{diff},\oplus}$ from this family obtained by considering set-product as the aggregation function. We evaluate it with respect to three criteria: logical properties, strategy-proofness and complexity. Other operators from this family are presented in (Everaere, Konieczny, & Marquis 2008).

The rest of the paper is as follows. In the following section, we give some formal preliminaries. Then, we define the family of model-based merging operators based on the diff measure of closeness, and make precise the specific operator $\Delta^{\mathsf{diff},\oplus}$ we focus on. In the next section, we report on the logical properties of $\Delta^{\mathsf{diff},\oplus}$ and we discuss the strategy-proofness issues for it. The computational complexity of $\Delta^{\mathsf{diff},\oplus}$ is given after, just before a discussion about some related work. The paper ends with some perspectives.

Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the usual connectives.

An interpretation (or world) is a total function from $\mathcal{P}$ to $\{0, 1\}$, denoted by a bit vector whenever a strict total order on $\mathcal{P}$ is specified. The set of all interpretations is noted $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[\phi]$ denotes the set of models of formula ϕ, i.e., $[\phi] = \{\omega \in \mathcal{W} \mid \omega \models \phi\}$.

A *base* K denotes the set of beliefs or goals of an agent, it is a finite and consistent set of propositional formulas, interpreted conjunctively. Unless stated otherwise, we identify K with the conjunction of its elements.

A *profile* E denotes the group of agents involved in the merging process. It is a multi-set (bag) of belief/goal bases $E = \{K_1, \dots, K_n\}$ (hence two agents are allowed to exhibit identical bases). We note $\sqcup$ the union of multi-sets. We denote by $\bigwedge E$ the conjunction of bases of E, i.e., $\bigwedge E = K_1 \wedge \dots \wedge K_n$. A profile E is said to be consistent if and only if $\bigwedge E$ is consistent. We say that two profiles are equivalent, noted $E_1 \equiv E_2$, if there exists a bijection f from E_1 to E_2 such that for every $\phi \in E_1$, ϕ and $f(\phi)$ are logically equivalent.

The result of the merging of the bases of a profile E, under the integrity constraints μ, is the *merged base* denoted $\Delta_\mu(E)$. The *integrity constraints* consist of a formula the merged base has to satisfy.

Diff-Based Merging Operators

As a gentle introduction to diff-based merging operators, let us first recall how distance-based merging operators are defined. This calls for a notion of (pseudo-)distance between interpretations and a notion of aggregation function.

Definition 1 *A* (pseudo-)distance between interpretations *is a total function d from $\mathcal{W} \times \mathcal{W}$ to $I\!N$ such that for every ω_1, $\omega_2 \in \mathcal{W}$:*

- $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$*, and*
- $d(\omega_1, \omega_2) = 0$ *if and only if* $\omega_1 = \omega_2$.

Any distance between interpretations d induces a "distance" between an interpretation ω and a base K defined by $d(\omega, K) = \min_{\omega' \models K} d(\omega, \omega')$.

Definition 2 *An* aggregation function *is a total function f associating a non-negative integer to every finite tuple of non-negative integers and verifying* (non-decreasingness)*,* (minimality) *and* (identity)*.*

- *if* $x \leq y$*, then* $f(x_1, \dots, x, \dots, x_n) \leq f(x_1, \dots, y, \dots, x_n)$. (non-decreasingness)
- $f(x_1, \dots, x_n) = 0$ *if and only if* $x_1 = \dots = x_n = 0$. (minimality)
- *for every non-negative integer x, $f(x) = x$.* (identity)

We can now define distance-based merging operators:

Definition 3 *Let d be a distance and f be an aggregation function. The* distance-based merging operator *induced by d and f is defined by: for any profile $E = \{K_1, \dots, K_n\}$ and any integrity constraint μ,*

$$[\Delta_\mu^{d,f}(E)] = \{\omega \models \mu \mid f(d(\omega, K_1), \dots, d(\omega, K_n)) \text{ is minimal}\}.$$

Such operators have been extensively studied, and many "standard" merging operators belong to this class (Revesz 1997; Liberatore & Schaerf 1998). Their logical properties are stated in (Konieczny & Pino Pérez 2002a), their strategy-proofness is studied in (Everaere, Konieczny, & Marquis 2007), and their computational complexity in (Konieczny, Lang, & Marquis 2004).

