Disjunctive Merging: Quota and Gmin Merging Operators

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Abstract

When aggregating information from a group of agents, accepting the pieces of information shared by all agents is a natural requirement. In this paper, we investigate such a unanimity condition in the setting of propositional merging. We discuss two interpretations of the unanimity condition. We show that the first interpretation is captured by existing postulates for merging. But the second interpretation is not, and this leads to the introduction of a new disjunction postulate **(Disj)**. It turns out that existing operators satisfying **(Disj)** do not perform well with respect to the standard criteria used to evaluate merging operators: logical properties, computational complexity and strategy-proofness. To fill this gap, we introduce two new families of propositional merging operators, quota operators and GMIN operators, which satisfy **(Disj)**, and achieve interesting trade-offs with respect to the logical, computational, and strategy-proofness criteria.

Key words: Belief Merging.

1 Introduction

Merging operators aim at defining the beliefs (resp. goals) of a group of agents from their individual beliefs (resp. goals) and some integrity constraints. The merging problem in the propositional setting has been considered in many works, both from the artificial intelligence community and the database community (see e.g.

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[10,26,20,21,3,4,18]). What makes the problem difficult is that agents often have conflicting pieces of information.

Propositional merging is close to important issues considered in social choice theory [1,23,2], especially vote and preference aggregation. Indeed, each agent can be viewed as a voter and her belief/goal base can be considered as a compact representation of a preference relation, which is such that the models of the base are the most preferred alternatives and the countermodels are strictly less preferred than the models. The output of the aggregation step (namely, the merged base) consists of the most preferred alternatives for the group. The set of models of the integrity constraint plays the role of an agenda (a set of available alternatives). In propositional merging, the **(IC)** postulates [18], complemented with the majority postulate **(Maj)** are used as criteria to characterize several meaningful families of operators, like the IC merging operators (those satisfying the **(IC)** postulates) and the IC majority operators (the IC operators satisfying **(Maj)**).

Now, there are several requirements that aggregation methods (including merging techniques and voting rules) are expected to satisfy, and which have been identified as conditions for voting rules and/or rationality postulates for merging. Among them is *unanimity*, asking to accept at the group level the pieces of information shared by all agents. When voting rules are concerned, it simply means that if candidate a is preferred to candidate b by each voter, then candidate a has to be preferred to candidate b for the group.

We can find at least two interpretations of unanimity in the merging framework.

The first one consists in viewing each base as the set of its models, representing the most preferred alternatives of the associated agent. This interpretation of the unanimity principle amounts to keeping as models of the merged base each model of the integrity constraint which satisfies all the bases. This property is ensured by every merging operator satisfying postulate **(IC2)** (one of the **(IC)** postulates), which is strictly more demanding (**(IC2)** requires that, when non empty, the set of models of the merged base consists precisely of the models of the integrity constraint satisfying all the bases).

In the other possible interpretation, when each base is viewed as the set of its logical consequences (i.e., the deductive closure of the base), the unanimity condition states that the set of consequences shared by all agents, must hold for the group. The formal characterization of this condition is what we call the disjunction postulate **(Disj)** for merging since it requires to select the models of the merged base among the models of the agents' bases (unless there is no model of the integrity constraint among them).

(Disj) is expected in some belief merging scenarios, especially when it is assumed that one of the agents is right (her beliefs hold in the actual world). For instance, consider a group of physicians, each of them reporting a prescription for the same patient; it could be harmful for the patient to ”mix” the individual prescriptions in order to obtain a prescription at the group level; requiring **(Disj)** prevents from such trade-offs between prescriptions. On the other hand, **(Disj)** should be avoided when a form of compromise between agents is desired. For instance, suppose that John and Mary want to spend their evening together, but while John would like to go to the sushi bar, then watch a movie, Mary would prefer to eat a risotto, then go to the theater. Requiring **(Disj)** to be satisfied would make some compromises (like going to the sushi bar then to the theater) out of reach.

In the following, we consider the family of *disjunctive* merging operators, i.e., those satisfying **(Disj)**. At the interpretation level, we show that **(Disj)** corresponds to an interpretation of the unanimity condition for countermodels: if all agents agree that some interpretations are countermodels, then the group must also agree on it. We also show that **(Disj)** is independent of the **(IC)** postulates.

Now, existing merging operators satisfying **(Disj)** are typically formula-based merging operators, i.e., operators which select subsets of the union of the given bases. Such operators do not perform well with respect to the standard criteria used to evaluate merging operators, namely logical properties, computational complexity and strategy-proofness. To fill this gap, we introduce two new families of propositional merging operators, quota operators and GMIN operators, which satisfy **(Disj)**, and achieve interesting trade-offs with respect to the logical, computational, and strategy-proofness criteria.

Quota operators rely on a simple idea: any possible world is viewed as a model of the merged base when it satisfies *“sufficiently many”* bases from the given profile (the collection of agents’ bases). *“Sufficiently many”* means either *“at least k”* (any integer, absolute quota), or *“at least $k\%$”* (a relative quota), or finally *“as many as possible”*, and each interpretation gives rise to a specific merging operator. The full family of quota operators is obtained by letting the quota vary. We show that quota operators exhibit good logical properties, have low computational complexity and are strategy-proof.

Each GMIN operator is parameterized by a pseudo-distance, and the family is obtained by letting it vary. Each GMIN operator refines all quota operators, has good logical properties, is mildly complex (i.e., the inference problem is at the first level of the polynomial hierarchy) but is not strategy-proof in the general case.

The rest of the paper is as follows. The next section gives some formal preliminaries. Section 3 discusses the main criteria for evaluating merging operators, and presents some expected logical properties for a merging operator. In Section 4, we formalize the unanimity condition in the propositional merging setting. In Section 5, quota operators are defined and their properties are presented. In Section 6, we define $\triangle^{k_{\max}}$, which is the operator obtained when optimizing the value of the quota. In Section 7, GMIN operators are defined and their properties are presented.

Finally, we conclude this paper in Section 8. Proofs are reported in appendix.

2 Formal Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the usual connectives, including $\top$ (the Boolean constant true) and $\bot$ (the Boolean constant false).

An interpretation (or world) is a total function from $\mathcal{P}$ to $\{0, 1\}$, denoted by a bit vector whenever a strict total order on $\mathcal{P}$ is specified. The set of all interpretations is noted $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $\models$ denotes logical entailment and $\equiv$ denotes logical equivalence. $[\phi]$ denotes the set of models of formula ϕ, i.e., $[\phi] = \{\omega \in \mathcal{W} \mid \omega \models \phi\}$. Conversely, let M be a set of interpretations, φ_M denotes the logical formula (unique up to logical equivalence) whose models are M.

A *base* K denotes the set of beliefs/goals of an agent, it is a finite and consistent set of propositional formulas, interpreted conjunctively. Unless stated otherwise, we identify K with the conjunction of its elements.

A *profile* E represents a group of n agents involved in the merging process. It is a non-empty multi-set of bases $E = \{K_1, \ldots, K_n\}$, hence different agents are allowed to exhibit identical bases. We denote by $\bigwedge E$ the conjunction of bases of E, i.e., $\bigwedge E = K_1 \wedge \ldots \wedge K_n$, and similarly $\bigvee E$ is the disjunction of the bases of E, i.e., $\bigvee E = K_1 \vee \ldots \vee K_n$. A profile E is said to be consistent if and only if $\bigwedge E$ is consistent. Multi-set union is noted $\sqcup$, and multi-set containment relation is noted $\sqsubseteq$. The cardinality of a finite set (or a finite multi-set) A is noted $\#(A)$. We say that two profiles are equivalent, noted $E_1 \equiv E_2$, if there exists a bijection f from E_1 to E_2 such that for every $K \in E_1$, K and $f(K)$ are logically equivalent.

A *merging operator* Δ associates any profile E and some integrity constraints μ to a merged base $\Delta_\mu(E)$. The *integrity constraints* μ consist of a consistent formula the merged base has to satisfy (it may represent some physical laws, some norms, etc.).

A preorder $\leq$ on $\mathcal{W}$ is a reflexive and transitive relation. A preorder on $\mathcal{W}$ is total if $\forall \omega, \omega' \in \mathcal{W}$, $\omega \leq \omega'$ or $\omega' \leq \omega$. Let $\leq$ be a preorder on $\mathcal{W}$, we define the corresponding strict ordering $<$ on $\mathcal{W}$ as $\omega < \omega'$ if and only if $\omega \leq \omega'$ and $\omega' \not\leq \omega$, and the induced equivalence relation (indifference) $\simeq$ on $\mathcal{W}$ is given by $\omega \simeq \omega'$ if and only if $\omega \leq \omega'$ and $\omega' \leq \omega$. We write $\omega \in \min(A, \leq)$ if and only if $\omega \in A$ and there does not exist $\omega' \in A$ s.t. $\omega' < \omega$.

We assume the reader familiar with the complexity classes P, NP and coNP and we consider the following classes located at the first level of the polynomial hierarchy (see [25] for an introduction to complexity theory):

- $\mathsf{BH}(2)$ (also known as DP) is the class of all languages L such that $L = L_1 \cap L_2$, where L_1 is in NP and L_2 in coNP. $\mathsf{BH}(3)$ is the class of all languages L such that $L = L_1 \cup L_2$, where L_1 is in $\mathsf{BH}(2)$ and L_2 in NP. $\mathsf{coBH}(3)$ is the class of all languages L such that $\overline{L} \in \mathsf{BH}(3)$.
- $\Delta_2^p = \mathsf{P}^{\mathsf{NP}}$ is the class of all languages that can be recognized in polynomial time by a deterministic Turing machine equipped with an NP oracle, where an NP oracle solves whatever instance of a problem from NP in unit time.
- $\Theta_2^p = \Delta_2^p[\mathcal{O}(log\ n)]$ is the class of all languages that can be recognized in polynomial time by a deterministic Turing machine using a number of calls to an NP oracle bounded by a logarithmic function of the size of the input.

3 Expected Properties of Merging Operators

Many merging operators have been defined so far. A distinction between model-based operators [26,18,16], which select some interpretations that are the *"closest"* to the bases encoding the beliefs/goals of agents, and formula-based ones [3,4,15], which pick some formulas in the union of the bases is often made [16].

3.1 How to Choose a "Good" Merging Operator?

Each existing merging operator is more or less suited to the various merging scenarios which can be considered. Subsequently, when facing an application for which merging is required, a first difficulty is the choice of a specific merging operator. Among the criteria which can be used to make a valuable choice, are the following ones:

Rationality: A main requirement for adhering to a merging method is that it offers the expected properties of what intuitively *"merging"* means. This calls for sets of rationality postulates and this issue has been addressed in several papers [26,20,18]. In the following, we focus on the rationality postulates given in [18], because they extend other proposals.

Computational complexity: When one looks for a merging operator for an autonomous multi-agent system, a natural requirement is computational efficiency. In the worst case, merging is not a computationally easy task [16], and query answering typically lies at the first or even the second level of the polynomial hierarchy. Computationally easier operators can be obviously preferred to more complex ones. Identifying the computational complexity of the query answering problem for an operator, and restrictions under which it decreases, are important issues to be investigated.

Strategy-proofness: It is usually expected for merging that agents report truthfully their beliefs/goals. For many applications, this assumption can easily be made, in particular when the agents have limited reasoning abilities. However, when rational agents with full inference power are considered, such an assumption must be questioned: agents can be tempted to misreport their beliefs/goals in order to achieve a better merging result from their point of view. Strategy-proof operators must be preferred in such a case.

How much existing merging operators fit the criteria above has been investigated in a number of previous papers. As to rationality, one can look at [26,20,21,15,18,16]. As to computational complexity, see [16,24], and for a study of strategy-proofness of many merging operators see [13] (see also [22] for a related study concerning merging operators for ordinal conditional functions).

The main result of [13] is that strategy-proofness is hard to achieve for merging operators. This result is not so surprising since, in social choice theory, an impossibility theorem (the Gibbard-Sattertwhaite theorem), states that this strategy-proofness task is not achievable, in the general case, when one aggregates preferences [14,27,23]. In [13], it is shown that even under very restrictive assumptions, most of the propositional merging operators from the literature are not strategy-proof.

In the light of these results, it appears that while no merging operator is better than any other operator with respect to all the above criteria, model-based operators [26,18,16] are typically better than formula-based operators [3,4,15]. To be more precise, while operators from both families are typically not strategy-proof, model-based operators are often computationally easier (inference is typically Θ_2^p-complete or Δ_2^p-complete) than formula-based ones (inference can be Π_2^p-hard) [16]. In addition, model-based operators also typically satisfy more rationality postulates than formula-based ones (see [18,15]).

It turns out that the main argument for making use of formula-based operators instead of model-based operators is their disjunctive behavior. [2] A main contribution of this paper is to show that disjunctive merging operators which are much better performers than formula-based ones with respect to the three criteria exist. Especially, we point out two new families of such disjunctive merging operators.

3.2 Logical Properties

The following set of logical properties for merging operators has been presented and discussed in [17,18]:

[2] See Section 4 for a discussion on disjunction.

Definition 1 (IC merging operators) *Let $\triangle$ be a propositional merging operator, E, E_1, E_2 be profiles, K_1, K_2 be bases and μ, μ_1, μ_2 be integrity constraints. Let n be an integer. $\triangle$ is an* IC merging operator *if and only if it satisfies the following postulates:*

(IC0) $\triangle_\mu(E) \models \mu$
(IC1) *If μ is consistent, then $\triangle_\mu(E)$ is consistent*
(IC2) *If $\bigwedge E$ is consistent with μ, then $\triangle_\mu(E) \equiv \bigwedge E \wedge \mu$*
(IC3) *If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\triangle_{\mu_1}(E_1) \equiv \triangle_{\mu_2}(E_2)$*
(IC4) *If $K_1 \models \mu$ and $K_2 \models \mu$, then $\triangle_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent*
(IC5) $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \models \triangle_\mu(E_1 \sqcup E_2)$
(IC6) *If $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$ is consistent, then $\triangle_\mu(E_1 \sqcup E_2) \models \triangle_\mu(E_1) \wedge \triangle_\mu(E_2)$*
(IC7) $\triangle_{\mu_1}(E) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(E)$
(IC8) *If $\triangle_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(E) \models \triangle_{\mu_1}(E) \wedge \mu_2$*

An IC merging operator is said to be an IC majority operator *if it satisfies* **(Maj)**

(Maj) $\exists n \ \triangle_\mu (E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \triangle_\mu(E_2)$

The intuitive meaning of the properties is the following: **(IC0)** ensures that the merged base satisfies the integrity constraints. **(IC1)** states that, if the integrity constraints are consistent, then the merged base has to be consistent. **(IC2)** states that if possible, the merged base is simply the conjunction of the bases with the integrity constraints. **(IC3)** is the principle of irrelevance of syntax: the result of merging has to depend only on the expressed opinions and not on their syntactical presentation. **(IC4)** is a fairness postulate meaning that when one merges *two* bases, one should not give preference to one of them (if the merged base is consistent with one of them, it has to be consistent with the other one.) It is a symmetry condition, which aims to rule out operators which give priority to one of the bases. **(IC5)** expresses the following idea: if profiles are viewed as expressing the beliefs/goals of the agents of a group, then if E_1 (corresponding to a first group) compromises on a set of alternatives which A belongs to, and E_2 (corresponding to a second group) compromises on another set of alternatives which contains A too, then A has to be in the chosen alternatives if we join the two groups. **(IC5)** and **(IC6)** together state that if one could find two subgroups of agents which agree on at least one alternative, then the merged base must be exactly those alternatives the two groups agree on. **(IC7)** and **(IC8)** state that the notion of closeness is well-behaved, i.e., an alternative that is preferred among the possible alternatives ($[\mu_1]$), will remain preferred if one restricts the possible choices ($[\mu_1 \wedge \mu_2]$). The majority postulate **(Maj)** just means that repeating sufficiently many times a subgroup of agents allows it to impose its view to the whole group.

See [17,19] for more explanations about these postulates and the behaviour of the corresponding operators.

4 Unanimity and Disjunction

As explained in the introduction, the unanimity condition for voting rules requires that if a candidate is chosen by every voter from a group then the group should also choose her. In the merging setting, at the interpretation level, available alternatives are the models of the integrity constraint; accordingly, such a **Una**nimity condition on **M**odels can be formalized by

(UnaM) If $\omega \models \mu$ and $\forall K \in E, \omega \models K$, then $\omega \models \triangle_\mu(E)$

It is easy to show that every merging operator satisfying **(IC2)** also satisfies **(UnaM)**.

Now, each propositional base can also be viewed as the (conjunctive) set of its logical consequences. This view gives rise to another interpretation of the **Una**nimity condition, at the **F**ormula level this time.

(UnaF) If $\exists K \in E$ s.t. $\mu \wedge K$ is consistent, then if $\forall K \in E$, $K \models \alpha$, then $\triangle_\mu(E) \models \alpha$

Roughly, this condition states that every formula which is a logical consequence of each base of the given profile E should also be a logical consequence of the merged base. Nevertheless, since one wants to preserve the basic postulates **(IC0)** and **(IC1)**,[3] we require this condition only when there exists at least one base K of E that is consistent with μ. **(UnaF)** turns out to be equivalent to the following (and simpler) **(Disj)** postulate:

(Disj) If $\bigvee E$ is consistent with μ, then $\triangle_\mu(E) \models \bigvee E$

This property clearly states that each model of the merged base must be chosen among the models of the disjunction of the bases, whenever this disjunction is consistent with the constraints.

Proposition 1 **(UnaF)** *and* **(Disj)** *are equivalent.*

Let us call *disjunctive operators* the operators satisfying the condition **(Disj)**.