Let us now turn to diff-based merging operators. Basically, the idea consists in evaluating closeness between two interpretations ω and ω' as the set of variables on which they differ:

$$\mathsf{diff}(\omega, \omega') = \{p \in \mathcal{P} \mid \omega(p) \neq \omega'(p)\}.$$

This definition has already been used in the belief revision/update literature in order to define a number of operators (Katsuno & Mendelzon 1991; Weber 1986; Satoh 1988; Borgida 1985; Winslett 1988).

As for distances, we can straightforwardly define, using diff, a notion of closeness between an interpretation and a base, as the minimum closeness between the interpretation and the models of the base. Of course, since diff gives as output a set instead of a number, set-inclusion has to be considered as minimality criterion:

$$\mathsf{diff}(\omega, K) = min(\{\mathsf{diff}(\omega, \omega') \mid \omega' \models K\}, \subseteq).$$

So the closeness between an interpretation ω and a base K is measured as the set of all minimal sets (for set inclusion) of propositional variables which have to be flipped in ω to make it a model of K.

Now, we need to aggregate these measures in order to define a global notion of closeness between an interpretation and a profile. This is the aim of the aggregation functions. Of course, usual functions at work for distance-based operators cannot be used here simply because we do not work with numbers, but with sets.

Several aggregation functions can be considered in our setting. We focus on a single one in this paper. We consider set-product $\oplus$ as aggregation function: for two sets of sets E and E', $E \oplus E' = \{c \cup c' \mid c \in E \text{ and } c' \in E'\}$.

ω	$\mathsf{diff}(\omega, K_1)$	$\mathsf{diff}(\omega, K_2)$	$\mathsf{diff}(\omega, \{K_1, K_2\})$
0000	$\{\{p,q,r\}\}$	$\{\{p,s\}\}$	$\{\{p,q,r,s\}\}$
0001	$\{\{p,q,r\}\}$	$\{\{p\}\}$	$\{\{p,q,r\}\}$
0010	$\{\{p,q\}\}$	$\{\{p,s\}\}$	$\{\{p,q,s\}\}$
0011	$\{\{p,q\}\}$	$\{\{p\}\}$	$\{\{p,q\}\}$
0100	$\{\{p,r\}\}$	$\{\{p,q,s\}\}$	$\{\{p,q,r,s\}\}$
0101	$\{\{p,r\}\}$	$\{\{p,q\}\}$	$\{\{p,q,r\}\}$
0110	$\{\{p\}\}$	$\{\{p,q,s\}\}$	$\{\{p,q,s\}\}$
0111	$\{\{p\}\}$	$\{\{p,q\}\}$	$\{\{p,q\}\}$
1000	$\{\{q,r\}\}$	$\{\{s\}\}$	$\{\{q,r,s\}\}$
1001	$\{\{q,r\}\}$	$\emptyset$	$\{\{q,r\}\}$
1010	$\{\{q\}\}$	$\{\{s\}\}$	$\{\{q,s\}\}$
1011	$\{\{q\}\}$	$\emptyset$	$\{\{q\}\}$
1100	$\{\{r\}\}$	$\{\{q,s\}\}$	$\{\{q,r,s\}\}$
1101	$\{\{r\}\}$	$\{\{q\}\}$	$\{\{q,r\}\}$
1110	$\emptyset$	$\{\{q,s\}\}$	$\{\{q,s\}\}$
1111	$\emptyset$	$\{\{q\}\}$	$\{\{q\}\}$

Table 1: Computation of $\Delta_{\top}^{\mathsf{diff},\oplus}(E)$

Definition 4 *Let $E = \{K_1, \ldots, K_n\}$ be a profile and ω an interpretation. The closeness between ω and E is given by:*

$$\mathsf{diff}(\omega, E) = \min(\{\oplus_{K_i \in E}\, \mathsf{diff}(\omega, K_i)\}, \subseteq).$$

By construction, each element of $\mathsf{diff}(\omega, E)$ is a minimal set c of variables (a conflict set) such that for each base K_i, ω can be transformed into a model of K_i by flipping in ω the variables of c.

Finally, we define a merging operator $\Delta^{\mathsf{diff},\oplus}$ which picks up the models of the integrity constraints whose closeness to the profile E contains at least one of the minimal (w.r.t. $\subseteq$) conflict set:

Definition 5 *Let $E = \{K_1, K_2, \ldots, K_n\}$ be a profile, μ an integrity constraint. Then:*

$$\mathsf{diff}_\mu(E) = \min(\{\mathsf{diff}(\omega, E) \mid \omega \models \mu\}, \subseteq)$$

and

$$[\Delta_\mu^{\mathsf{diff},\oplus}(E)] = \{\omega \models \mu \mid \exists c \in \mathsf{diff}(\omega, E) \text{ s.t. } c \in \mathsf{diff}_\mu(E)\}.$$

Example 1 *We consider a profile $E = \{K_1, K_2\}$ with $K_1 = \{p \wedge q \wedge r\}$ and $K_2 = \{p \wedge \neg q \wedge s\}$, there is no integrity constraint (i.e., $\mu \equiv \top$). $\mathsf{diff}_\mu(E) = \min(\{\{p,q,r,s\}, \{p,q,r\}, \{p,q,s\}, \{p,q\}, \{q,r,s\}, \{q,r\}, \{q,s\}, \{q\}\}, \subseteq) = \{\{q\}\}$. $[\Delta_\top^{\mathsf{diff},\oplus}(E)] = \{1111, 1011\}$ so $\Delta_\top^{\mathsf{diff},\oplus}(E) \equiv p \wedge r \wedge s$ (see Table 1).*

Just as many IC merging operators can be considered as generalizations of AGM revision operators (Konieczny & Pino Pérez 2002a), one can easily show that $\Delta^{\mathsf{diff},\oplus}$ can be viewed as a generalization of the well-known Satoh's revision operator (Satoh 1988), denoted $\circ_{\mathcal{S}}$:

Proposition 1 *Let K be a base and μ an integrity constraint. We have:*

$$\Delta_\mu^{\mathsf{diff},\oplus}(\{K\}) \equiv K \circ_{\mathcal{S}} \mu.$$

Logical Properties

Since we aim at investigating the logical properties of the merging operator $\Delta^{\mathsf{diff},\oplus}$, a set of properties must first be considered as a base line. The following set of postulates was proposed in (Konieczny & Pino Pérez 2002a):

Definition 6 *$\triangle$ is an* IC merging operator *if and only if it satisfies the following postulates:*

(IC0) $\triangle_\mu(E) \models \mu$

(IC1) *If μ is consistent, then $\triangle_\mu(E)$ is consistent*

(IC2) *If $\bigwedge E$ is consistent with μ, then $\triangle_\mu(E) \equiv \bigwedge E \wedge \mu$*

(IC3) *If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$*

(IC4) *If $K_1 \models \mu$ and $K_2 \models \mu$, then $\triangle_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent*

(IC5) $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \models \triangle_\mu(E_1 \sqcup E_2)$

(IC6) *If $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent, then $\triangle_\mu(E_1 \sqcup E_2) \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$*

(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(E)$

(IC8) *If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \models \triangle_{\mu_1}(E)$*

For explanations on these postulates see (Konieczny & Pino Pérez 2002a).

Proposition 2 *$\Delta^{\mathsf{diff},\oplus}$ satisfies* **(IC0)***,* **(IC1)***,* **(IC2)***,* **(IC3)***,* **(IC4)** *and* **(IC7)***. It does not satisfy* **(IC5)***,* **(IC6)** *and* **(IC8)**.

The reason why $\Delta^{\mathsf{diff},\oplus}$ does not satisfies **(IC8)** is that this property requires a total criterion (i.e., the corresponding syncretic assignment (Konieczny & Pino Pérez 2002b) must associates a **total** pre-order to each profile, so that any two interpretations can always be compared), whereas diff gives rise only to partial relations.