Interestingly, at the interpretation level, this property is also equivalent to the following **(UnaC)** postulate (**Una**nimity for **C**ountermodels):

(UnaC) If $\bigvee E$ is consistent with μ, then if $\forall K \in E, \omega \not\models K$, then $\omega \not\models \triangle_\mu(E)$

The rationale for **(UnaC)** is to discard from the models of the merged base all the

[3] From an aggregation point of view, **(IC0)** means that the chosen alternatives for the group are in the set of available alternatives and **(IC1)** means that there exist chosen alternatives for the group as soon as the set of available alternatives is not empty.

interpretations which are discarded by each agent of the group, which is a natural requirement.

Proposition 2 **(UnaC)** *and* **(Disj)** *are equivalent.*

The statements of **(UnaM)** and **(UnaC)** have quite a similar structure, but **(UnaM)** expresses a unanimity on models whereas **(UnaC)** is concerned with unanimity on countermodels.

Let us now extend the representation theorem for IC merging operators given in [19] to the case of disjunctive merging operators. Let us first recall the definition of syncretic assignments from [19]:

Definition 2 (syncretic assignments) *A* syncretic assignment *is a total function φ mapping each profile E to a relation $\leq_E$ over interpretations, such that for any $\omega, \omega' \in \mathcal{W}$:*

(1) If $\omega \models \bigwedge E$ and $\omega' \models \bigwedge E$, then $\omega \simeq_E \omega'$
(2) If $\omega \models \bigwedge E$ and $\omega' \not\models \bigwedge E$, then $\omega <_E \omega'$
(3) If $E_1 \equiv E_2$, then $\leq_{E_1} = \leq_{E_2}$
(4) $\forall \omega \in \mathcal{W}$, if $\omega \models K$, then $\exists \omega' \models K'$ s.t. $\omega' \leq_{\{K,K'\}} \omega$
(5) If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega \leq_{E_1 \sqcup E_2} \omega'$
(6) If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$, then $\omega <_{E_1 \sqcup E_2} \omega'$

Let us now introduce a condition which characterizes the disjunctive behavior:

Definition 3 (disjunctive syncretic assignments) *A* disjunctive syncretic assignment *is a syncretic assignment satisfying the following condition:*

(d) If $\omega \models \bigvee E$ and $\omega' \not\models \bigvee E$, then $\omega <_E \omega'$

We derived a representation theorem for disjunctive merging operators:

Proposition 3 *$\triangle$ is a disjunctive IC merging operator (i.e., it satisfies* **(IC0-IC8)** *and* **(Disj)***) if and only if there exists a disjunctive syncretic assignment which maps each profile E to a total preorder $\leq_E$ such that $[\triangle_\mu(E)] = \min([\mu], \leq_E)$.*

It turns out that the disjunction property **(Disj)** is not satisfied by many IC merging operators [18], since most of them allow for "*generating*" some new beliefs/goals from the ones in the bases of the profile (some interpretations which do not satisfy any of the bases can be chosen as models of the merged base). This is justified by the fact that merging operators are sometimes expected to find trade-offs between the agent's views. When this behaviour is unexpected, formula-based merging operators – which satisfy **(Disj)** – can be used, but such operators:

- do not satisfy many rationality postulates [15] (especially **(IC3)** is not satisfied),
- are often hard from a computational point of view [16],

- and are not strategy-proof [13].

Now, at a first glance, a straightforward idea to define disjunctive operators is to enforce the disjunction condition in the integrity constraints of (non-disjunctive) operators. To be more precise:

Definition 4 (**$\triangle^d$**) *Let $\triangle$ be any propositional merging operator. The* "disjunctive" merging operator $\triangle^d$ *induced by* $\triangle$ *is defined by:* $\forall E, \mu, \triangle^d_\mu(E) \equiv \triangle_{(\bigvee E) \wedge \mu}(E)$.

However, there is no guarantee that the resulting operator is valuable from a logical point of view. Indeed, even if one starts with an IC merging operator $\triangle$ (i.e., an operator satisfying all the **(IC)** postulates), one cannot ensure in the general case that $\triangle^d$ is also an IC operator:

Proposition 4 *Let $\triangle$ be an IC merging operator. Then $\triangle^d$ satisfies* **(IC0)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC7)**, **(IC8)** *and* **(Disj)**. *None of* **(IC1)**, **(IC5)**, **(IC6)** *is satisfied in the general case.*

So an important issue is to determine the impact of the new disjunction postulate on the existing families of operators. The key question concerns the independence of the postulate w.r.t. existing ones. The results presented in the following show that **(Disj)** is independent of the IC postulates in the sense that some IC operators (like $\triangle^{d_D,\Sigma}$ [19]) satisfies it, while other IC operators (like $\triangle^{d_H,\Sigma}$ [19]) do not. Since $\triangle^{d_D,\Sigma}$ and $\triangle^{d_H,\Sigma}$ also satisfy the majority postulate, **(Disj)** enables to split the family of IC majority operators into two non-empty subsets.

While **(Disj)** is compatible with the **(IC)** postulates, only few existing operators satisfy both conditions, especially because only few operators satisfy **(Disj)**; indeed, the standard model-based merging operators based on the Hamming distance between interpretations [19] satisfy all the **(IC)** postulates but do not satisfy **(Disj)**. Contrastingly, as explained previously, formula-based merging operators from the literature are typically disjunctive ones, but they do not satisfy all the **(IC)** postulates, they have a high complexity and they are not strategy-proof.

This calls for new disjunctive merging operators satisfying as many **(IC)** postulates as possible, and more generally, performing better than formula-based operators with respect to the complexity and strategy-proofness criteria. In the following we fill this gap by providing two families of new disjunctive merging operators which offer interesting alternatives to formula-based operators in this respect.

5 Quota Operators

A first family consists of *quota operators*. Quota operators rely on a simple idea: any possible world is viewed as a model of the merged base when it satisfies *"suf-*

ficiently many" bases from the given profile.

Definition 5 (quota operators) *Let k be an integer ≥ 0, $E = \{K_1, \ldots, K_n\}$ be a profile, and μ be an integrity constraint. The k-*quota merging operator*, noted $\triangle^k$, is defined in a model-theoretic way as:*

$$[\triangle^k_\mu(E)] = \begin{cases} \{\omega \in [\mu] \mid \forall K_i \in E\ \omega \models K_i\} \text{ if non empty,} \\ \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\} \text{ otherwise.} \end{cases}$$

Essentially, this definition states that the models of the result of the k-quota merging of profile E under constraints μ are the models of μ which satisfy at least k bases of E. When there is no conflict for the merging, i.e., $\bigwedge E \wedge \mu$ is consistent, the result of the merging is simply the conjunction of the bases with the integrity constraint.

Example 1 *Consider a set $\mathcal{P}$ consisting of three atoms and a profile $E = \{K_1, K_2, K_3, K_4\}$ with $[K_1] = \{100, 001, 010, 101\}$, $[K_2] = \{001, 101\}$, $[K_3] = \{100, 000, 011\}$, and $[K_4] = \{111\}$, and the integrity constraints $[\mu] = \mathcal{W} \setminus \{010, 011\}$.*

Using quota operators, we get:

- $[\triangle^1_\mu(E)] = \{000, 001, 100, 101, 111\}$*: the models of the merged base are the models of μ which satisfy at least one base.*
- $[\triangle^2_\mu(E)] = \{001, 100, 101\}$*: the models of the merged base are the models of μ which satisfy at least two bases.*
- $[\triangle^3_\mu(E)] = \emptyset$*: no model of μ satisfies at least three bases.*

Fixing the quota to 0 or 1 leads to operators close to operators known in the literature. Thus, $\triangle^0$ gives the conjunction of the bases with the constraints μ when consistent and μ otherwise. It is called *full meet merging operator* in [17]. This operator leads to giving up all the agents' beliefs as soon as they are conflicting. $\triangle^1$ gives the conjunction of the bases with μ when consistent and the conjunction of μ with the disjunction of the bases otherwise; it is close to the *basic merging operator* [17], and is also definable as a model-based merging operator obtained using the drastic distance and *Max* as aggregation function [16]. The only difference is that $\triangle^1$ gives an inconsistent result when the disjunction of the bases is not consistent with μ, whilst the basic merging operator gives μ in this case.

Here is an equivalent syntactical characterization of each quota operator $\triangle^k_\mu(E)$ (i.e., the result is directly given by a formula) that is obtained from subsets of E. [4] Let us first define the following notation:

[4] *"Subsets"* is to be considered here with respect to multi-set containment; *"Sub multi-sets"* would be more correct but it sounds too bad.

$$\ulcorner n_k \urcorner = \{C \subseteq \{1, \ldots, n\} \mid \#(C) = k\}.$$

Then the following proposition gives a characterization of quota operators:

Proposition 5 *Let k be an integer ≥ 0, $E = \{K_1, \ldots, K_n\}$ be a profile, and μ be an integrity constraint.*

$$\triangle_\mu^k(E) \equiv \begin{cases} \bigwedge E \wedge \mu & \textit{if consistent,} \\ (\bigvee_{C \in \ulcorner n_k \urcorner} (\bigwedge_{j \in C} K_j)) \wedge \mu & \textit{otherwise.} \end{cases}$$

Interestingly, the size of the formula equivalent to $[\triangle_\mu^k(E)]$ given by Proposition 5 is polynomial in $|E| + |\mu|$. Hence, merged bases can be easily compiled as propositional formulas, i.e., turned into an equivalent propositional formula in polynomial space (and even in polynomial time in this case). This property is not shared by many merging operators. Indeed, there are strong connections between belief merging operators (under integrity constraints) and belief revision operators, and it has been shown in [9] some (non-)compilability results for several belief revision operators.

5.1 Logical Properties

Quota merging operators exhibit good logical properties :

Proposition 6 *$\triangle^k$ operators satisfy* **(IC0)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC5)**, **(IC7)**, **(IC8)**, *and* **(Disj)** *if $k > 0$. They do not satisfy* **(IC1)**, **(IC6)** *and* **(Maj)** *in the general case.*

Only two properties of IC merging operators are not satisfied: **(IC1)** since the result of the quota merging can be inconsistent (see Example 1), and **(IC6)**.

Note that it is possible to make **(IC1)** satisfied by requiring that, when no interpretation reaches the quota (i.e., satisfies at least k bases), the merged base is equivalent to the integrity constraints. However, this alternative definition leads to operators which satisfy neither **(Disj)** nor the important postulate **(IC5)**, so we did not consider this option (we do not want to expand further on it here, for the sake of brevity; see nevertheless Proposition 22 in the appendix).

The other postulate which is not satisfied by quota operators, **(IC6)**, is one of the postulates that no formula-based operators satisfy [15]. From this point of view, quota operators have a similar behaviour to that of formula-based operators.

Two other interesting properties can be defined for characterizing more precisely quota operators; the first one is a weakening of **(Maj)**:

(Wmaj) If $\triangle_\mu(E_2)$ is consistent,

$$\text{then } \exists n \ \triangle_\mu\,(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \wedge \triangle_\mu(E_2) \text{ is consistent}$$

That **(Wmaj)** is satisfied by quota operators can be easily explained by the fact that duplicating some bases from a given profile can only weaken the resulting merged base when quota operators are considered.

The second property shows the prominence of the largest maximal consistent subsets of the profile with respect to the merged base. We first need to define maximal consistent subsets:

Definition 6 (maximal consistent subsets)

$$\textsc{maxcons}_\mu(E) = \{M \mid M \sqsubseteq E, \bigwedge M \wedge \mu \textit{ is consistent},$$
$$\textit{and if } M \sqsubset M' \sqsubseteq E, \textit{ then } \bigwedge M' \wedge \mu \textit{ is not consistent}\}$$

We are now ready to define the cardinality property **(Card)**:

(Card) If $M_1, M_2 \in \textsc{maxcons}_\mu(E)$, $\#(M_1) \leq \#(M_2)$, and $\triangle_\mu(E) \wedge M_1$ is consistent, then $\triangle_\mu(E) \wedge M_2$ is consistent

This property can be seen as a kind of majority property. The maximal consistent subsets of bases are the largest (with respect to multi-set inclusion) conflict-free sets of formulas from the bases, and, as such, they play a fundamental role in many approaches to reasoning under inconsistency (see e.g. [8,7,6]). **(Card)** states that the largest sets (with respect to cardinality) among these sets have to be considered so that if the merged base is consistent with a maximal consistent subset M_1, it has to be consistent with every maximal consistent subset M_2 which is larger than M_1.

Unlike **(Wmaj)**, the cardinality postulate **(Card)** is not a weakening of **(Maj)**, even under the **(IC)** conditions, but it is independent of it. Thus, in the following, we show that the $\triangle^{k_{max}}$ operator (cf. Section 6) is an IC majority merging operator satisfying **(Card)**, and that GMIN operators (cf. Section 7) are IC operators which do not satisfy any of **(Wmaj)** or **(Card)** in the general case. On the other hand, the IC majority merging operator $\triangle^{d_H,\Sigma}$ [19] does not satisfy **(Card)**. Indeed, consider the following counter-example: $\mathcal{P} = \{a, b\}$, $E = \{K_1, K_2, K_3\}$ with $K_1 = \{\neg a\}$, $K_2 = \{a \wedge \neg b\}$ and $K_3 = \{(\neg a \wedge b) \vee (a \wedge \neg b)\}$. $\textsc{maxcons}_\top(E)$ contains two elements: $M_1 = \{\neg a, (\neg a \wedge b) \vee (a \wedge \neg b)\}$ and $M_2 = \{a \wedge \neg b, (\neg a \wedge b) \vee (a \wedge \neg b)\}$. Clearly, $\#(M_1) = \#(M_2)$. $\triangle_\top^{d_H,\Sigma}(E) \equiv a \wedge \neg b$ is consistent with M_2 but not with M_1.

Proposition 7 $\triangle^k$ *operators satisfy* **(Card)** *and* **(Wmaj)**.

5.2 Computational Complexity

Let $\triangle$ be a propositional merging operator, we consider the following decision problem $\text{MERGE}(\triangle)$:

- **Input :** a triple $\langle E, \mu, \alpha \rangle$ where $E = \{K_1, \ldots, K_n\}$ is a profile, $\mu \in \mathcal{L}$ is an integrity constraint, and $\alpha \in \mathcal{L}$ is a formula.
- **Question :** Does $\triangle_\mu(E) \models \alpha$ hold?

For quota merging operators, we can prove that:

Proposition 8 $\text{MERGE}(\triangle^k)$ *is* $\mathsf{coBH(3)}$*-complete.*

This $\mathsf{coBH(3)}$-completeness result is obtained even in the restricted case when the query α is a propositional atom and there is no integrity constraints ($\mu \equiv \top$). Note that this complexity class is located at a low level of the Boolean hierarchy. Furthermore, the complexity of $\text{MERGE}(\triangle^k)$ decreases to coNP in the degenerate cases whenever k is not lower than the number of bases of E or under the restriction when $\bigwedge E \wedge \mu$ is known at start as inconsistent.

5.3 Strategy-Proofness

Let us now investigate how robust quota operators are with respect to manipulation. Intuitively, a merging operator is strategy-proof if and only if, given the beliefs/goals of the other agents, reporting untruthful beliefs/goals does not enable an agent to improve her satisfaction. A formal definition suited to this intuition is given in [13]:

Definition 7 (strategy-proofness) *Let* i *be a satisfaction index, i.e., a total function from* $\mathcal{L} \times \mathcal{L}$ *to* $\mathbb{R}$*. A merging operator* Δ *is* strategy-proof for i *if and only if there is no integrity constraint* μ*, no profile* $E = \{K_1, \ldots, K_n\}$*, no base* K *and no base* K' *such that* $i(K, \Delta_\mu(E \sqcup \{K'\})) > i(K, \Delta_\mu(E \sqcup \{K\}))$.

Clearly, there are numerous ways to define the satisfaction of an agent given a merged base. While many *ad hoc* definitions can be considered, the following three indexes from [13] are meaningful when no additional information is available:

Definition 8 (indexes) *Let* K*,* $K_\triangle$ *be two bases:*

- $i_{d_w}(K, K_\Delta) = \begin{cases} 1 & \textit{if } K \wedge K_\Delta \textit{ is consistent,} \\ 0 & \textit{otherwise.} \end{cases}$
- $i_{d_s}(K, K_\Delta) = \begin{cases} 1 & \textit{if } K_\Delta \models K, \\ 0 & \textit{otherwise.} \end{cases}$
- $i_p(K, K_\Delta) = \begin{cases} \frac{\#([K] \cap [K_\Delta])}{\#([K_\Delta])} & \textit{if } \#([K_\Delta]) \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$

For the weak drastic index (i_{d_w}), the agent is considered fully satisfied as soon as her beliefs/goals are consistent with the merged base. For the strong drastic index (i_{d_s}), in order to be fully satisfied, the agent must impose her beliefs/goals to the group. The probabilistic index i_p is not a Boolean one, leading to a more gradual notion of satisfaction. The more similar to the agent's base the merged base, the more satisfied the agent. The similarity degree of K with K_Δ is the (normalized) number of models of K that are models of K_Δ as well.

These three indexes are not fully independent: ensuring strategy-proofness for i_p is sufficient to ensure strategy-proofness for the two drastic indexes (provided that the merging operator satisfies **(IC1)**) [13].

Proposition 9 *Quota merging operators are strategy-proof for i_p, i_{d_w} and i_{d_s}.*

Strategy-proofness is hard to achieve, as illustrated in social choice theory, for the aggregation of preference relations, by the Gibbard-Satterthwaite impossibility theorem [14,27,23]. Accordingly, it has been shown in [13] that most of existing merging operators are not strategy-proof. So this result is an interesting one for quota operators.

5.4 Absolute and Relative Quotas

In the definition of quota merging operators, an absolute threshold, i.e., a fixed integer not depending on the number of bases in the profile, has been used. But it can prove also sensible to express quota in a relative manner, and to define the models of the merged base as the interpretations satisfying at least half (or two thirds, or any wanted ratio) of the initial bases. This technique is close to a well-known voting method used in social choice theory, namely voting in committees [5]. Let us call such operators k-ratio merging operators:

Definition 9 (ratio operators) *Let k be a real number such that $0 \leq k \leq 1$, $E = \{K_1, \ldots, K_n\}$ be a profile, and μ be an integrity constraint. The* k-ratio merging operator, *denoted* $\overline{\triangle}^k$*, is defined in a model-theoretic way as:*

$$[\overline{\triangle}_\mu^k(E)] = \begin{cases} \{\omega \in [\mu] \mid \forall K_i \in E\ \omega \models K_i\} \text{ if non empty,} \\ \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k \times n\} \text{ otherwise.} \end{cases}$$

Example 1 (continued) $E = \{K_1, K_2, K_3, K_4\}$ *with* $[K_1] = \{100, 001, 010, 101\}$, $[K_2] = \{001, 101\}$, $[K_3] = \{100, 000, 011\}$*, and* $[K_4] = \{111\}$*, and the integrity constraints* $[\mu] = \mathcal{W} \setminus \{010, 011\}$.
$[\overline{\triangle}_\mu^{0.25}(E)] = \{001, 100, 101, 000, 111\}$, $[\overline{\triangle}_\mu^{0.3}(E)] = [\overline{\triangle}_\mu^{0.5}(E)] = \{001, 100, 101\}$.

One can quickly figure out the close connections between the two families of quota merging operators (the one based on absolute quota and the other one on relative quota, or ratio). Each ratio merging operator corresponds to a family of quota merging operators (one for each possible cardinality of the profile). And for each cardinality of a profile, each (absolute) quota merging operator corresponds to a family of ratio merging operators. The exact correspondence between absolute quotas and ratios is made precise by the following proposition:

Proposition 10 *Let* E *be any profile such that* $\#(E) = n$ *and let* μ *be an integrity constraint.*

(1) Let k *be a real number such that* $0 \leq k \leq 1$*. We have* $\overline{\triangle}_\mu^k(E) \equiv \triangle_\mu^{\lfloor k \times n \rfloor}(E)$.
(2) Let k *be an integer* ≥ 0*. If* $k < n$ *then for any* $\overline{k} \in [\frac{k}{n}, \frac{k+1}{n})$*, we have* $\triangle_\mu^k(E) \equiv \overline{\triangle}_\mu^{\overline{k}}(E)$*; otherwise, we have* $\triangle_\mu^k(E) \equiv \overline{\triangle}_\mu^1(E)$.

Although the intuitive motivations of the two definitions of these families look different, it turns out that ratio merging operators have exactly the same properties with respect to computational complexity and strategy-proofness as (absolute) quota merging operators (this is a direct consequence of Proposition 10). Only some logical properties are different.

Proposition 11 $\overline{\triangle}^k$ *operators satisfy* **(IC0)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC5)**, **(IC7)**, **(IC8)**, *and* **(Card)**. *They satisfy* **(Maj)** *if* $k > 0$ *and* **(Disj)** *if* $k \geq \frac{1}{\#(E)}$. *They do not satisfy* **(IC1)** *and* **(IC6)** *in the general case.*

Proposition 11 shows that all ratio merging operators satisfy **(Maj)**, except $\overline{\triangle}^0$, which coincides with $\triangle^0$, and is trivial (as explained before). This highly contrasts with quota operators which do not satisfy **(Maj)**.

6 The $\triangle^{k_{\max}}$ Operator

Now, regardless of whether the chosen quota is absolute or not, an important point is the choice of its value. Let us first observe that quota merging operators lead to a sequence of merged bases that is monotonic with respect to logical entailment:

Proposition 12 *Let E be a profile, μ be an integrity constraint. We have $\triangle_{\mu}^{k+1}(E) \models \triangle_{\mu}^{k}(E)$ for all integers $k \geq 0$.*

Each time k is increased, the resulting merged base is either equivalent to the one obtained for the previous value of k or is logically stronger. In our finite propositional framework, the sequence $(\triangle_{\mu}^{k}(E))_{(k \geq 0)}$ is obviously stationary from some stage. The value for which it becomes stationary is not interesting in itself, since the corresponding merged base is either equivalent to the conjunction of the bases of the profile (with the constraints), or to the inconsistent base. But an interesting value of k is the one leading to the last nontrivial merged base.

Definition 10 ($\triangle^{k_{max}}$) *Let $E = \{K_1, \ldots, K_n\}$ be a profile, μ be an integrity constraint. Let $k_{\max} = \max(\{i \leq \#(E) \mid \triangle_{\mu}^{i}(E) \not\models \bot\})$. $\triangle^{k_{max}}$ is defined in a model-theoretic way as:*

$$[\triangle_{\mu}^{k_{max}}(E)] = \begin{cases} \{\omega \in [\mu] \mid \forall K_i \in E \; \omega \models K_i\} \text{ if non empty,} \\ \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) = k_{\max}\} \text{ otherwise.} \end{cases}$$

While very close to quota operators, the resulting operator $\triangle^{k_{\max}}$ is not a true quota operator since the value of k_{max} is not given *a priori*, but depends on E and μ.

Example 1 (continued) $[\triangle_{\mu}^{k_{\max}}(E)] = \{001, 100, 101\}$.

At first glance, $\triangle^{k_{\max}}$ looks similar to the formula-based operator Δ^{C4} which selects cardinality-maximal subsets in the union of the bases from the profile [15,3,4]. However, $\triangle^{k_{\max}}$ and Δ^{C4} are distinct: while both operators satisfy **(Disj)**, $\triangle^{k_{\max}}$ satisfies **(IC3)** and **(Maj)** (see Proposition 1) and Δ^{C4} satisfies none of them [15]. Indeed, $\triangle^{k_{\max}}$ belongs to two important families of model-based merging operators, namely the $\triangle^{\Sigma}$ family and the $\triangle^{\text{GMAX}}$ family when the *drastic distance* [5] d_D is used [19]:

Proposition 13 $\triangle^{k_{\max}} = \triangle^{d_D,\Sigma} = \triangle^{d_D,\text{GMAX}}$.

Accordingly, $\triangle^{k_{\max}}$ exhibits many expected logical properties:

Lemma 1 $\triangle^{k_{\max}}$ *satisfies* **(IC0 - IC8)***,* **(Maj)***,* **(Disj)** *and* **(Card)**.

[5] For any $\omega_1, \omega_2 \in \mathcal{W}$, $d_D(\omega_1, \omega_2) = 0$ if $\omega_1 = \omega_2$ and $d_D(\omega_1, \omega_2) = 1$ otherwise.

Since $\triangle^{k_{\max}}$ is obtained by considering the problem of optimizing the quota (for quota operators, k is given, so it does not need to be computed), the corresponding inference problem is computationally harder than the inference problem for quota operators (under the standard assumptions of complexity theory):

Lemma 2 MERGE$(\triangle^{k_{\max}})$ *is* Θ_2^p*-complete.*

Clearly enough, if k_{max} is computed during an off-line pre-processing stage and becomes part of the input afterwards, the complexity falls down to coNP.

Now, as to strategy-proofness, the $\triangle^{k_{\max}}$ operator exhibits all the good properties of quota operators:

Lemma 3 $\triangle^{k_{\max}}$ *is strategy-proof for the three indexes* i_p, i_{d_w} *and* i_{d_s}.

The result directly follows from the fact that $\triangle_\mu^{k_{max}}$ coincides with $\triangle_\mu^{d_D,\Sigma}$ (Proposition 13), that is known to be strategy-proof [13].

7 GMIN Operators

Starting from $\triangle^{k_{\max}}$, one could wonder whether it is possible to constrain further the quota operators so as to get operators with a higher inferential power, i.e,. allowing more conclusions to be obtained. In this section we provide a family GMIN of such operators. As far as we know, this family has never been considered up to now in a propositional merging context.

Each operator $\triangle^{d,\text{GMIN}}$ of the GMIN family is parameterized by a pseudo-distance d:

Definition 11 (pseudo-distances) *A pseudo-distance between interpretations is a function* d *from* $\mathcal{W} \times \mathcal{W}$ *to* $\mathbb{N}$ *such that for every* ω_1, $\omega_2 \in \mathcal{W}$:

- $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$, *and*
- $d(\omega_1, \omega_2) = 0$ *if and only if* $\omega_1 = \omega_2$.

Any pseudo-distance between interpretations d *induces a "distance" between an interpretation* ω *and a formula* K *given by* $d(\omega, K) = \min_{\omega' \models K} d(\omega, \omega')$.

Examples of such pseudo-distances are the drastic distance d_D (cf. Footnote 5), and the *Dalal distance* [11], noted d_H, that is the Hamming distance between interpretations ($d(\omega_1, \omega_2)$ is equal to the number of atoms on which ω_1 and ω_2 differ).

Then $\triangle^{d,\text{GMIN}}$ operators are defined as:

Definition 12 (GMIN operators) *Let* d *be a pseudo-distance,* μ *an integrity constraint,* $E = \{K_1, \ldots, K_n\}$ *a profile and let* ω *be an interpretation. The "distance"*

ω	K_1	K_2	K_3	K_4	$d_{d_H,\text{GMIN}}(\omega, E)$
000	1	1	0	3	$(0,1,1,3)$
001	0	0	1	2	$(0,0,1,2)$
100	0	1	0	2	$(0,0,1,2)$
101	**0**	**0**	**1**	**1**	$(\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{1})$
110	1	2	1	1	$(1,1,1,2)$
111	1	1	1	0	$(0,1,1,1)$

Table 1
$\triangle^{d_H,\text{GMIN}}$ operator.

$d_{d,\text{GMIN}}(\omega, E)$ is defined as the list of numbers $(d_1, \ldots, d_n)$ obtained by sorting in increasing order the multi-set $\{d(\omega, K_i) \mid K_i \in E\}$. The models of $\triangle_\mu^{d,\text{GMIN}}(E)$ are the models ω of μ such that $d_{d,\text{GMIN}}(\omega, E)$ is minimal with respect to the lexicographic ordering $\leq_{lex}$ induced by the natural order, [6] *i.e.,*

$$\omega \leq_E^{d,\text{GMIN}} \omega' \textit{ iff } d_{d,\text{GMIN}}(\omega, E) \leq_{lex} d_{d,\text{GMIN}}(\omega', E)$$

and

$$[\triangle_\mu^{d,\text{GMIN}}(E)] = \min([\mu], \leq_E^{d,\text{GMIN}}).$$

Example 1 (continued) $[\triangle_\mu^{d_D,\text{GMIN}}(E)] = \{001, 100, 101\}$. $[\triangle_\mu^{d_H,\text{GMIN}}(E)] = \{101\}$. *The computations are reported in Table 1. Each row corresponds to a model ω of the constraint μ. Each column K_i gives the distance $d_H(\omega, K_i)$ between a model ω of μ and the base K_i. The boldface row corresponds to the model of μ which minimizes $d_{d_H,\text{GMIN}}(., E)$.*

As stated by the following proposition, each GMIN operator refines $\Delta^{k_{\max}}$. As a consequence, each of them refines also every quota merging operator which does not lead to an inconsistent merged base, thanks to Proposition 12.

Proposition 14 *For any pseudo-distance d, any integrity constraint μ and any profile E, $\triangle_\mu^{d,\text{GMIN}}(E) \models \Delta_\mu^{k_{\max}}(E)$.*

The choice of the drastic distance leads exactly to $\Delta^{k_{\max}}$:

Proposition 15 $\triangle^{d_D,\text{GMIN}} = \Delta^{k_{\max}}$.

Furthermore, GMIN operators are IC merging operators:

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[6] We give here the definition of GMIN by means of lists of numbers. Using Ordered Weighted Averages, one could define it directly from distances (numbers) so as to fit the definition of model-based operators (see [16]).

Proposition 16 *Let d be any pseudo-distance. $\triangle^{d,\text{GMIN}}$ satisfies* **(IC0 - IC8)***, and* **(Disj)***. It does not satisfy* **(Card)***,* **(Maj)** *and* **(Wmaj)** *in the general case.*

The significance of Proposition 14 is improved by the fact that $\triangle^{d,\text{GMIN}}$ satisfies **(IC1)**; Indeed, together with Proposition 16, it shows that $\triangle^{d,\text{GMIN}}$ preserves at least all the information from the bases as those preserved by $\Delta^{k_{\max}}$, without leading to an inconsistent merged base.

As shown by the previous proposition, each operator $\triangle^{d,\text{GMIN}}$ satisfies **(Disj)**. This is also the case of formula-based merging operators. However, GMIN operators appear as much better operators than formula-based ones with respect to logical properties. Indeed, while formula-based merging operators typically fail to satisfy important logical properties [15], $\triangle^{d,\text{GMIN}}$ operators are IC merging operators (i.e., they satisfy **(IC0)-(IC8)**).

It is also interesting to observe that each $\triangle^{d,\text{GMIN}}$ satisfies a second weakening of **(Maj)**:

(Wmaj2) If $(\bigvee E_2) \wedge \mu$ is consistent, then $\exists n \ \triangle_\mu(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \triangle_\mu(E_2)$

(Wmaj2) adds just a precondition to the usual **(Maj)** property. It asks to listen to the majority when at least one base of this majority is consistent with the integrity constraints.

Note that the two weakenings of the postulate **(Maj)**, namely **(Wmaj)** and **(Wmaj2)**, are independent of each other. For example, quota operators satisfy **(Wmaj)** but they do not satisfy **(Wmaj2)**, and GMIN operators satisfy **(Wmaj2)** but they do not satisfy **(Wmaj)**.

Proposition 17 *Let d be any pseudo-distance. $\triangle^{d,\text{GMIN}}$ satisfies* **(Wmaj2)**.

To conclude with the logical properties, while at the definition level, GMIN operators are close to the well-known GMAX arbitration operators [19]: the difference between them just lies in the choice of distinct aggregation functions from the lexicographic family, GMIN vs. GMAX. However, the behaviours of $\triangle^{d,\text{GMIN}}$ and $\triangle^{d,\text{GMAX}}$ from a logical point of view are quite different in general. Thus, though both operators are IC merging ones, $\triangle^{d_H,\text{GMIN}}$ satisfies **(Disj)** but does not satisfy the arbitration postulate **(Arb)** (see [19] for details about **(Arb)**), while $\triangle^{d_H,\text{GMAX}}$ satisfies **(Arb)** but does not satisfy **(Disj)**. Accordingly, each operator is suited to one of the merging scenarios sketched in Section 1, but not to both of them.

Let us now investigate the strategy-proofness issue for GMIN operators. In the general case, strategy-proofness of quota merging operators is lost. As shown in [13], even if an operator is not strategy-proof in the general case, it may happen that strategy-proofness is achievable under some restrictions. It turns out that strategy-proofness can be guaranteed for GMIN operators, but only in some very specific cases:

Proposition 18 *Let d be any pseudo-distance.*

- $\triangle^{d,\text{GMIN}}$ *is strategy-proof for i_p if every base from the profile E is complete (i.e., each base has a unique model).*
- $\triangle^{d,\text{GMIN}}$ *is strategy-proof for the indexes i_{d_w} and i_{d_s} if every base from the profile E is complete, or if $\#(E) = 2$ and $\mu \equiv \top$.*

Considering some specific distances, additional strategy-proofness results can be obtained:

Proposition 19

- $\triangle^{d_D,\text{GMIN}}$ *is strategy-proof for the three indexes i_{d_w}, i_p and i_{d_s}.*
- $\triangle^{d_H,\text{GMIN}}$ *is strategy-proof for i_p if and only if every base from the profile E is complete.*
- $\triangle^{d_H,\text{GMIN}}$ *is strategy-proof for i_{d_w} and i_{d_s} if and only if every base from the profile E is complete or if $\#(E) = 2$ and $\mu \equiv \top$.*

Although GMIN operators can be seen as improvements of quota operators in the sense that they allow to draw more conclusions, this gain in inferential power has to be paid by the lost of most of the strategy-proofness properties, which are a main advantage of quota operators.

As to the strategy-proofness criterion, the behaviour of GMIN operators is quite good compared to other model-based operators [13].

Finally, let us turn to the computational complexity criterion. The next proposition is a direct consequence of a result from [16]:

Proposition 20 *Assume that the pseudo-distance $d(\omega_1, \omega_2)$ of any pair of interpretations ω_1 and ω_2 can be computed in time polynomial in $|\omega_1| + |\omega_2|$. Then* MERGE$(\triangle^{d,\text{GMIN}})$ *is in Δ_2^p.*

For specific choices of d, more precise results can be derived:

Proposition 21

- MERGE$(\triangle^{d_D,\text{GMIN}})$ *is Θ_2^p-complete.*
- MERGE$(\triangle^{d_H,\text{GMIN}})$ *is Δ_2^p-complete.*

As one may expect, the complexity of inference for GMIN operators is slightly higher than the complexity of inference for quota operators (under the usual assumptions of complexity theory). However, it remains at the first level of the polynomial hierarchy under reasonable requirements on the pseudo-distance, and is comparable to the complexity of model-based operators. This shows that GMIN operators are typically better merging operators than formula-based ones with respect to the computational dimension (for many formula-based operators inference is at the second level of the polynomial hierarchy).

8 Conclusion

In this paper, we have considered the standard unanimity condition for preference aggregation in the setting of propositional merging. We have shown that this unanimity condition can be interpreted in two different ways in the merging framework.

The first one is about the models of the bases. It is already captured by the usual merging postulates, and many existing merging operators satisfy it.

The second one is about the countermodels of the bases. While it is very natural, it is not captured by existing postulates. This led us to introduce a new **(Disj)**unction postulate.