$\Delta^{\mathsf{diff},\oplus}$ also does not satisfy **(IC5)** and **(IC6)**, which are postulates capturing aggregation properties. This is not surprising since, unlike distance-based operators (as the ones based on Hamming distance), $\Delta^{\mathsf{diff},\oplus}$ keeps a justification of the minimality of an interpretation (as a conflict set). So, when joining two groups, it may happen that the justifications needed to motivate this choice become too weak. As an example, assume that two profiles E_1 and E_2 select independently as the result of the merging a formula with only two models ω and ω' (i.e., $[\Delta_\mu^{\mathsf{diff},\oplus}(E_1)] = [\Delta_\mu^{\mathsf{diff},\oplus}(E_2)] = \{\omega, \omega'\}$), and suppose that the conflict sets between the models and the profiles are the following ones: $\mathsf{diff}(\omega, E_1) = \mathsf{diff}(\omega', E_1) = \mathsf{diff}(\omega, E_2) = \{\{a, b\}\}$, and $\mathsf{diff}(\omega', E_2) = \{\{a, c\}\}$. Then, if we join the two groups we obtain $\mathsf{diff}(\omega, E_1 \sqcup E_2) = \{\{a, b\}\}$ and $\mathsf{diff}(\omega', E_1 \sqcup E_2) = \{\{a, b, c\}\}$. The conflict set associated to ω' is not minimal anymore. Since $\mathsf{diff}(\omega, E_1 \sqcup E_2) \subset \mathsf{diff}(\omega', E_1 \sqcup E_2)$, we have $\omega' \not\models \Delta_\mu^{\mathsf{diff},\oplus}(E_1 \sqcup E_2)$, whereas $\omega' \models \Delta_\mu^{\mathsf{diff},\oplus}(E_1) \wedge \Delta_\mu^{\mathsf{diff},\oplus}(E_2)$, which contradicts **(IC6)**.

On the following example, we illustrate how $\Delta^{\mathsf{diff},\oplus}$ can prove better than usual distance-based merging operators:

Example 2 *Consider four bases $[K_1] = \{0010, 0100\}$, $[K_2] = \{0001, 0100\}$, $[K_3] = \{0111, 0100\}$, and $[K_4] = \{1011, 0100\}$. $E = \{K_1, K_2, K_3, K_4\}$. The only possible worlds are $[\mu] = \{0011, 0000\}$.*

ω		0011		0000	
$\mathsf{diff}(\omega, K_1)$	$d_H(\omega, K_1)$	$\{\{d\}\}$	1	$\{\{b\},\{c\}\}$	1
$\mathsf{diff}(\omega, K_2)$	$d_H(\omega, K_2)$	$\{\{c\}\}$	1	$\{\{b\},\{d\}\}$	1
$\mathsf{diff}(\omega, K_3)$	$d_H(\omega, K_3)$	$\{\{b\}\}$	1	$\{\{b\}\}$	1
$\mathsf{diff}(\omega, K_4)$	$d_H(\omega, K_4)$	$\{\{a\},\{b,c,d\}\}$	1	$\{\{b\},\{a,c,d\}\}$	1
$\mathsf{diff}(\omega, E)$	$d_{H,\Sigma}(\omega, E)$	$\{\{b,c,d\}\}$	4	$\{\{b\}\}$	4

Table 2: Computations of $\Delta_\mu^{\mathsf{diff},\oplus}(E)$ and $\Delta_\mu^{d_H,\Sigma}(E)$

Computations of the merged bases for operators $\Delta^{\mathsf{diff},\oplus}$ and $\Delta_\mu^{d_H,\Sigma}$ are summed up in Table 2 ($\Delta_\mu^{d_H,\Sigma}$ is the distance-based merging operator relying on the Hamming distance and using sum as an aggregation function (Revesz 1997; Konieczny & Pino Pérez 2002a)).
We get $[\Delta_\mu^{\mathsf{diff},\oplus}(E)] = \{0000\}$, while $[\Delta_\mu^{d_H,\Sigma}(E)] = \{0011, 0000\}$.
Clearly the Hamming distance d_H does not discriminate between the two possible worlds, which can be problematic. Here all the agents agree on what they disagree with 0000 *(i.e., the conflict is on b), while this is not the case for* 0011*. Operators based on the Hamming distance cannot make this distinction. As one can check in Table 2 the Hamming distances of the interpretations to the bases are all identical and equal to* 1*, whereas the* diff *distance exhibits the fact that there is less conflict on* 0000 *than on* 0011 *(while flipping the variable b in* 0000 *is enough to obtain a model of all the bases, it is not the case with* 0011*).*

Beyond the IC postulates, $\Delta^{\mathsf{diff},\oplus}$ satisfies also an interesting additional logical property:

Definition 7 *A merging operator Δ satisfies the* temperance *property iff for every profile $\{K_1, \ldots, K_n\}$:*

$\Delta_\top(\{K_1, \ldots, K_n\})$ *is consistent with each K_i* **(temperance)**

Proposition 3 $\Delta^{\mathsf{diff},\oplus}$ *satisfies* **(temperance)**.

This proposition shows that the merged base obtained using $\Delta^{\mathsf{diff},\oplus}$ is consistent with every base of the profile (when there is no integrity constraint). This proposition also gives an additional explanation to the fact that $\Delta^{\mathsf{diff},\oplus}$ does not satisfy **(IC6)**, since temperance is not compatible with this postulate.

Proposition 4 *There is no merging operator satisfying all of* **(IC2)**, **(IC6)**, *and* **(temperance)**.

It is worth noting that the temperance property is not satisfied by many merging operators. In particular, as implied by the previous proposition, none of the IC merging operators satisfies temperance. Interestingly, the temperance property shows that $\Delta^{\mathsf{diff},\oplus}$ can be viewed as a kind of negotiation operator, which can be used for determining the most consensual parts of the bases of all agents. This can prove useful for defining new negotiation operators, as studied for instance in (Zhang *et al.* 2004; Meyer *et al.* 2004b; 2004a; Booth 2001; 2006; Konieczny 2004).

Strategy-Proofness

Let us now investigate how robust $\Delta^{\mathsf{diff},\oplus}$ is with respect to manipulation. Intuitively, a merging operator is strategy-proof if and only if, given the beliefs/goals of the other agents, reporting untruthful beliefs/goals does not enable an agent to improve her satisfaction. A formal counterpart of this idea is given in (Everaere, Konieczny, & Marquis 2004; 2007):

Definition 8 *Let i be a satisfaction index, i.e., a total function from $\mathcal{L} \times \mathcal{L}$ to $I\!R$. A merging operator Δ is* strategy-proof for i *if and only if there is no integrity constraint μ, no profile $E = \{K_1, \ldots, K_n\}$, no base K and no base K' such that*

$$i(K, \Delta_\mu(E \sqcup \{K'\})) > i(K, \Delta_\mu(E \sqcup \{K\})).$$

Clearly, there are numerous different ways to define the satisfaction of an agent given a merged base. While many *ad hoc* definitions can be considered, the following three indexes are meaningful when no additional information are available (Everaere, Konieczny, & Marquis 2004; 2007):

Definition 9

- $i_{d_w}(K, K_\Delta) = \begin{cases} 1 & \text{if } K \wedge K_\Delta \text{ is consistent,} \\ 0 & \text{otherwise.} \end{cases}$
- $i_{d_s}(K, K_\Delta) = \begin{cases} 1 & \text{if } K_\Delta \models K, \\ 0 & \text{otherwise.} \end{cases}$
- $i_p(K, K_\Delta) = \begin{cases} \frac{\#([K]\cap[K_\Delta])}{\#([K_\Delta])} & \text{if } \#([K_\Delta]) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$

For the weak drastic index (i_{d_w}), the agent is considered satisfied as soon as its beliefs/goals are consistent with the merged base. For the strong drastic index (i_{d_s}), in order to be satisfied, the agent must impose her beliefs/goals to the whole group. The last index ("probabilistic index" i_p) is not a Boolean one, leading to a more gradual notion of satisfaction. The more compatible the merged base with the agent's base the more satisfied the agent. The compatibility degree of K with K_Δ is the (normalized) number of models of K that are models of K_Δ as well.