Unfortunately, only few operators satisfy **(Disj)**, and they are typically formula-based operators. As such, they perform badly with respect to the standard criteria used to evaluate merging operators, namely, logical properties, computational complexity and strategy-proofness. Actually, the very argument to make use of formula-based operators for a merging issue is that they are disjunctive in essence. This is not very satisfying and this calls for disjunctive operators achieving better trade-offs with respect to the three criteria.

In order to fill this gap, we have introduced two new families of disjunctive model-based merging operators, namely quota operators and GMIN operators. Investigating their properties, we have shown that these operators are interesting alternatives to formula-based merging operators. Thus, both quota and GMIN operators have a complexity lying at the first level of the polynomial hierarchy, while the family of formula-based merging operators does not offer this property. Quota operators are strategy-proof unlike the great majority of other existing merging operators. Furthermore, even if GMIN operators are not strategy-proof in the general case, these operators perform quite well with respect to this criterion compared to other model-based operators. Finally, GMIN operators are IC merging operators while formula-based merging operators typically fail to satisfy IC constraints. Accordingly, our results show that formula-based merging operators can be profitably replaced by GMIN operators.

A Proofs

Proof of Proposition 1: (UnaF) is equivalent to **(Disj)**.

- Let us prove that **(UnaF)** implies **(Disj)**. Suppose that $\bigvee E$ is consistent with μ, then take the formula $\alpha = \bigvee E$. Then clearly $\forall K \in E$, $K \models \alpha$. Since $\bigvee E$ is consistent with μ there is at least one K s.t. K is consistent with μ. So by **(UnaF)** we get $\triangle_\mu(E) \models \alpha$, that is exactly the conclusion of **(Disj)**.
- Let us show that **(Disj)** implies **(UnaF)**. Suppose that $\exists K \in E$ s.t. $\mu \wedge K$ is consistent and that $\forall K \in E$, $K \models \alpha$. As $\exists K \in E$ s.t. $\mu \wedge K$ is consistent, we have $\bigvee E$ consistent with μ. So by **(Disj)** $\triangle_\mu(E) \models \bigvee E$. Since the hypothesis $\forall K \in E$, $K \models \alpha$ implies that $\bigvee E \models \alpha$, by transitivity we obtain $\triangle_\mu(E) \models \alpha$.

$\square$

Proof of Proposition 2: (Disj) is equivalent to **(UnaC)**.

- Let us prove that **(Disj)** implies **(UnaC)**. Suppose that $\bigvee E$ is consistent with μ. Let ω be an interpretation such that $\forall K \in E, \omega \not\models K$. Then $\omega \not\models \bigvee E$. Since $\Delta_\mu(E) \models \bigvee E$, if $\omega \not\models \bigvee E$, then $\omega \not\models \Delta_\mu(E)$: this proves **(UnaC)**.
- Let us prove that **(UnaC)** implies **(Disj)**. Suppose that $\bigvee E$ is consistent with μ. Let ω be an interpretation such that $\omega \models \Delta_\mu(E)$. If $\forall K \in E, \omega \not\models K$, then ω cannot be a model of $\Delta_\mu(E)$, so $\exists K \in E$, $\omega \models K$, and consequently $\omega \models \bigvee E$: this proves **(Disj)**.

$\square$

Proof of Proposition 3: The fact that conditions (1-6) of syncretic assignments corresponds to IC postulates **(IC0-IC8)** is a consequence of the representation theorem given in [19]. So it remains to show that condition (d) corresponds to postulate **(Disj)**.

(If) Consider a merging operator $\triangle$ defined from a disjunctive syncretic assignment. Let us show that $\triangle$ satisfies **(Disj)**. Suppose that $\bigvee E$ is consistent with μ. This means that $\exists \omega \in \mathcal{W}$ such that $\omega \models \bigvee E$ and $\omega \models \mu$. Towards a contradiction, suppose that $\triangle_\mu(E) \not\models \bigvee E$, that is $\exists \omega' \in \mathcal{W}$ such that $\omega' \models \triangle_\mu(E)$ and $\omega' \not\models \bigvee E$. $\omega' \models \triangle_\mu(E)$ implies that $\omega' \in \min([\mu], \leq_E)$, that is $\nexists \omega'' \models \mu$ such that $\omega'' <_E \omega'$. Since $\omega' \not\models \bigvee E$ and $\omega \models \bigvee E$ we obtain by (d) that $\omega <_E \omega'$. Contradiction.

(Only If) Let $\triangle$ be a disjunctive IC merging operator (i.e., $\triangle$ satisfies **(IC0-IC8)** and **(Disj)**). Then we define a syncretic assignment in the usual way [19], as follows: $\forall \omega, \omega' \in \mathcal{W}$, $\omega \leq_E \omega'$ if and only if $\omega \models \triangle_{\varphi_{\{\omega,\omega'\}}}(E)$. Let us show that condition (d) holds. Suppose that $\omega \models \bigvee E$ and $\omega' \not\models \bigvee E$. This implies that $\bigvee E$ is consistent with $\varphi_{\{\omega,\omega'\}}$. By **(Disj)** we get that $\triangle_{\varphi_{\{\omega,\omega'\}}}(E) \models \bigvee E$. By **(IC0)** we also have that $\triangle_{\varphi_{\{\omega,\omega'\}}}(E) \models \varphi_{\{\omega,\omega'\}}$. Hence we have $\triangle_{\varphi_{\{\omega,\omega'\}}}(E) \models \bigvee E \wedge \varphi_{\{\omega,\omega'\}}$. By assumption $\bigvee E \wedge \varphi_{\{\omega,\omega'\}} \equiv \varphi_{\{\omega\}}$. Therefore $\triangle_{\varphi_{\{\omega,\omega'\}}}(E) \models \varphi_{\{\omega\}}$. By **(IC1)** we get that $\triangle_{\varphi_{\{\omega,\omega'\}}}(E)$ is consistent, hence $\triangle_{\varphi_{\{\omega,\omega'\}}}(E) \equiv \varphi_{\{\omega\}}$. By definition of the assignment we finally get that $\omega <_E \omega'$.

$\square$

Proof of Proposition 4:

(IC0) If $\triangle$ satisfies **(IC0)**, then by construction, $\triangle^d$ satisfies **(IC0)**.

(IC1) If $(\bigvee E) \wedge \mu$ is not consistent, then $\triangle^d_\mu(E)$ is not consistent, so $\triangle^d$ does not satisfy **(IC1)**, even if $\triangle$ satisfies **(IC1)**.

(IC2) If $(\bigwedge E) \wedge \mu$ is consistent, then $(\bigwedge E) \wedge (\bigvee E) \wedge \mu$ is consistent as well. Hence, if $\triangle$ satisfies **(IC2)**, we get that $\triangle^d_{IC}(E) \equiv (\bigwedge E) \wedge (\bigvee E) \wedge \mu \equiv (\bigwedge E) \wedge \mu$.

(IC3) Obviously satisfied.

(IC4) Suppose that $K_1 \models \mu$, that $K_2 \models \mu$, and that $\triangle$ satisfies **(IC4)**. To show that $\triangle^d$ satisfies **(IC4)**, we must prove that $\triangle^d_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\triangle^d_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent. So suppose that $\triangle^d_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent. Then $\triangle^d_{(K_1 \vee K_2) \wedge \mu}(\{K_1, K_2\}) \wedge K_1$ is consistent. In order to simplify the notations, let us note μ' the formula $(K_1 \vee K_2) \wedge \mu$. Because $K_1 \models \mu$ and $K_2 \models \mu$, we have $K_1 \models \mu'$ and $K_2 \models \mu'$ and $\triangle^d_{\mu'}(\{K_1, K_2\}) \wedge K_1$ consistent. Since $\triangle$ satisfies **(IC4)**, $\triangle^d_{\mu'}(\{K_1, K_2\}) \wedge K_2$ is also consistent and $\triangle^d$ satisfies **(IC4)**.

(IC5) and **(IC6)**: As a counter-example, we consider four bases: $[K_1] = \{000\}$, $[K_2] = \{111\}$, $[K_3] = \{000, 011, 110, 101\}$, $[K_4] = \{001, 010, 100\}$ and two profiles $E_1 = \{K_1, K_2\}$ and $E_2 = \{K_3, K_4\}$. Then $[\triangle^{d_H,\Sigma}_{\bigvee E_1}(E_1)] = \{000, 111\}$ and $[\triangle^{d_H,\Sigma}_{\bigvee E_2}(E_2)] = \{000, 001, 010, 011, 100, 101, 110\}$. So, as $\triangle^{d_H,\Sigma}_{\bigvee E_1}(E_1)$ and $\triangle^{d_H,\Sigma}_{\bigvee E_2}(E_2)$ are consistent, $[\triangle^{d_H,\Sigma}_{\bigvee E_1}(E_1) \wedge \triangle^{d_H,\Sigma}_{\bigvee E_2}(E_2)] = \{000\}$. With $E = E_1 \sqcup E_2$, we have $[\triangle^{d_H,\Sigma}_{\bigvee E_1 \vee \bigvee E_2}(E)] = \{001, 010, 011, 100, 101, 110\}$. Consequently, $\triangle^{d_H,\Sigma}_{\bigvee E_1}(E_1) \wedge \triangle^{d_H,\Sigma}_{\bigvee E_2}(E_2) \not\models \triangle^{d_H,\Sigma}_{\bigvee E_1 \vee \bigvee E_2}(E_1 \sqcup E_2)$, which contradicts **(IC5)** and $\triangle^{d_H,\Sigma}_{\bigvee E_1 \vee \bigvee E_2}(E_1 \sqcup E_2) \not\models \triangle^{d_H,\Sigma}_{\bigvee E_1}(E_1) \wedge \triangle^{d_H,\Sigma}_{\bigvee E_2}(E_2)$, which contradicts **(IC6)**.

(IC7) We consider a profile E, two integrity constraints μ_1 and μ_2. We suppose that $\triangle$ satisfies **(IC7)**. We have to show that $\triangle^d_{\mu_1}(E) \wedge \mu_2 \models \triangle^d_{\mu_1 \wedge \mu_2}(E)$, i.e., $\triangle_{(\bigvee E) \wedge \mu_1}(E) \wedge \mu_2 \models \triangle_{(\bigvee E) \wedge \mu_1 \wedge \mu_2}(E)$. Let us note μ'_1 the formula $(\bigvee E) \wedge \mu_1$. We have $\triangle_{(\bigvee E) \wedge \mu_1}(E) \wedge \mu_2 \equiv \triangle_{\mu'_1}(E) \wedge \mu_2$. As $\triangle$ satisfies **(IC7)**, $\triangle_{\mu'_1}(E) \wedge \mu_2 \models \triangle_{\mu'_1 \wedge \mu_2}(E)$. As $\triangle_{\mu'_1 \wedge \mu_2}(E) \equiv \triangle_{(\bigvee E) \wedge \mu_1 \wedge \mu_2}(E)$, the result holds.

(IC8) Suppose $\triangle^d_{\mu_1}(E) \wedge \mu_2$ consistent. We must show that $\triangle^d_{\mu_1 \wedge \mu_2}(E) \models \triangle^d_{\mu_1}(E) \wedge \mu_2$, that means $\triangle_{(\bigvee E) \wedge \mu_1 \wedge \mu_2}(E) \models \triangle_{(\bigvee E) \wedge \mu_1}(E) \wedge \mu_2$. With $\mu'_1 \equiv (\bigvee E) \wedge \mu_1$, we have $\triangle_{(\bigvee E) \wedge \mu_1 \wedge \mu_2}(E) \equiv \triangle_{\mu'_1 \wedge \mu_2}(E)$ consistent. As $\triangle$ satisfies **(IC8)** and $\triangle_{\mu'_1 \wedge \mu_2}(E)$ is consistent, we get $\triangle_{\mu'_1 \wedge \mu_2}(E) \models \triangle_{\mu'_1}(E) \wedge \mu_2$. Since $\triangle_{\mu'_1}(E) \wedge \mu_2 \equiv \triangle_{(\bigvee E) \wedge \mu_1 \wedge \mu_2}(E)$, the result holds: $\triangle^d$ satisfies **(IC8)** if $\triangle$ does.

(Disj) If $\triangle$ satisfies **(IC0)**, then by construction, $\triangle^d$ satisfies **(Disj)**.

$\square$

Proof of Proposition 5: Immediate from the two following equalities:

- $[\bigwedge E \wedge \mu] = \{\omega \in [\mu] \mid \forall K_i \in E\ \omega \models K_i\}$.
- $[\bigvee_{C \in \ulcorner n_k \urcorner}(\bigwedge_{j \in C} K_j) \wedge \mu] = \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$.

$\square$

Proof of Proposition 6:

(IC0) Obvious from the definition of $\triangle^k$.
(IC1) Consider the following counter-example: $E = \{K_1, K_2\}$; $K_1 = \{a\}$, $K_2 = \{\neg a\}$, $k = 2$ and $\mu = \top$. μ is consistent but $\triangle^k_\mu(E)$ is not.
(IC2) Obvious from the definition of $\triangle^k$.
(IC3) Obvious from the definition of $\triangle^k$.
(IC4) We have to show that if $K_1 \models \mu$, $K_2 \models \mu$, and $\triangle^k_\mu(\{K_1, K_2\}) \wedge K_1 \not\models \perp$, then $\triangle^k_\mu(\{K_1, K_2\}) \wedge K_2 \not\models \perp$.
Let $E = \{K_1, K_2\}$. Assume that $K_1 \models \mu$ and $K_2 \models \mu$. There are two cases:
- $K_1 \wedge K_2 \wedge \mu$ is consistent. Then $\triangle^k_\mu(\{K_1, K_2\}) \equiv K_1 \wedge K_2 \wedge \mu$. Since $\triangle^k_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent, **(IC4)** is satisfied.
- $K_1 \wedge K_2 \wedge \mu$ is inconsistent.
  Then $[\triangle^k_\mu(\{K_1, K_2\})] = \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$. Four cases for k have to be considered:

$k \geq 3$. No interpretation can satisfy k bases of E since $\#(E) = 2$, hence $\triangle^k_\mu(\{K_1, K_2\}) \equiv \perp$ and **(IC4)** trivially holds.

$k = 2$. Again, no interpretation can satisfy k bases of E since $\#(E) = 2$ and (by assumption) $K_1 \wedge K_2 \wedge \mu \models \perp$. **(IC4)** trivially holds as well.

$k = 1$. The models of the merged base are the models of μ satisfying one base of E. Hence, the result of the merging process is equivalent to $(K_1 \vee K_2) \wedge \mu$. Therefore, $\triangle^k_\mu(\{K_1, K_2\}) \wedge K_2$ is equivalent to K_2, which is consistent (remind that every base from a profile is consistent). Again, **(IC4)** holds.

$k = 0$. We have $\triangle^k_\mu(\{K_1, K_2\}) \equiv \mu$. Since $K_2 \wedge \mu$ is equivalent to K_2 that is consistent, **(IC4)** is satisfied.

(IC5) In order to show that **(IC5)** holds, we have to prove that $\triangle_\mu(E_1) \wedge \triangle_\mu(E_2) \models \triangle_\mu(E_1 \sqcup E_2)$. We first prove the following lemma:
Lemma 4 *Let E, E' and F be three profiles, with $E' = E \sqcup F$. Then:*
- *if $\bigwedge E' \wedge \mu$ is consistent, then $\triangle^k_\mu(E') \models \triangle^k_\mu(E)$.*
- *if $\bigwedge E \wedge \mu$ is not consistent, then $\triangle^k_\mu(E) \models \triangle^k_\mu(E')$.*

Proof of Lemma 4:
- If $\bigwedge E' \wedge \mu$ is consistent, then $\bigwedge E \wedge \mu$ is consistent since $E' = E \sqcup F$. Hence

$\triangle^k_\mu(E) \equiv \bigwedge E \wedge \mu$, and $\triangle^k_\mu(E') \equiv \bigwedge E' \wedge \mu$. Since $\bigwedge E' \wedge \mu \equiv \bigwedge E \wedge \bigwedge F \wedge \mu$, we get $\triangle^k_\mu(E') \models \triangle^k_\mu(E)$.

- If $\bigwedge E \wedge \mu$ is inconsistent, then $[\triangle^k_\mu(E)] = \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$; there are two cases:

  $k > \#(E)$**.** We have $\triangle^k_\mu(E) \equiv \bot$, since no model of μ can satisfy k bases of E. As a consequence, we get $\triangle^k_\mu(E) \models \triangle^k_\mu(E')$.

  $k \leq \#(E)$**.** We have $[\triangle^k_\mu(E)] = \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ and $[\triangle^k_\mu(E')] = \{\omega \in [\mu] \mid \#(\{K_i \in E' \mid \omega \models K_i\}) \geq k\}$. Since every model of μ satisfying at least k bases from E also satisfies at least k bases from its superset $E' = E \sqcup F$, we obtain that $\triangle^k_\mu(E) \models \triangle^k_\mu(E')$.

□

With $E = E_1$ and $E' = E_1 \sqcup E_2$, Lemma 4 shows that if $\bigwedge E_1 \wedge \mu$ is inconsistent, then $\triangle^k_\mu(E_1) \models \triangle^k_\mu(E_1 \sqcup E_2)$. Similarly, we also get that $\triangle^k_\mu(E_2) \models \triangle^k_\mu(E_1 \sqcup E_2)$ (E_1 and E_2 play symmetric roles here). As a consequence, if $\bigwedge E_1 \wedge \mu$ is inconsistent or $\bigwedge E_2 \wedge \mu$ is inconsistent, we have that $\triangle^k_\mu(E_1) \wedge \triangle^k_\mu(E_2) \models \triangle^k_\mu(E_1 \sqcup E_2)$, since classical entailment is monotonic. Hence **(IC5)** is satisfied.