Proposition 5 *In the general case $\Delta^{\mathsf{diff},\oplus}$ is not strategy-proof for any of the three indexes i_{d_w}, i_{d_s} and i_p. When there is no integrity constraint (i.e., $\mu \equiv \top$), $\Delta^{\mathsf{diff},\oplus}$ is strategy-proof for i_{d_w}, but still not strategy-proof for i_{d_s} or i_p.*

Most of the model-based operators are not strategy-proof, even in very restricted situations (Everaere, Konieczny, & Marquis 2007). For example, $\Delta^{d_H,\Sigma}$ or $\Delta^{d_H,Gmin}$, which are the best model-based operators with respect to strategy-proofness, are not strategy-proof for i_{d_w}, even if $\mu \equiv \top$. $\Delta^{\mathsf{diff},\oplus}$ performs slightly better than any of them with this respect.

Complexity Issues

Let us consider now the complexity issue for the inference problem from a $\Delta^{\mathsf{diff},\oplus}$-merged base. We assume the reader acquainted with basics of complexity theory (see (Papadimitriou 1994)).

Formally, let us consider the following decision problem MERGE($\Delta^{\mathsf{diff},\oplus}$):

- **Input:** A triple $\langle E,\mu,\alpha\rangle$ where $E=\{K_1,\dots,K_n\}$ is a profile, $\mu\in\mathcal{L}$ is a formula, and $\alpha\in\mathcal{L}$ is a formula.
- **Question:** Does $\Delta^{\mathsf{diff},\oplus}_{\mu}(E)\models\alpha$ hold?

Proposition 6 MERGE($\Delta^{\mathsf{diff},\oplus}$) *is* Π^p_2*-complete.*

This result shows that $\Delta^{\mathsf{diff},\oplus}$ is computationally harder than usual distance-based operators, but is at the same complexity level as many formula-based operators (Konieczny, Lang, & Marquis 2004), and as complex as the corresponding revision operator (see Proposition 1) (Eiter & Gottlob 1992).

Related Work: Consistency-Based Operators

In (Delgrande & Schaub 2007) two consistency-based merging operators, based on a default inference relation, are proposed. The idea is to use a specific language for each of the bases (disjoint from all other), so as to make their union consistent, and then to add as much default equivalence as possible in order to identify the corresponding variables of the different languages.

At a first glance, these operators seem very close to $\Delta^{\mathsf{diff},\oplus}$, since they try to maximise the agreement between the bases at the variable level, whereas $\Delta^{\mathsf{diff},\oplus}$ tries to minimize the conflict. Furthermore, these two operators satisfy also the temperance property. However one can show that all three operators are actually distinct (and even incomparable as to their inferential power).

Let us first give a brief refresher on Delgrande and Schaub's operators. A i-renaming of a language $\mathcal{L}$ is the language $\mathcal{L}^i$, built from the set of propositional variables $\mathcal{P}^i=\{p^i\mid p\in\mathcal{P}\}$, where for each $\alpha\in\mathcal{L}$, α^i is the result of replacing in α each propositional variable $p\in\mathcal{P}$ by the corresponding propositional variable $p^i\in\mathcal{P}^i$. Given a base K, the i-renaming of (the formulas of) K, is denoted K^i.