The case when $\bigwedge E_1 \wedge \mu$ is consistent and $\bigwedge E_2 \wedge \mu$ is consistent remains to be considered. In this case, we have $\triangle^k_\mu(E_1) \equiv \bigwedge E_1 \wedge \mu$ and $\triangle^k_\mu(E_2) \equiv \bigwedge E_2 \wedge \mu$ by definition of the quota merging operator. Hence, $\triangle^k_\mu(E_1) \wedge \triangle^k_\mu(E_2) \equiv \bigwedge E_1 \wedge \bigwedge E_2 \wedge \mu$. Now, every quota operator is such that, for any profile E and any integrity constraint μ, $\bigwedge E \wedge \mu \models \triangle^k_\mu(E)$ (this is a direct consequence of the definition of $\triangle^k$). Taking $\bigwedge E$ equivalent to $\bigwedge E_1 \wedge \bigwedge E_2$ gives that **(IC5)** also holds in this case.

(IC6) Consider the following counter-example: $\mathcal{P} = \{a\}$, $E_1 = \{\{a\}, \{a\}, \{\neg a\}\}$, $E_2 = \{\{a\}, \{a\}, \{\neg a\}\}$ and $\mu = \top$. We have $\triangle^2_\mu(E_1) \equiv a$ and $\triangle^2_\mu(E_2) \equiv a$, hence the conjunction $\triangle^2_\mu(E_1) \wedge \triangle^2_\mu(E_2)$ is consistent. We also have $\triangle^2_\mu(E_1 \sqcup E_2) \equiv \top$, which does not entail $\triangle^2_\mu(E_1)$.

(IC7) We have to show that $\triangle^k_{\mu_1}(E) \wedge \mu_2 \models \triangle^k_{\mu_1 \wedge \mu_2}(E)$. We consider two cases: (1) If $\bigwedge E \wedge \mu_1$ is consistent, then $\triangle^k_{\mu_1}(E) \wedge \mu_2 \equiv \bigwedge E \wedge \mu_1 \wedge \mu_2$. Since we have $\bigwedge E \wedge \mu_1 \wedge \mu_2 \models \triangle^k_{\mu_1 \wedge \mu_2}(E)$, **(IC7)** trivially holds.
(2) If $\bigwedge E \wedge \mu_1$ is inconsistent, then we have

$$[\triangle^k_{\mu_1}(E) \wedge \mu_2] = \{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\} \cap [\mu_2].$$

Furthermore, when $\bigwedge E \wedge \mu_1$ is inconsistent, we also have that $\bigwedge E \wedge \mu_1 \wedge \mu_2$ is inconsistent and

$$[\triangle^k_{\mu_1 \wedge \mu_2}(E)] = \{\omega \in [\mu_1 \wedge \mu_2] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}.$$

Therefore $\triangle^k_{\mu_1}(E) \wedge \mu_2 \equiv \triangle^k_{\mu_1 \wedge \mu_2}(E)$, and **(IC7)** is satisfied.

(IC8) We have to show that if $\triangle^k_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\triangle^k_{\mu_1 \wedge \mu_2}(E) \models \triangle^k_{\mu_1}(E) \wedge \mu_2$. We consider three cases:
(1) If $\mu_1 \wedge \mu_2 \wedge \bigwedge E$ is consistent, then $\mu_1 \wedge \bigwedge E$ is consistent as well and we have $\triangle^k_{\mu_1 \wedge \mu_2}(E) \equiv \mu_1 \wedge \mu_2 \wedge \bigwedge E$. Hence $\triangle^k_{\mu_1 \wedge \mu_2}(E) \models \triangle^k_{\mu_1}(E) \wedge \mu_2$, and **(IC8)** is

satisfied.
(2) If $\mu_1 \wedge \bigwedge E$ is inconsistent, then $\mu_1 \wedge \mu_2 \wedge \bigwedge E$ is inconsistent. In this case:

$$[\triangle^k_{\mu_1}(E)] = \{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}.$$

Since $[\triangle^k_{\mu_1 \wedge \mu_2}(E)] = \{\omega \in [\mu_1 \wedge \mu_2] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$, we have:

$$\triangle^k_{\mu_1 \wedge \mu_2}(E) \models \triangle^k_{\mu_1}(E) \wedge \mu_2,$$

and **(IC8)** holds.
(3) The remaining case is when $\mu_1 \wedge \mu_2 \wedge \bigwedge E$ is inconsistent and $\mu_1 \wedge \bigwedge E$ is consistent. In this case, $\triangle^k_{\mu_1}(E) \wedge \mu_2 \equiv \mu_1 \wedge \bigwedge E \wedge \mu_2$ is inconsistent, hence **(IC8)** trivially holds.

(Disj) There are two cases:
(1) If $(\bigwedge E) \wedge \mu$ is consistent, then $\triangle^k_\mu(E) \equiv (\bigwedge E) \wedge \mu$ and $\triangle^k_\mu(E) \models (\bigvee E) \wedge \mu$.
(2) If $(\bigwedge E) \wedge \mu$ is not consistent, then the models of $\triangle^k_\mu(E)$ are the models of μ which satisfy at least k bases ($k \geq 1$) of the profile E. So they also are models of $(\bigvee E) \wedge \mu$, and the result holds.

(Maj) Consider the following counter-example: $\mathcal{P} = \{a\}$, $E_1 = \{K_1\}$, $E_2 = \{K_2\}$, $K_1 = \{a\}$, $K_2 = \{\neg a\}$, $k = 1$, $\mu = \top$. The interpretation $\omega = (a = 1)$ is a model of $\triangle^1_\mu(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ for every $n \geq 0$ since it satisfies K_1. Contrastingly, ω is not a model of $\triangle^1_\mu(E_2) \equiv \neg a$.

□

Proof of Proposition 7:

(Card) Let $M_1, M_2 \in \textsc{maxcons}_\mu(E)$ such that $\#(M_1) \leq \#(M_2)$. By hypothesis $\triangle^k_\mu(E) \wedge M_1$ is consistent. There are two cases:
(1) $(\bigwedge E) \wedge \mu$ is consistent. Then $\triangle^k_\mu(E) \equiv (\bigwedge E) \wedge \mu$. As a consequence, $\textsc{maxcons}_\mu(E) = \{E\}$. So $\triangle^k_\mu(E) \wedge M_2$ is consistent.
(2) $(\bigwedge E) \wedge \mu$ is not consistent. Let ω be any model of $\triangle^k_\mu(E) \wedge M_1$; ω is a model of M_1 which satisfies at least k bases of the profile E (since $\triangle^k_\mu(E)$ is consistent when $\triangle^k_\mu(E) \wedge M_1$ is consistent). Since $M_1 \in \textsc{maxcons}_\mu(E)$, we can deduce that $\#(M_1) \geq k$. Furthermore, since $\#(M_1) \leq \#(M_2)$, any model ω' of M_2 satisfies μ and at least k bases of E. Subsequently, ω' is a model of $\triangle^k_\mu(E)$ as well, and $\triangle^k_\mu(E) \wedge M_2$ is consistent.

(Wmaj) Suppose that $\triangle^k_\mu(E_2)$ is consistent. There are two cases:
(1) $(\bigwedge E_2) \wedge \mu$ is consistent. Then $\triangle^k_\mu(E_2) \equiv (\bigwedge E_2) \wedge \mu$.

- If $\bigwedge(E_1 \sqcup E_2) \wedge \mu$ is consistent, then $\triangle^k_\mu(E_1 \sqcup E_2) \equiv \bigwedge(E_1 \sqcup E_2) \wedge \mu$, which is trivially consistent with $(\bigwedge E_2) \wedge \mu$, hence with $\triangle^k_\mu(E_2)$. Thus **(Wmaj)** holds with $n = 1$.
- If $\bigwedge(E_1 \sqcup E_2) \wedge \mu$ is not consistent, then for any integer $n \geq 0$, $\bigwedge(E_1 \sqcup$

$\underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}$) is not consistent. The models of $\triangle^k_\mu(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ are by definition the models of μ that satisfies at least k bases of $E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}$.

Let ω be a model of $\triangle^k_\mu(E_2)$. Since $\#(E_2) \geq 1$, ω satisfies μ and at least one base of E_2. Hence, for any $n \geq k$, ω satisfies μ and at least k bases of $E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}$. Subsequently, ω is a model of $\triangle^k_\mu(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ and of $\triangle^k_\mu(E_2)$.

(2) $(\bigwedge E_2) \wedge \mu$ is not consistent. Let us consider any model ω of $\triangle^k_\mu(E_2)$; ω satisfies μ and at least k bases of E_2. Hence ω satisfies μ and at least k bases of $E_1 \sqcup E_2$, so ω is a model of $\triangle^k_\mu(E_1 \sqcup E_2) \wedge \triangle^k_\mu(E_2)$. Thus **(Wmaj)** holds with $n = 1$.

□

Proof of Proposition 8:

- Membership: we give a polynomial reduction from MERGE($\triangle^k$) to UNSAT(3), the language defined by UNSAT(3) = $\{\langle \phi_1, \phi_2, \phi_3\rangle \mid \phi_1, \phi_2, \phi_3 \in \mathcal{L}$ and $\phi_1 \in$ UNSAT or ($\phi_2 \in$ SAT and $\phi_3 \in$ UNSAT)$\}$.
  We have shown in Proposition 5 that when $\bigwedge E \wedge \mu$ is inconsistent, we have $\triangle^k_\mu(E) \equiv (\bigvee_{C \in \ulcorner n_k \urcorner} \bigwedge_{j \in C} K_j) \wedge \mu$ where $\ulcorner n_k \urcorner = \{C \subseteq \{1, \ldots, n\} \mid \#(C) = k\}$. Moreover, $(\bigvee_{C \in \ulcorner n_k \urcorner} \bigwedge_{j \in C} K_j) \wedge \mu$ has a size polynomial in $|E| + |\mu|$. Let f be the polynomial reduction which assigns to every instance $\langle E, \mu, \alpha\rangle$ of MERGE($\triangle^k$) the instance $\langle \phi_1 = (\bigvee_{C \in \ulcorner n_k \urcorner} \bigwedge_{j \in C} K_j) \wedge \mu \wedge \neg\alpha, \phi_2 = \bigwedge E \wedge \mu, \phi_3 = \bigwedge E \wedge \mu \wedge \neg\alpha\rangle$ of UNSAT(3). As $\bigwedge E \wedge \mu \models \triangle^k_\mu(E)$, we have $\langle E, \mu, \alpha\rangle \in$ MERGE($\triangle^k$) if and only if $(\bigvee_{C \in \ulcorner n_k \urcorner} \bigwedge_{j \in C} K_j) \wedge \mu \wedge \neg\alpha \in$ UNSAT or ($\bigwedge E \wedge \mu \in$ SAT and $\bigwedge E \wedge \mu \wedge \neg\alpha \in$ UNSAT).
- Hardness: we do not give a direct polynomial reduction of UNSAT(3) to MERGE($\triangle^k$), but give a faithful and modular polynomial traduction of the full-meet inference problem (which is equivalent to the inference problem from a merged base using the full meet merging operator or, equivalently, the quota operator with quota 0) to the inference from a merged base using any quota operator. The *full-meet* inference problem can be defined by: for all $\phi_1, \phi_2, \phi_3 \in \mathcal{L}$, we have $\phi_1 \circ_{FM} \phi_2 \models \phi_3$ if and only if:

  if $\phi_1 \wedge \phi_2$ is consistent
  then $\phi_1 \wedge \phi_2 \models \phi_3$
  else $\phi_2 \models \phi_3$

  Inference from a merged base using any quota operator can be defined as follows; for any profile E, integrity constraint μ and formula α, we have $\triangle^k_\mu(E) \models \alpha$ if and only if:

if $\bigwedge E \wedge \mu$ is consistent
then $\bigwedge E \wedge \mu \models \alpha$
else $(\bigvee_{C \in \ulcorner n_k \urcorner} \bigwedge_{j \in C} K_j) \wedge \mu \models \alpha$

To any integer $k \geq 0$ and triple of formulas $\langle \phi_1, \phi_2, \phi_3 \rangle$ of $\mathcal{L}$, we can associate in polynomial time the triple $\langle E = \{\phi_1\} \sqcup \{\phi_2\}^k, \mu = \phi_2, \alpha = \phi_3 \rangle$ where $\{\phi_2\}^k$ is the multi-set in which ϕ_2 appears k times (in particular, the empty multi-set when $k = 0$). We have $\phi_1 \circ_{FM} \phi_2 \models \phi_3$ if and only if $\triangle^k_\mu(E) \models \alpha$. Since the full-meet inference problem is $\mathsf{coBH(3)}$-hard (cf. Proposition 4.3 from [24]), this concludes the proof.

□

Proof of Proposition 9: We first consider the probabilistic index i_p.
Reductio ad absurdum. Assume that there exists an integer $k \geq 0$ and an integrity constraint μ such that $\triangle^k_\mu$ is not strategy-proof for i_p. Hence there exists a profile $E = \{K_2, \ldots, K_n\}$, two bases K and K' such that

$$i_p(K, \triangle^k_\mu(E \sqcup \{K\})) < i_p(K, \triangle^k_\mu(E \sqcup \{K'\})) \tag{A.1}$$

There are two cases:

(1) $\triangle^k_\mu(E \sqcup \{K\})$ is inconsistent.Then $\forall \omega \models K$, we have $\omega \not\models \triangle^k_\mu(E \sqcup \{K\})$. Hence, $\forall \omega \models K$, ω does not satisfy μ or ω satisfies strictly less than $k-1$ bases of E. In those two cases, ω cannot satisfy $\triangle^k_\mu(E \sqcup \{K'\})$ since it satisfies at most $k-1$ bases from $E \sqcup \{K'\}$ or it does not satisfy μ.

Therefore, $\forall \omega \models K, \omega \not\models \triangle^k_\mu(E \sqcup \{K'\})$. Hence $\#([K] \cap [\triangle^k_\mu(E \sqcup \{K'\})]) = 0$. As a consequence, $i_p(K, \Delta(E \sqcup \{K'\}) = 0$, which prevents from any manipulation for i_p.

(2) $\triangle^k_\mu(E \sqcup \{K\})$ is consistent. Hence we have from Inequation (A.1):

$$\frac{\#([K] \cap [\triangle^k_\mu(E \sqcup \{K\})])}{\#([\triangle^k_\mu(E \sqcup \{K\})])} < \frac{\#([K] \cap [\triangle^k_\mu(E \sqcup \{K'\})])}{\#([\triangle^k_\mu(E \sqcup \{K'\})])} \tag{A.2}$$

Two cases have to be considered:

- $\bigwedge E \wedge K \wedge \mu$ is consistent. Then $\triangle^k_\mu(E \sqcup \{K\}) \equiv \bigwedge E \wedge K \wedge \mu$. Hence each model of $\triangle^k_\mu(E \sqcup \{K\})$ is a model of K, which implies that the value $i_p(K, \triangle^k_\mu(E \sqcup \{K\})) = 1$ is maximum, so it cannot be improved, and no manipulation is possible in this case.
- $\bigwedge E \wedge K \wedge \mu$ is inconsistent. Then $\triangle^k_\mu(E \sqcup \{K\}) \equiv (\bigvee_{C \in \ulcorner n_k \urcorner} (\bigwedge_{j \in C} K_j)) \wedge \mu$, where $K_1 = K$ and $E = \{K_2, \ldots, K_n\}$. As $\triangle^k_\mu(E \sqcup \{K\})$ is consistent, there are two cases:

– $\bigwedge E \wedge K' \wedge \mu$ is consistent. Then $\triangle^k_\mu(E \sqcup \{K'\}) \equiv \bigwedge E \wedge K' \wedge \mu$. No model of K is a model of $\triangle^k_\mu(E \sqcup \{K'\})$. Indeed, if it were not the case, there would exist an interpretation ω such that $\omega \models K$ and $\omega \models \bigwedge(E \sqcup \{K'\}) \wedge \mu$. Then we would have $\omega \models \bigwedge(E \sqcup \{K\}) \wedge \mu$ which is impossible since $\bigwedge E \wedge K \wedge \mu$ is inconsistent. Hence $[K] \cap [\triangle^k_\mu(E \sqcup \{K'\})] = \emptyset$ and $i_p(K, \triangle^k_\mu(E \sqcup \{K'\}) = 0$, which prevents from any manipulation for i_p.

– $\bigwedge E \wedge K' \wedge \mu$ is inconsistent. Then $\triangle^k_\mu(E \sqcup \{K'\}) \equiv (\bigvee_{C \in \ulcorner n_k \urcorner} (\bigwedge_{j \in C} K'_j)) \wedge \mu$, where $K'_1 = K'$ and $K'_i = K_i$ for $i > 1$.

  If $\omega \models K$ and $\omega \not\models \triangle^k_\mu(E \sqcup \{K\})$, then ω does not satisfy μ or ω satisfies strictly less than $k - 1$ bases K_i with $i > 1$. In the two cases, ω cannot be a model of $\triangle^k_\mu(E \sqcup \{K'\})$. As a consequence:

$$\#([K] \cap [\triangle^k_\mu(E \sqcup \{K\})]) \geq \#([K] \cap [\triangle^k_\mu(E \sqcup \{K'\})]). \tag{A.3}$$

  On the other hand, if $\omega \not\models K$ and $\omega \models \triangle^k_\mu(E \sqcup \{K\})$, then there exist at least k bases K_i with $i > 1$ such that $\omega \models K_i \wedge \mu$. Then $\omega \models \triangle^k_\mu(E \sqcup \{K'\})$, and subsequently:

$$\#([\neg K] \cap [\triangle^k_\mu(E \sqcup \{K\})]) \leq \#([\neg K] \cap [\triangle^k_\mu(E \sqcup \{K'\})]). \tag{A.4}$$

  In order to simplify the notations, we set:
  $x = \#([K] \cap [\triangle^k_\mu(E \sqcup \{K\})]), y = \#([\neg K] \cap [\triangle^k_\mu(E \sqcup \{K\})]),$
  $x' = \#([K] \cap [\triangle^k_\mu(E \sqcup \{K\})]), y' = \#([\neg K] \cap [\triangle^k_\mu(E \sqcup \{K\})]),$

  Inequation (A.2) becomes: $\frac{x}{x+y} < \frac{x'}{x'+y'}$. Since $y \leq y'$ from (A.4), we have: $\frac{x}{x+y} < \frac{x'}{x'+y}$. From (A.3), we know that $x \geq x'$. Hence we can write $x' = x - z$, with $z \geq 0$. We get $\frac{x}{x+y} < \frac{x-z}{x+y-z}$, which is equivalent to:

$$\frac{(x)(x+y-z)}{(x+y)(x+y-z)} < \frac{(x-z)(x+y)}{(x+y-z)(x+y)},$$

  hence $zy < 0$ with y, and z positive: this is impossible.

A manipulation for i_{d_w} entails a manipulation for i_p, even if the operator does not satisfy **(IC1)** (see [13]). So the strategy-proofness of quota merging operators for i_{d_w} comes from the above proof for i_p.

Finally, the last case concerns the strong drastic index i_{d_s}. Let us suppose that there is a manipulation for this index: assume that there exists an integer $k \geq 0$ and an integrity constraint μ such that $\triangle^k_\mu$ is not strategy-proof for i_{d_s}. Hence there exist a profile $E = \{K_2, \ldots, K_n\}$, two bases K and K' such that

$$i_{d_s}(K, \triangle^k_\mu(E \sqcup \{K\})) < i_{d_s}(K, \triangle^k_\mu(E \sqcup \{K'\}).$$

This inequation implies that:

$$i_{d_s}(K, \triangle_\mu^k(E \sqcup \{K\})) = 0 \text{ so } \triangle_\mu^k(E \sqcup \{K\}) \not\models K$$

and

$$i_{d_s}(K, \triangle_\mu^k(E \sqcup \{K'\})) = 1 \text{ so } \triangle_\mu^k(E \sqcup \{K'\}) \models K.$$

If $\triangle_\mu^k(E \sqcup \{K'\})$ is consistent, this implies a manipulation for the index i_p (see [13]), and we have seen that it is impossible in the first part of the proof. Now, if $\triangle_\mu^k(E \sqcup \{K'\})$ is not consistent, then there are two cases:

- $k > \#(E \sqcup \{K'\})$. In this case, we have $k > \#(E \sqcup \{K\})$. Then $\triangle_\mu^k(E \sqcup \{K\})$ is not consistent and $\triangle_\mu^k(E \sqcup \{K\}) \models K$, which contradicts the assumption.
- $k \leq \#(E \sqcup \{K'\})$. In this case, there is no model of μ which satisfies k bases among $\{K_2, \ldots, K_n, K'\}$. Since $\triangle_\mu^k(E \sqcup \{K\})$ is consistent, the models of $\triangle_\mu^k(E \sqcup \{K\})$ are models of K, which contradicts the assumption.

Hence, no manipulation is possible when $\triangle_\mu^k(E \sqcup \{K'\})$ is not consistent, and quota merging operators are strategy-proof for i_{d_s}.

□

Proof of Proposition 10: There are two cases:
(1) If $\bigwedge E \wedge \mu$ is consistent, $\triangle_\mu^k(E) \equiv \bigwedge E \wedge \mu \equiv \overline{\triangle}_\mu^{\overline{k}}(E)$ for every integer $k \geq 0$ and every real number $\overline{k} \in [0, 1]$.
(2) If $\bigwedge E \wedge \mu$ is not consistent, then we consider two cases:

- $\triangle_\mu^k(E)$ is consistent. Let ω be a model of $\triangle_\mu^k(E)$. Then ω satisfies μ and at least k bases of E. So ω satisfies μ and a ratio of bases of E greater or equal to $\overline{k} = \frac{k}{n}$. Hence $\omega \models \overline{\triangle}_\mu^{\overline{k}}(E)$, and as a consequence, $\triangle_\mu^k(E) \models \overline{\triangle}_\mu^{\overline{k}}(E)$. Conversely, if $\overline{\triangle}_\mu^{\overline{k}}(E)$ is consistent and ω is a model of $\overline{\triangle}_\mu^{\overline{k}}(E)$, then ω satisfies μ and a ratio of bases of E greater or equal to $\overline{k}$. So ω satisfies μ and at least $k = \lfloor \overline{k} \times n \rfloor$ bases of E. Hence $\omega \models \triangle_\mu^k(E)$, and as a consequence, $\overline{\triangle}_\mu^{\overline{k}}(E) \models \triangle_\mu^k(E)$. This completes the proof.
- $\triangle_\mu^k(E)$ is not consistent. Then no model of μ satisfies at least k bases of E. If $k \geq n$, no model of μ satisfies a ratio of bases of E equal to 1 (since $\bigwedge E \wedge \mu$ is not consistent). Hence $\overline{\triangle}_\mu^{1}(E)$ is inconsistent, and as a consequence, we have $\overline{\triangle}_\mu^{1}(E) \equiv \triangle_\mu^k(E)$. If $k < n$, no model of μ satisfies a ratio of bases of E greater or equal to $\overline{k} = \frac{k}{n}$ (otherwise it would satisfy at least k bases of E). Hence $\overline{\triangle}_\mu^{\overline{k}}(E)$ is inconsistent, and as a consequence, $\overline{\triangle}_\mu^{\overline{k}}(E) \equiv \triangle_\mu^k(E)$.

□

Proof of Proposition 11: Thanks to Proposition 10, many proofs for the ratio operators can be deduced from the proofs for the corresponding absolute quota operators. More precisely, each time the cardinality of the profile E can be fixed at the beginning of the proof, i.e. for **(IC0)**, **(IC1)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC7)**, **(IC8)**, **(Disj)** for a ratio $\geq \frac{1}{\#(E)}$ and **(Card)**, the corresponding proof for the ratio operators can be obtained from the proof for the absolute quota operators by making the following changes; let n be the cardinality of the initial profile E, and let k and $\overline{k}$ be two numbers linked as explained in Proposition 10; replace

- $\langle \triangle^k_\mu \rangle$ by $\langle \overline{\triangle}^{\overline{k}}_\mu \rangle$,
- $\langle k$ bases$\rangle$ by $\langle$ a ratio $\overline{k} = \frac{k}{n}$ of bases$\rangle$,
- $\langle \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}\rangle$by $\langle \{\omega \in [\mu] \mid \frac{\#(\{K_i \in E \mid \omega \models K_i\})}{n} \geq \overline{k}\}\rangle$

Thus only three proofs are missing :

- **(IC5):** We consider two profiles E_1 with $\#(E_1) = n_1$ and E_2 with $\#(E_2) = n_2$. If $\overline{k}$ is the given ratio, we note $k_1 = \overline{k} \times n_1$, $k_2 = \overline{k} \times n_2$ and $k = \overline{k} \times (n_1 + n_2) = k_1 + k_2$.
  If $\overline{\triangle}^{\overline{k}}_\mu(E_1) \wedge \overline{\triangle}^{\overline{k}}_\mu(E_2)$ is not consistent, then the implication is obvious. Otherwise, let us consider any model ω of $\overline{\triangle}^{\overline{k}}_\mu(E_1) \wedge \overline{\triangle}^{\overline{k}}_\mu(E_2)$. There are three cases:
  - $(\bigwedge E_1) \wedge \mu$ and $(\bigwedge E_2) \wedge \mu$ are consistent. Then $\overline{\triangle}^{\overline{k}}_\mu(E_1) \equiv (\bigwedge E_1) \wedge \mu$ and $\overline{\triangle}^{\overline{k}}_\mu(E_2) \equiv (\bigwedge E_2) \wedge \mu$. In that case, $\omega \models (\bigwedge E_1) \wedge (\bigwedge E_2) \wedge \mu$, so $\bigwedge(E_1 \sqcup E_2) \wedge \mu$ is consistent. Hence $\overline{\triangle}^{\overline{k}}_\mu(E_1 \sqcup E_2) \equiv (\bigwedge E_1) \wedge (\bigwedge E_2) \wedge \mu$, and we have $\overline{\triangle}^{\overline{k}}_\mu(E_1) \wedge \overline{\triangle}^{\overline{k}}_\mu(E_2) \models \overline{\triangle}^{\overline{k}}_\mu(E_1 \sqcup E_2)$.
  - One but not both of $(\bigwedge E_1) \wedge \mu$ and $(\bigwedge E_2) \wedge \mu$ is consistent. Assume that $(\bigwedge E_1) \wedge \mu$ is consistent and $(\bigwedge E_2) \wedge \mu$ is not (the remaining case is similar by symmetry). Let us consider ω that satisfies μ, n_1 bases of E_1, and at least k_2 bases of E_2. So it satisfies at least $n_1 + k_2$ bases of $E_1 \sqcup E_2$. Subsequently, ω satisfies μ and a ratio greater or equal to $\frac{n_1+k_2}{n_1+n_2}$ of bases of $E_1 \sqcup E_2$. Since $n_1 \geq k_1$, we have:
  $$\frac{n_1 + k_2}{n_1 + n_2} \geq \frac{k_1 + k_2}{n_1 + n_2} \geq \frac{k}{n_1 + n_2} \geq \overline{k}.$$
  Hence ω satisfies μ and a ratio greater or equal to $\overline{k}$ bases of $E_1 \sqcup E_2$. So $\omega \models \overline{\triangle}^{\overline{k}}_\mu(E_1 \sqcup E_2)$.
  - $(\bigwedge E_1) \wedge \mu$ is not consistent and $(\bigwedge E_2) \wedge \mu$ is not consistent. Then, ω satisfies μ, at least k_1 bases of E_1, and at least k_2 bases ok E_2. Hence it satisfies at least $k_1 + k_2 = k$ bases of $E_1 \sqcup E_2$. So ω satisfies μ and a ratio greater or equal to $\frac{k}{n_1+n_2}$ bases of $E_1 \sqcup E_2$; subsequently, $\omega \models \overline{\triangle}^{\overline{k}}_\mu(E_1 \sqcup E_2)$.
- **(IC6):** Consider the following counter-example: $\mathcal{P} = \{a\}$, $E_1 = \{\{a\}, \{a\}, \{a\}, \{\neg a\}\}$, $E_2 = \{\{a\}, \{\neg a\}, \{\neg a\}\}$ and $\mu = \top$. We have $\overline{\triangle}^{\frac{1}{3}}_\mu(E_1) \equiv a$ and $\overline{\triangle}^{\frac{1}{3}}_\mu(E_2) \equiv \top$, hence the conjunction $\overline{\triangle}^{\frac{1}{3}}_\mu(E_1) \wedge \overline{\triangle}^{\frac{1}{3}}_\mu(E_2)$ is consistent. We also have $\overline{\triangle}^{\frac{1}{3}}_\mu(E_1 \sqcup E_2) \equiv \top$, which does not entail $\overline{\triangle}^{\frac{1}{3}}_\mu(E_1)$.

- **(Maj):** We want to show that $\exists n \in I\!N, \overline{\triangle}^{\overline{k}}_{\mu}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \overline{\triangle}^{\overline{k}}_{\mu}(E_2)$.

  In order to simplify the proof, let us introduce the following notations:
  $k_1^\omega = \#(\{K \mid K \in E_1 \text{ and } \omega \models K\})$,
  $k_2^\omega = \#(K \mid K \in E_2 \text{ and } \omega \models K\})$,
  $n_1 = \#(E_1)$,
  $n_2 = \#(E_2)$,

  We consider two cases:

  (1) $(\bigwedge E_2) \wedge \mu$ is consistent. Then there are two cases:

  - $\bigwedge(E_1 \sqcup E_2) \wedge \mu$ is consistent. Then $\overline{\triangle}^{\overline{k}}_{\mu}(E_1 \sqcup E_2) \equiv (\bigwedge E_1) \wedge (\bigwedge E_2) \wedge \mu$ and $\overline{\triangle}^{\overline{k}}_{\mu}(E_2) \models (\bigwedge E_2) \wedge \mu$. Hence the property holds with $n = 1$.
  - $\bigwedge(E_1 \sqcup E_2) \wedge \mu$ is not consistent. Then for any $n \geq 1$, $\bigwedge(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ $\wedge\mu$ is also not consistent. *Reductio ad absurdum*: suppose that there is a world ω such that, for any integer $n \geq 0$, $\omega \models \overline{\triangle}^{\overline{k}}_{\mu}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ and $\omega \not\models \overline{\triangle}^{\overline{k}}_{\mu}(E_2)$. Then ω is a model of μ which satisfies a ratio greater or equal to $\overline{k}$ of bases of $E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}$, and ω does not satisfy $(\bigwedge E_2) \wedge \mu$. Hence ω is a model of $(\bigvee E_1) \wedge \mu$ and it is not a model of $(\bigvee E_2) \wedge \mu$ (since $\bigwedge(E_1 \sqcup E_2) \wedge \mu$ is not consistent). Subsequently ω satisfies exactly k_1^ω bases of $E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}$ among the $n_1 + n \times n_2$ bases of this profile, for any $n > \frac{k_1^\omega - \overline{k} \times n_1}{\overline{k} \times n_2}$. So ω satisfies a ratio lower than $\overline{k}$ of bases of $E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}$. Contradiction. So $\overline{\triangle}^{\overline{k}}_{\mu}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \overline{\triangle}^{\overline{k}}_{\mu}(E_2)$.

  (2) $(\bigwedge E_2) \wedge \mu$ is not consistent. *Reductio ad absurdum*. Suppose that there is a world ω such that, for any integer $n \geq 0$, $\omega \models \overline{\triangle}^{\overline{k}}_{\mu}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ and $\omega \not\models \overline{\triangle}^{\overline{k}}_{\mu}(E_2)$.

  Since $\omega \not\models \overline{\triangle}^{\overline{k}}_{\mu}(E_2)$, we have $\frac{k_2^\omega}{n_2} < \overline{k}$, so $k_2^\omega < \overline{k} \times n_2$; hence, we can note $k_2^\omega = \overline{k} \times n_2 - \epsilon_2^\omega$ for some $\epsilon_2^\omega > 0$.

  Since $\omega \models \overline{\triangle}^{\overline{k}}_{\mu}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$, we also have $\frac{k_1^\omega + n \times k_2^\omega}{n_1 + n \times n_2} \geq \overline{k}$. Let n_ω be any integer ≥ 0 such that $n_\omega > \frac{k_1^\omega}{\epsilon_2^\omega}$. Then: $\frac{k_1^\omega}{\epsilon_2^\omega} - n_\omega < 0$, so $k_1^\omega - n_\omega \times \epsilon_2^\omega < 0$, hence $\frac{k_1^\omega - n_\omega \times \epsilon_2^\omega}{n_1 + n_\omega \times n_2} < 0$. So we have

$$\frac{n_\omega \times \overline{k} \times n_2}{n_1 + n_\omega \times n_2} + \frac{k_1^\omega - n_\omega \times \epsilon_2^\omega}{n_1 + n_\omega \times n_2} < \frac{n_\omega \times \overline{k} \times n_2}{n_1 + n_\omega \times n_2}$$

and, since $n_1 > 0$:

$$\frac{n_\omega \times \overline{k} \times n_2}{n_1 + n_\omega \times n_2} < \frac{n_\omega \times \overline{k} \times n_2}{n_\omega \times n_2} = \frac{\overline{k} \times n_2}{n_2} = \overline{k}.$$

So:

$$\frac{n_\omega \times \overline{k} \times n_2 + k_1^\omega - n_\omega \times \epsilon_2^\omega}{n_1 + n_\omega \times n_2} = \frac{k_1^\omega + n_\omega \times k_2^\omega}{n_1 + n_\omega \times n_2} < \overline{k}.$$

Hence $\omega \not\models \overline{\triangle}_\mu^{\overline{k}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n_\omega})$.

Furthermore, increasing further the number of copies of E_2 in the profile cannot enforce ω to satisfy the merged base: $\forall n \geq n_\omega, \omega \not\models \overline{\triangle}_\mu^{\overline{k}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$.

So we know that for every ω such that $\exists n \in I\!N, \omega \models \overline{\triangle}_\mu^{\overline{k}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ and $\omega \not\models \overline{\triangle}_\mu^{\overline{k}}(E_2)$, one can find an integer $n_\omega \geq 0$ such that $\omega \not\models \overline{\triangle}_\mu^{\overline{k}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n_\omega})$. To conclude the proof, it is sufficient to consider $N = max_{\omega \in \mathcal{W}} n_\omega$: for every $n \geq N$, we have $\overline{\triangle}_\mu^{\overline{k}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \overline{\triangle}_\mu^{\overline{k}}(E_2)$.

□

Proof of Proposition 12: If $\bigwedge E \wedge \mu$ is consistent, then $\triangle_\mu^{k+1}(E) \equiv \triangle_\mu^{k}(E) \equiv \bigwedge E \wedge \mu$, and the property holds. If $\bigwedge E \wedge \mu$ is inconsistent, then whenever an interpretation ω satisfies at least $k+1$ bases from E, it satisfies at least k elements from E, and the conclusion follows.