Definition 10 *Let $E=\{K_1,K_2,\dots,K_n\}$ be a profile.*

- *Let EQ be a maximal (w.r.t $\subseteq$) subset of $\{p^k\Leftrightarrow p^l\mid p\in\mathcal{L}$ and $k,l\in\{1\dots n\}\}$ such that $(\bigwedge_{K_i\in E}K^i_i)\wedge EQ$ is consistent.*
  Then $\{\alpha\mid\forall j\in\{1\dots n\}\ (\bigwedge_{K_i\in E}K^i_i)\wedge EQ\models\alpha^j\}$ is a consistent symmetric belief change extension *of E.*
  The skeptical merging *$\Delta_s(E)$ of E is the intersection of all the consistent symmetric belief change extensions of E.*

ω	$\mathsf{diff}(\omega,K_1)$	$\mathsf{diff}(\omega,K_2)$	$\mathsf{diff}(\omega,K_3)$	$\mathsf{diff}(\omega,E)$
000	$\{\{p,q\},\{q,r\}\}$	$\{\{p\},\{q\}\}$	$\{\emptyset\}$	$\{\{p,q\},\{q,r\}\}$
001	$\{\{q\}\}$	$\{\{p,r\},\{q,r\}\}$	$\{\{r\}\}$	$\{\{q,r\}\}$
010	$\{\{p\},\{r\}\}$	$\{\emptyset\}$	$\{\{q\}\}$	$\{\{p,q\},\{q,r\}\}$
011	$\{\emptyset\}$	$\{\{r\}\}$	$\{\{q,r\}\}$	$\{\{q,r\}\}$
100	$\{\{q\}\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\{q\}\}$
101	$\{\{p,q\},\{q,r\}\}$	$\{\{r\}\}$	$\{\{r\}\}$	$\{\{q,r\}\}$
110	$\{\emptyset\}$	$\{\{q\},\{p\}\}$	$\{\{q\}\}$	$\{\{q\}\}$
111	$\{\{p\},\{r\}\}$	$\{\{q,r\},\{p,r\}\}$	$\{\{q,r\}\}$	$\{\{q,r\}\}$

Table 3: Example 2 - Computation of $\Delta^{\mathsf{diff},\oplus}_{\top}(E)$

- *Let EQ be a maximal (w.r.t $\subseteq$) subset of $\{p^j\Leftrightarrow p\mid p\in\mathcal{L}$ and $j\in\{1\dots n\}\}$ such that $(\bigwedge_{K_i\in E}K^i_i)\wedge EQ$ is consistent.*
  Then $(\bigwedge_{K_i\in E}K^i_i)\wedge EQ$ is a consistent projected belief change extension *of E.*
  The skeptical merging *$\nabla_s(E)$ of E is the intersection of all the consistent projected belief change extensions of E.*

Example 3 *We consider the profile $E=\{K_1,K_2,K_3\}$, with $K_1=(p\wedge q\wedge\neg r)\vee(\neg p\wedge q\wedge r)$, $K_2=(p\wedge\neg q\wedge\neg r)\vee(\neg p\wedge q\wedge\neg r)$ and $K_3=\neg q\wedge\neg r$.*
The computation of $\Delta^{\mathsf{diff},\oplus}_{\top}(E)$ is described in Table 3. We have $\Delta^{\mathsf{diff},\oplus}_{\top}(E)\equiv p\wedge\neg r$.
There are four maximal sets of equivalences for $\Delta_s(E)$:

EQ1 = $\{p^1\Leftrightarrow p^2, p^1\Leftrightarrow p^3, p^2\Leftrightarrow p^3, r^1\Leftrightarrow r^2, r^1\Leftrightarrow r^3, r^2\Leftrightarrow r^3, q^2\Leftrightarrow q^3\}$
EQ2 = $\{p^1\Leftrightarrow p^3, r^1\Leftrightarrow r^2, r^1\Leftrightarrow r^3, r^2\Leftrightarrow r^3, q^1\Leftrightarrow q^2\}$
EQ3 = $\{p^2\Leftrightarrow p^3, r^1\Leftrightarrow r^2, r^1\Leftrightarrow r^3, r^2\Leftrightarrow r^3, q^1\Leftrightarrow q^2\}$
EQ4 = $\{p^1\Leftrightarrow p^2, p^1\Leftrightarrow p^3, p^2\Leftrightarrow p^3, r^2\Leftrightarrow r^3, q^1\Leftrightarrow q^2\}$