□

Proof of Proposition 13: By definition, the models of $\triangle_\mu^{d_D,\Sigma}(E)$ are exactly the models ω of μ minimizing $d_D(\omega, E) = \sum_{K \in E} d_D(\omega, K)$. Since d_D is drastic, the number of bases of E satisfied by ω is exactly $\#(E) - d_D(\omega, E)$, hence the minimal value of $d_D(\omega, E)$ when ω varies among the models of μ is equal to $\#(E) - k_{max}$, and this shows that $\triangle_\mu^{k_{max}}(E) \equiv \triangle_\mu^{d_D,\Sigma}(E)$. Finally, the fact that $\triangle_\mu^{d_D,\Sigma} = \triangle_\mu^{d_D,\text{GMAX}}$ (Theorem 4 from [18]) concludes the proof.

□

Proof of Lemma 1: Since $\triangle_\mu^{k_{max}}$ coincides with the IC merging operator $\triangle_\mu^{d_D,\Sigma}$ (Proposition 13), the fact that it satisfies **(IC0) - (IC8)** and **(Maj)** is a direct consequence of Theorem 2 from [19]. Now, the fact that it also satisfies **(Disj)** and **(Card)** comes from Proposition 6 and the fact that $\triangle_\mu^{k_{max}}$ can be associated to an equivalent quota merging operator (such a quota merging operator is the one with $k = k_{max}$: while the prior computation of k_{max} is necessary to get a quota operator, its unique impact concerns the computational aspects (but not the logical ones)).

□

Proof of Lemma 2: The result directly follows from the fact that $\triangle_\mu^{k_{max}}$ coincides with $\triangle_\mu^{d_D,\Sigma}$ (Proposition 13), which can be considered as the DA^2 merging operator $\triangle_\mu^{d_D,\textsc{Max},\Sigma}$, where each base consists of a single formula (see [16]). Theorem 3 from [16] completes the proof.

□

Proof of Lemma 3: The result directly follows from the fact that $\triangle_\mu^{k_{max}}$ coincides with $\triangle_\mu^{d_D,\Sigma}$ (Proposition 13), and Theorem 2 from [12].

□

Proof of Proposition 14: Let us show that if ω and ω' are two models of μ such that ω satisfies k_{max} bases from E and ω' satisfies k' bases from E, with $k' < k_{max}$, then $d_{d,\textsc{Gmin}}(\omega, E)$ is strictly lower than $d_{d,\textsc{Gmin}}(\omega', E)$ with respect to the lexicographic ordering $\leq_{lex}$. This is easy since (1) when ω satisfies k_{max} bases from E and d is a pseudo-distance, the k_{max} first coordinates of $d_{d,\textsc{Gmin}}(\omega, E)$ are equal to 0, and (2) when ω' satisfies stricly less bases from E and d is a pseudo-distance, the k_{max}^{th} coordinate of $d_{d,\textsc{Gmin}}(\omega', E)$ is not equal to 0.

□

Proof of Proposition 15: From Proposition 14, we know that $\triangle_\mu^{d_D,\textsc{Gmin}}(E) \models \triangle_\mu^{k_{max}}(E)$. So it remains to show that $\triangle_\mu^{k_{max}}(E) \models \triangle_\mu^{d_D,\textsc{Gmin}}(E)$. Let us consider a model ω of $\triangle_\mu^{k_{max}}(E)$, where $E = \{K_1, \ldots, K_n\}$.

First suppose that $(\bigvee E) \wedge \mu$ is consistent. Then ω satisfies μ and a maximal number k of bases K_i (i.e., there is no ω' that satisfies more than k bases), and k is strictly greater than 0. Hence, the k first elements of the list $d_{d_D,\textsc{Gmin}}(\omega, \{K_1, \ldots, K_n\})$ are 0, and the $n-k$ following ones are 1. Since this list is necessarily minimal with respect to the lexicographic ordering among the lists induced by the models of μ and E (since for all other ω', $d_{d_D,\textsc{Gmin}}(\omega', \{K_1, \ldots, K_n\})$ is a list of at most k 0s, followed by 1s), ω is a model of $\triangle_\mu^{d_D,\textsc{Gmin}}(E)$. Subsequently, $\triangle_\mu^{k_{max}}(E) \models \triangle_\mu^{d_D,\textsc{Gmin}}(E)$.

Now, suppose that $(\bigvee E) \wedge \mu$ is inconsistent. Then we have $\triangle_\mu^{k_{max}}(E) \equiv \mu$, so for every model ω of μ and every base K_i of E we have $d_D(\omega, K_i) = 1$. So for every model ω of μ, $d_{d_D,\textsc{Gmin}}(\omega, E)$ is $(1, 1, \ldots, 1)$, and $\triangle_\mu^{d_D,\textsc{Gmin}}(E) \equiv \mu \equiv \triangle_\mu^{k_{max}}(E)$.

□

Proof of Proposition 16: In order to show that $\triangle^{d,\textsc{Gmin}}$ satisfies **(IC0 - IC8)** we first show that the function which associates to each profile E the preorder $\leq_E^{d,\textsc{Gmin}}$ is a syncretic assignment, and conclude by Theorem 11 of [18]. Let us first state two useful lemmata:

Definition 13 ($\odot$) *Let v_1 and v_2 be two lists of integers. We note $v_1 \odot v_2$ the list of integers obtained by sorting in increasing order the concatenation of v_1 and v_2.*

Lemma 5 *Let v_1, v_1', v_2, v_2' be four lists of integers sorted in increasing order. If*

$v_1 \leq_{lex} v'_1$ *and* $v_2 \leq_{lex} v'_2$*, then* $v_1 \odot v_2 \leq_{lex} v'_1 \odot v'_2$.

Proof of Lemma 5: Suppose that $v_1 \leq_{lex} v'_1$ and $v_2 \leq_{lex} v'_2$, it is easy to show that: $v_1 \odot v_2 \leq_{lex} v'_1 \odot v_2$ and $v'_1 \odot v_2 \leq_{lex} v'_1 \odot v'_2$. Then by transitivity of $\leq_{lex}$, we get $v_1 \odot v_2 \leq_{lex} v'_1 \odot v'_2$.

□

Lemma 6 *Let* v_1, v'_1, v_2, v'_2 *be four lists of integers sorted in increasing order. If* $v_1 \leq_{lex} v'_1$ *and* $v_2 <_{lex} v'_2$*, then* $v_1 \odot v_2 <_{lex} v'_1 \odot v'_2$ *(where* $<_{lex}$ *designed the strict relation associated to* $\leq_{lex}$*).*

Proof of Lemma 6: Under the assumptions of the lemma, it is easy to show that: $v_1 \odot v_2 \leq_{lex} v'_1 \odot v_2$ and $v'_1 \odot v_2 <_{lex} v'_1 \odot v'_2$. Then by transitivity of $\leq_{lex}$, we get $v_1 \odot v_2 <_{lex} v'_1 \odot v'_2$.

□

Now let us check the conditions of syncretic assignements:

1. If $\omega \models E$ and $\omega' \models E$, then $\forall K_i \in E, \omega \models E$ and $\omega' \models E$, so $d_{d,\text{GMIN}}(\omega, E) = (0, 0, \ldots, 0)$ and $d_{d,\text{GMIN}}(\omega', E) = (0, 0, \ldots, 0)$, so $\omega \simeq_E^{d,\text{GMIN}} \omega'$.
2. If $\omega \models E$ and $\omega' \not\models E$, then $d_{d,\text{GMIN}}(\omega, E) = (0, 0, \ldots, 0)$ and $d_{d,\text{GMIN}}(\omega', E) \neq (0, 0, \ldots, 0)$, so $\omega <_E^{d,\text{GMIN}} \omega'$.
3. If $E_1 \equiv E_2$, then $\leq_{E_1}^{d,\text{GMIN}} = \leq_{E_2}^{d,\text{GMIN}}$.
4. We want to show that $\forall \omega \models K \exists \omega' \models K'$ such that $\omega' \leq_{\{K,K'\}}^{d,\text{GMIN}} \omega$. We have that $d(\omega, K) = 0$ and $d(\omega, K') = \min_{\omega'' \models K'} d(\omega, \omega'')$. Consider any $\omega' \models K'$ such that $d(\omega, \omega') = d(\omega, K')$. Then $d(\omega', K) = \min_{\omega'' \models K} d(\omega', \omega'') \leq d(\omega', \omega)$, and $d(\omega', K') = 0$. So $d_{d,\text{GMIN}}(\omega', \{K, K'\}) \leq_{lex} d_{d,\text{GMIN}}(\omega, \{K, K'\})$. So by definition $\omega' \leq_{\{K,K'\}}^{d,\text{GMIN}} \omega$.
5. We want to show that if $d_{d,\text{GMIN}}(\omega, E_1) \leq_{lex} d_{d,\text{GMIN}}(\omega', E_1)$ and $d_{d,\text{GMIN}}(\omega, E_2) \leq_{lex} d_{d,\text{GMIN}}(\omega', E_2)$, then $d_{d,\text{GMIN}}(\omega, \{E_1, E_2\}) \leq_{lex} d_{d,\text{GMIN}}(\omega', \{E_1, E_2\})$. This is a direct consequence of Lemma 5.
6. We want to show that if $d_{d,\text{GMIN}}(\omega, E_1) <_{lex} d_{d,\text{GMIN}}(\omega', E_1)$ and $d_{d,\text{GMIN}}(\omega, E_2) \leq_{lex} d_{d,\text{GMIN}}(\omega', E_2)$, then $d_{d,\text{GMIN}}(\omega, \{E_1, E_2\}) <_{lex} d_{d,\text{GMIN}}(\omega', \{E_1, E_2\})$. This is a direct consequence of Lemma 6.

So the function is a syncretic assignment, and by Theorem 11 of [18] this shows that $\triangle^{d,\text{GMIN}}$ satisfies **(IC0 - IC8)**.

(Disj): Direct consequence of Proposition 14 since $\triangle_\mu^{d,\text{GMIN}}(E) \models \Delta_\mu^{k_{\max}}(E)$.

(Card): Consider the following counter-example: $\mathcal{P} = \{a, b\}, E = \{K_1, K_2, K_3\}$ with $K_1 = \{\neg a\}$, $K_2 = \{a \wedge \neg b\}$ and $K_3 = \{(\neg a \wedge b) \vee (a \wedge \neg b)\}$. $\text{MAXCONS}_\top(E)$ contains two elements: $M_1 = \{\neg a, (\neg a \wedge b) \vee (a \wedge \neg b)\}$ and $M_2 = \{a \wedge \neg b, (\neg a \wedge b) \vee (a \wedge \neg b)\}$. Clearly, $\#(M_1) = \#(M_2)$. $\triangle_\mu^{d_H,\text{GMIN}}(E) \equiv a \wedge \neg b$ is consistent with M_2 but not with M_1.

(Wmaj) and (Maj): Consider the following counter-example: $\mathcal{P} = \{a, b\}$, $E_1 =$

$\{a \wedge b\}$, $E_2 = \{\neg a \wedge \neg b\}$ and $\mu \equiv b$. Then, for any $n \geq 1$, $\triangle_\mu^{d,\textsc{Gmin}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ has a single model $\omega = (a = 1; b = 1)$ (the first element of $d_{d,\textsc{Gmin}}(\omega, E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ is 0, so the distance is minimal, and ω is the only world in this case). So $\triangle_\mu^{d,\textsc{Gmin}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ is not consistent with $\triangle_\mu^{d,\textsc{Gmin}}(E_2)$ which single model is $\omega' = (a = 0, b = 1)$. Hence **(Wmaj)** (and then **(Maj)**) is not satisfied.

□

Proof of Proposition 17: If $\bigvee E_2$ is consistent with the constraint μ, then for any $n > \#(E_1)$, no model of E_1 which is not a model of E_2 can be in $[\triangle_\mu^{d,\textsc{Gmin}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})]$ because the list $d_{d,\textsc{Gmin}}(\omega, E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$ contains at least n zero when ω is a model of $\triangle_\mu^{d,\textsc{Gmin}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n})$. Hence $\exists n \in I\!N$, $\triangle_\mu^{d,\textsc{Gmin}}(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \triangle_\mu^{d,\textsc{Gmin}}(E_2)$.

□

Proof of Proposition 18:

- $\triangle^{d,\textsc{Gmin}}$ is strategy-proof for i_p if every base from the profile E is complete (i.e., each base has a unique model).

  We first show that if a merging operator $\triangle_\mu^{d,\textsc{Gmin}}$ is not strategy-proof for i_p, then it is not strategy-proof by erosion (i.e., when a manipulation is possible by reporting a base which entails the actual one). Clearly, no such manipulation is possible when each base from the profile is complete, so we can conclude that $\triangle_\mu^{d,\textsc{Gmin}}$ is not strategy-proof for i_p, hence for the two other indexes as well ($\triangle_\mu^{d,\textsc{Gmin}}$ satisfies **(IC1)**). *Reductio ad absurdum.* Suppose that there exists a profile $E = \{K_2, \ldots, K_n\}$, an integrity constraint μ and two bases K and K' with $K' \not\models K$, such that

$$i_p(K, \triangle_\mu^{d,\textsc{Gmin}}(\{K\} \sqcup E) < i_p(K, \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E).$$

  Equivalently:

$$\frac{\#([K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K\} \sqcup E)])}{\#([\triangle_\mu^{d,\textsc{Gmin}}(\{K\} \sqcup E)])} < \frac{\#([K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)])}{\#([\triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)])}.$$

  We define K'' by $[K''] = [K \wedge K' \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)]$. We show in the rest of the proof that a manipulation can be achieved by reporting K'' instead of K, (hence a manipulation by erosion since $K'' \models K$).

  1. First, we show by *reductio ad absurdum* that $K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E) \models K'$.

Let us suppose that $\exists \omega \not\models K'$ such that $\omega \models K \wedge \triangle_{\mu}^{d,\textsc{Gmin}}(\{K'\} \sqcup E)$. Since ω is a model of $\triangle_{\mu}^{d,\textsc{Gmin}}(\{K'\} \sqcup E)$, ω satisfies μ and a maximal number of bases of $\{K'\} \sqcup E$, say k bases. Since $\omega \not\models K'$, ω satisfies k bases of E. So, ω satisfies $k+1$ bases of $\{K\} \sqcup E$. Suppose that $\exists \omega' \not\models K$ such that $\omega' \models \triangle_{\mu}^{d,\textsc{Gmin}}(\{K\} \sqcup E)$. Then ω' satisfies μ and satisfies at least $k+1$ bases of $\{K\} \sqcup E$. Since $\omega' \not\models K$, ω satisfies at least $k+1$ bases of E, so at least $k+1$ bases of $\{K'\} \sqcup E$. This contradicts the fact that ω is a model of $\triangle_{\mu}^{d,\textsc{Gmin}}(\{K'\} \sqcup E)$ by satisfying a maximal number k of bases of $\{K'\} \sqcup E$. So every model of $\triangle_{\mu}^{d,\textsc{Gmin}}(\{K\} \sqcup E)$ is a model of K, and $i_p(K, \triangle_{\mu}^{d,\textsc{Gmin}}(\{K\} \sqcup E)) = 1$ is maximal, which contradicts the assumption.

2. Second, we have $[K''] \neq \emptyset$, since otherwise there would be no model of K in $\triangle_{\mu}^{d,\textsc{Gmin}}(\{K'\} \sqcup E)$, which contradicts the manipulability of E for K'.
3. Let us now consider a model ω_1 of $K \wedge K' \wedge \triangle_{\mu}^{d,\textsc{Gmin}}(\{K'\} \sqcup E)$. In order to simplify the notations, we note d instead of $d_{d,\textsc{Gmin}}$ in this proof. We have $\omega_1 \models \mu$ and $d(\omega_1, \{K'', K_2, \ldots, K_n\}) = d(\omega_1, \{K', K_2, \ldots, K_n\})$, because $d(\omega_1, K'') = d(\omega_1, K') = 0$. Moreover:

$$d(\omega_1, \{K', K_2, \ldots, K_n\}) = \min(\{d(\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}).$$

So:

$$d(\omega_1, \{K'', K_2, \ldots, K_n\}) = \min(\{d(\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}). \tag{A.5}$$

and

$$\min(\{d(\omega, \{K'', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}) \leq \min(\{d(\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}).$$

Besides, since $K'' \models K'$, we have that $\forall \omega \in \mathcal{W}, d(\omega, K') \leq d(\omega, K'')$. So $\forall \omega \in \mathcal{W}, d(\omega, \{K', K_2, \ldots, K_n\}) \leq_{lex} d(\omega, \{K'', K_2, \ldots, K_n\})$, and

$$\min(\{\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}) \leq \min(\{\omega, \{K'', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}).$$

With (A.5), we get:

$$\min(\{d(\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}) = \min(\{d(\omega, \{K'', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}). \tag{A.6}$$

4. Consider now a model ω_1 of $K \wedge \triangle_{\mu}^{d,\textsc{Gmin}}(\{K'\} \sqcup E)$. We have $\omega_1 \models \mu$ and $\omega_1 \models K'$ from point 1. of the proof. Then $\omega_1 \models K''$, and since $d(\omega_1, K') = d(\omega_1, K'') = 0$, we have $d(\omega_1, \{K'', K_2, \ldots, K_n\}) = d(\omega_1, \{K', K_2, \ldots, K_n\})$. Furthermore, since:

$$d(\omega_1, \{K', K_2, \ldots, K_n\}) = \min(\{d(\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}),$$

(A.6) gives that:

$d(\omega_1, \{K'', K_2, \ldots, K_n\}) = \min(\{d(\omega, \{K'', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex}))$.

So ω_1 is a model of $\triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E)$ and we have:

$$\#([K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)]) \leq \#([K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E)]).$$

5. Finally, if we consider $\omega_1 \models \neg K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E)$, then $\omega_1 \models \mu$ and:

   $d(\omega_1, \{K'', K_2, \ldots, K_n\}) = \min(\{d(\omega, \{K'', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex})$.