So, $\Delta_s(E)\equiv\neg r\vee(\neg p\wedge q)$, and $\Delta_s(E)\not\models\Delta^{\mathsf{diff},\oplus}_{\top}(E)$.
For ∇_s, the maximal sets of equivalences are the following ones ($p\Leftrightarrow p^1\Leftrightarrow p^2\Leftrightarrow p^3$ is used as a concise notation for $p\Leftrightarrow p^1, p\Leftrightarrow p^2, p\Leftrightarrow p^3$ and similarly for the other variables):

$EQ_1=\{p\Leftrightarrow p^1\Leftrightarrow p^2\Leftrightarrow p^3, r\Leftrightarrow r^1\Leftrightarrow r^2\Leftrightarrow r^3, q\Leftrightarrow q^2\Leftrightarrow q^3\}$
$EQ'_1=\{p\Leftrightarrow p^1\Leftrightarrow p^2\Leftrightarrow p^3, r\Leftrightarrow r^1\Leftrightarrow r^2\Leftrightarrow r^3, q\Leftrightarrow q^1\}$,
$EQ_2=\{p\Leftrightarrow p^1\Leftrightarrow p^3, r\Leftrightarrow r^1\Leftrightarrow r^2\Leftrightarrow r^3, q\Leftrightarrow q^1\Leftrightarrow q^2\}$
$EQ_3=\{p\Leftrightarrow p^2\Leftrightarrow p^3, r\Leftrightarrow r^1\Leftrightarrow r^2\Leftrightarrow r^3, q\Leftrightarrow q^1\Leftrightarrow q^2\}$
$EQ_4=\{p\Leftrightarrow p^1\Leftrightarrow p^2\Leftrightarrow p^3, r\Leftrightarrow r^2\Leftrightarrow r^3, q\Leftrightarrow q^1\Leftrightarrow q^2\}$
$EQ'_4=\{p\Leftrightarrow p^1\Leftrightarrow p^2\Leftrightarrow p^3, r\Leftrightarrow r^1, q\Leftrightarrow q^1\Leftrightarrow q^2\}$

So, $\nabla_s(E)\equiv(p\wedge\neg r)\vee(\neg p\wedge q)$, and $\nabla_s(E)\not\models\Delta^{\mathsf{diff},\oplus}_{\top}(E)$.

More generally, we can prove the following statement:

Proposition 7 *$\Delta^{\mathsf{diff},\oplus}_{\top}$, Δ_s, ∇_s are pairwise incomparable with respect to inferential power, i.e., it is not the case that for every profile E, the merged base obtained using one of these operators implies the merged base obtained using another operator among these three ones.*

Conclusion and Perspectives

In this paper we have introduced a family of model-based merging operators, relying on a set-theoretic measure of conflict. We focused on set-product as an aggregation function and considered the corresponding operator $\Delta^{\mathsf{diff},\oplus}$. A feature of this operator, typically not shared by existing model-based operators, is that it satisfies the temperance property, and as a consequence, it is strategy-proof for the weak drastic index when there are no integrity constraints. The price to be paid is a higher complexity than usual model-based operators (but similar to the one of formula-based merging operators (Everaere, Konieczny, & Marquis 2007)).

An important point of this work is that it illustrates the fact that the widely used Hamming distance (and more generally all distance-based operators whatever the distance), can be criticized for aggregation. We show through examples in this paper that using diff can allow to find subtler results.

The main perspective opened by this work is to characterize the merging scenarios requiring such subtler information, and to improve existing merging operators by taking it into account. This work calls for a number of other perspectives. Especially, there are several parameters used in the definition of $\Delta^{\mathsf{diff},\oplus}$ for which alternative choices could be made (especially, other aggregation functions, other minimality criteria for characterizing the models of the merged base). It would be interesting to determine whether some specific choices for these parameters would lead to majority-like operators or arbitration-like operators (Konieczny & Pino Pérez 2002a). Another issue for further research consists in determining rationality conditions on aggregation functions (as it has been achieved for distance-based merging operators). More generally, investigating the properties of the whole family of *diff*-based operators is an interesting issue.

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