   Since $K'' \models K'$, we have that $d(\omega_1, K') \leq d(\omega_1, K'')$. So we get:

   $$d(\omega_1, \{K', K_2, \ldots, K_n\}) \leq_{lex} d(\omega_1, \{K'', K_2, \ldots, K_n\}).$$

   Hence:

   $d(\omega_1, \{K', K_2, \ldots, K_n\}) \leq_{lex} \min(\{d(\omega, \{K'', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex})$.

   From (A.6), we get:

   $d(\omega_1, \{K', K_2, \ldots, K_n\}) \leq_{lex} \min(\{d(\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex})$.

   So we can deduce that:

   $d(\omega_1, \{K', K_2, \ldots, K_n\}) = \min(\{d(\omega, \{K', K_2, \ldots, K_n\}) \mid \omega \models \mu\}, \leq_{lex})$

   and ω_1 is a model of of $\triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)$ and:

   $$\#([\neg K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E)]) \leq \#([\neg K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)]).$$

   Then,

   $$\frac{\#([K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)])}{\#([\triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E)])} \leq \frac{\#([K \wedge \triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E)])}{\#([\triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E)])}.$$

So,

$$i_p(K, \triangle_\mu^{d,\textsc{Gmin}}(\{K'\} \sqcup E) \leq i_p(K, \triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E).$$

And finally,

$$i_p(K, \triangle_\mu^{d,\textsc{Gmin}}(\{K\} \sqcup E) < i_p(K, \triangle_\mu^{d,\textsc{Gmin}}(\{K''\} \sqcup E),$$

which concludes the proof.

- $\triangle^{d,\textsc{Gmin}}$ is strategy-proof for the indexes i_{d_w} and i_{d_s} if every base from the profile E is complete, or if $\#(E) = 2$ and $\mu \equiv \top$.

  We know that a merging operator satisfying **(IC1)** and strategy-proof for i_p is also strategy-proof for the drastic indexes i_{d_w} and i_{d_s}. Here, with the first point of the proof, we know that $\triangle^{d,\text{GMIN}}$ is strategy-proof for i_p if every base of the profile is complete, and then the result for i_{d_w} and i_{d_s} follows ($\triangle^{d,\text{GMIN}}$ satisfies **(IC1)**).

The second case is when $\#(E) = 2$ and $\mu \equiv \top$. The result under these assumptions is a direct consequence of the following lemma:

Lemma 7 $\triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\}) \wedge K_1$ *is consistent.*

Proof of Lemma 7: *Reductio ad absurdum.* Let us suppose that $\triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\})$ is inconsistent with K_1. We have that:

$$\exists\omega' \models \neg K_1, \forall\omega \models K_1, d(\omega, \{K_1, K_2\}) >_{lex} d(\omega', \{K_1, K_2\}).$$

Since $\forall\omega \models K_1, d(\omega, K_1) = 0$, we get:

$$\exists\omega' \models \neg K_1, \forall\omega \models K_1, (0, d(\omega, K_2)) >_{lex} d_{d,\text{GMIN}}(\omega', \{K_1, K_2\}). \tag{A.7}$$

As $\omega' \models \neg K_1$, we have $d(\omega', K_1) \neq 0$; hence for Inequation (A.7) to hold we must have $d(\omega', K_2) = 0$. So

$$\exists\omega' \models \neg K_1, \forall\omega \models K_1, (0, d(\omega, K_2)) >_{lex} (0, d(\omega', K_1)).$$

In particular, if we consider $\omega_1 \models K_1$ such that $d(\omega', K_1) = d(\omega', \omega_1)$, we get:

$$(0, d(\omega_1, K_2)) >_{lex} (0, d(\omega', \omega_1)).$$

This requires that $d(\omega_1, K_2) > d(\omega', \omega_1)$ with $\omega' \models K_2$, but this is impossible. Contradiction.

$\square$

Let us now prove the main proposition:

i_{d_w} : Since $\triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\}) \wedge K_1$ is consistent (Lemma 7), we always have $i_{d_w}(K_1, \triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\})) = 1$, so no manipulation is possible (i_{d_w} is maximal).

i_{d_s} : *Reductio ad absurdum.* If $\triangle_{\top}^{d,\text{GMIN}}$ is not strategy-proof, then we can find K_1' such that $i_{d_s}(K_1, \triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\})) < i_{d_s}(K_1, \triangle_{\top}^{d,\text{GMIN}}(\{K_1', K_2\}))$. For the strong drastic index, this means exactly that $i_{d_s}(K_1, \triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\})) = 0$ and $i_{d_s}(K_1, \triangle_{\top}^{d,\text{GMIN}}(\{K_1', K_2\})) = 1$. So we have:

$$\triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\}) \not\models K_1 \tag{A.8}$$

$$\triangle_{\top}^{d,\text{GMIN}}(\{K_1', K_2\}) \models K_1. \tag{A.9}$$

Since $\triangle_{\top}^{d,\text{GMIN}}(\{K_1', K_2\}) \wedge K_2$ is consistent (Lemma 7), we can find $\omega_2 \models K_2$ such that $\omega_2 \models \triangle_{\top}^{d,\text{GMIN}}(\{K_1', K_2\})$. With (A.9), we can conclude that $\omega_2 \models K_1$ as well.

Since $\omega_2 \models K_1 \wedge K_2$, then $d(\omega_2, \{K_1, K_2\}) = (0, 0)$. So for every model ω of $\triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\})$, we have $d(\omega, \{K_1, K_2\}) = (0, 0)$. This implies that $\triangle_{\top}^{d,\text{GMIN}}(\{K_1, K_2\}) \equiv K_1 \wedge K_2$. This contradicts (A.8), so no manipulation is possible.

$\square$

Proof of Proposition 19:

- $\triangle^{d_D,\text{GMIN}}$ is strategy-proof for the three indexes i_{d_w}, i_p and i_{d_s}. This is a consequence of Proposition 15 ($\triangle^{d_D,\text{GMIN}} = \Delta^{k_{\max}}$) and Proposition 3 ($\Delta^{k_{\max}}$ is strategy-proof for the three indexes).
- $\triangle^{d_H,\text{GMIN}}$ is strategy-proof for i_p if and only if every base from the profile E is complete.
  - If every base from the profile E is complete, from Proposition 18, it comes that $\triangle^{d_H,\text{GMIN}}$ is strategy-proof for i_p.
  - As to the converse, the following example shows that $\triangle_\mu^{d_H,\text{GMIN}}$ is not strategy-proof for i_p, even when $\mu = \top$ and two bases are to be merged. Let us consider $[K_1] = \{0000, 0111, 1011, 1101, 1110\}$, $[K_2] = \{1000, 0100, 0010, 0001\}$, and $\mu = \top$. Then $[\triangle_\top^{d_H,\text{GMIN}}(\{K_1, K_2\}] = \{0000, 0001, 0010, 0100, 1000\}$, and $i_p(K_1, \triangle_\top^{d_H,\text{GMIN}}(\{K_1, K_2\})) = \frac{1}{5}$.

    If agent 1 gives K'_1 with $[K'_1] = \{0111, 1011, 1101, 1110\}$ instead of K_1, then $[\triangle_\top^{d_H,\text{GMIN}}(\{K'_1, K_2\})] = \{0001, 0010, 0100, 0111, 1000, 1011, 1101, 1110\}$ and $i_p(K_1, \triangle_\top^{d_H,\text{GMIN}}(\{K'_1, K_2\})) = \frac{1}{2}$, showing the manipulability.
- $\triangle_\mu^{d_H,\text{GMIN}}$ is strategy-proof for i_{d_w} and i_{d_s} if and only if every base from the profile E is complete, or if $\#(E) = 2$ and $\mu \equiv \top$.
  - If every base from the profile E is complete or if $\#(E) = 2$ and $\mu \equiv \top$, from Proposition 18, it comes that $\triangle^{d_H,\text{GMIN}}$ is strategy-proof for the drastic indexes i_{d_w} and i_{d_s}.
  - As to the converse, by case analysis:

    i_{d_w} : Suppose that $\mu \not\equiv \top$ and $\#(E) = 2$. Then consider $\mathcal{P} = \{a, b\}$, $[K_1] = \{00, 01\}$, $[K_2] = \{11\}$, and $\mu = a \vee \neg b$. We have $[\triangle_\mu^{d_H,\text{GMIN}}(\{K_1, K_2\}] = \{11\}$, and $i_{d_w}(K_1, \triangle_\mu^{d_H,\text{GMIN}}(\{K_1, K_2\})) = 0$. If agent 1 gives $[K'_1] = \{00\}$ instead of K_1, then $[\triangle_\mu^{d_H,\text{GMIN}}(\{K'_1, K_2\})] = \{00, 11\}$ and we have $i_{d_w}(K_1, \triangle_\mu^{d_H,\text{GMIN}}(\{K'_1, K_2\})) = 1$.

    Suppose now that $\mu \equiv \top$ and $\#(E) \neq 2$. Then consider $[K_1] = \{000, 001\}$, $[K_2] = \{100, 111\}$, $[K_3] = \{011\}$ and $\mu = \top$. We obtain $[\triangle_\top^{d_H,\text{GMIN}}(\{K_1, K_2, K_3\}] = \{011\}$, and $i_{d_w}(K_1, \triangle_\top^{d_H,\text{GMIN}}(\{K_1, K_2, K_3\})) = 0$. If agent 1 gives K'_1 with $[K'_1] = \{000\}$ instead of K_1, then $[\triangle_\top^{d_H,\text{GMIN}}(\{K'_1, K_2, K_3\})] = \{000, 011\}$ and $i_{d_w}(K_1, \triangle_\top^{d_H,\text{GMIN}}(\{K'_1, K_2, K_3\})) = 1$.

    i_{d_s} : Suppose that $\mu \not\equiv \top$ and $\#(E) = 2$, and consider $\mathcal{P} = \{a, b, c\}$, $[K_1] = \{000, 011\}$, $[K_2] = \{001, 111\}$, and $\mu = a \vee b \vee \neg c$. Then $[\triangle_\mu^{d_H,\text{GMIN}}(\{K_1, K_2\}] = \{000, 011, 111\}$, and $i_{d_s}(K_1, \triangle_\mu^{d_H,\text{GMIN}}(\{K_1, K_2\})) = 0$. If agent 1 gives $[K'_1] = \{000\}$ instead of K_1, then $[\triangle_\mu^{d_H,\text{GMIN}}(\{K'_1, K_2\})] = \{000\}$ and $i_{d_s}(K_1, \triangle_\mu^{d_H,\text{GMIN}}(\{K'_1, K_2\})) = 1$.

    Finally, suppose $\mu \equiv \top$ and $\#(E) \neq 2$; consider $[K_1] = \{000, 011\}$, $[K_2] = \{000, 111\}$, $[K_3] = \{001, 111\}$ and $\mu = \top$. We have $[\triangle_\top^{d_H,\text{GMIN}}(\{K_1, K_2, K_3\}] = \{000, 111\}$, and $i_{d_s}(K_1, \triangle_\top^{d_H,\text{GMIN}}(\{K_1, K_2, K_3\})) = 0$. If agent 1 gives K'_1 with $[K'_1] = \{000\}$ instead of K_1, then the result is $[\triangle_\top^{d_H,\text{GMIN}}(\{K'_1,$

$K_2, K_3\})] = \{000\}$ and $i_{d_s}(K_1, \triangle_{\top}^{d_H,\text{GMIN}}(\{K_1', K_2, K_3\})) = 1$.

□

Proof of Proposition 20: Immediate from Theorem 2(1) from [16] and the fact that each $\triangle^{d,\text{GMIN}}$ operator coincides with the DA^2 merging operator $\triangle^{d,\text{MAX},\text{GMIN}}$, where each base consists of a single formula.

□

Proof of Proposition 21:

- Immediate from Proposition 2 and Proposition 15.
- Membership comes directly from Proposition 20. As to hardness, we consider the following polynomial reduction f from MAX-SAT-ASG$_{odd}$ to MERGE($\triangle^{d_H,\text{GMIN}}$). Let Σ be a propositional formula such that $Var(\Sigma) = \{x_1, \ldots, x_n\}$. Let $f(\Sigma) =$

$$\langle E = \{K_i = \{x_i \wedge \bigwedge_{j=1}^{2(i-1)} new_j\} \mid i \in 1, \ldots, n\}, \mu = \Sigma \wedge \bigwedge_{j=1}^{2(n-1)} \neg new_j, \alpha = x_n\rangle$$

  where each new_j ($j \in 1, \ldots, 2n-2$) is a new variable (not occurring in Σ.) Now, for every model ω of μ and for every $i \in 1, \ldots, n-1$, we have

$$d_H(\omega, K_i) < d_H(\omega, K_{i+1}).$$

  This shows that the lists $d_{d_H,\text{GMIN}}(\omega, E)$ obtained by sorting the set $\{d_H(\omega, K_i) \mid i \in 1, \ldots, n\}$ in increasing order are always sorted in the same way (independently of ω): the first element is $d_H(\omega, K_1)$, the second one is $d_H(\omega, K_2)$, etc. Furthermore, whenever a model ω_1 of μ is strictly lower than a model ω_2 of μ with respect to the lexicographic ordering $\preccurlyeq$ induced by $x_1 < x_2 < \ldots < x_n$, then $d_{d_H,\text{GMIN}}(\omega_1, E)$ is strictly greater than $d_{d_H,\text{GMIN}}(\omega_2, E)$ (with respect to $\leq_{lex}$). Since the models of μ are totally ordered with respect to $\preccurlyeq$, exactly one model of μ is minimal with respect to the preference ordering induced by E: this is the model of μ that is maximal with respect to $\preccurlyeq$. Accordingly, x_n is true in this model if and only if $\triangle_{\mu}^{d_H,\text{GMIN}}(E) \models \alpha$ holds. This concludes the proof.

□

Proposition 22 *Let k be an integer ≥ 0, $E = \{K_1, \ldots, K_n\}$ be a profile, and μ be an integrity constraint. The* alternative k-quota merging operator*, denoted* $\widehat{\triangle^k}$, *is defined in a model-theoretic way as:*

$$[\widehat{\triangle^k_\mu}(E)] = \begin{cases} \{\omega \in [\mu] \mid \forall K_i \in E\ \omega \models K_i\} \textit{ if non empty, else} \\ \{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\} \textit{ if non empty, else} \\ \{\omega \in [\mu]\} \end{cases}$$

$\widehat{\triangle^k}$ *operators satisfy* **(IC0)**, **(IC1)**, **(IC2)**, **(IC3)**, **(IC4)**, **(IC7)** *and* **(IC8)**. *They do not satisfy* **(IC5)**, **(IC6)**, **(Disj)** *and* **(Maj)** *in the general case.*

Proof of Proposition 22:

(IC0), (IC1), (IC2), (IC3) Obvious from the definition of $\widehat{\triangle^k}$.

(IC4) We have to show that if $K_1 \models \mu$, $K_2 \models \mu$, and $\widehat{\triangle^k_\mu}(\{K_1, K_2\}) \wedge K_1 \not\models \bot$, then $\widehat{\triangle^k_\mu}(\{K_1, K_2\}) \wedge K_2 \not\models \bot$.
Let $E = \{K_1, K_2\}$. Assume that $K_1 \models \mu$ and $K_2 \models \mu$. If $K_1 \wedge K_2 \wedge \mu$ is consistent or $\{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is not empty, then the definition of $\widehat{\triangle^k}$ is the same as the one of $\triangle^k$, so from Proposition 6, **(IC4)** holds. In the remaining case, i.e., if $K_1 \wedge K_2 \wedge \mu$ is not consistent and $\{\omega \in [\mu] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is empty, then $\widehat{\triangle^k_\mu}(\{K_1, K_2\}) \equiv \mu$. In this case, as $K_2 \models \mu$, $\widehat{\triangle^k_\mu}(\{K_1, K_2\}) \wedge K_2$ is consistent and **(IC4)** is satisfied.

(IC5) If $k = 2$ and $E_1 = \{K_1\}$, $E_2 = \{K_2\}$, then we have $\widehat{\triangle^2_\mu}(E_1) = \widehat{\triangle^2_\mu}(E_2) \equiv \mu$. But $\widehat{\triangle^2_\mu}(E_1 \sqcup E_2) \equiv K_1 \wedge K_2 \wedge \mu$ if consistent, and $\widehat{\triangle^2_\mu}(E_1) \wedge \widehat{\triangle^2_\mu}(E_2) \equiv \mu \not\models K_1 \wedge K_2 \wedge \mu$.

(IC6) The counter-example used in Proposition 6 to show that $\triangle^k$ dos not satisfy **(IC6)** still applies here.

(IC7) We have to show that $\widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2 \models \widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E)$.
If $\bigwedge E \wedge \mu_1$ is consistent or $\{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is not empty, then the definition of $\widehat{\triangle^k}$ is the same as the one of $\triangle^k$, so from Proposition 6, **(IC7)** holds. In the remaining case, i.e., if $\bigwedge E \wedge \mu_1$ is not consistent and $\{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is empty, then $\widehat{\triangle^k_{\mu_1}}(E) \equiv \mu_1$. In this case, $\widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2 \equiv \mu_1 \wedge \mu_2$. As $\bigwedge E \wedge \mu_1$ is not consistent, $\bigwedge E \wedge (\mu_1 \wedge \mu_2)$ is not consistent as well.
Now, since $\{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is empty, there is no model ω of $\mu_1 \wedge \mu_2$ such that $\#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ so $\{\omega \in [\mu_1 \wedge \mu_2] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is also empty.
Therefore $\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E) \equiv \mu_1 \wedge \mu_2$ and $\widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2 \models \widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E)$, showing that **(IC7)** is satisfied.

(IC8) We have to show that if $\widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2$ is consistent, then $\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E) \models \widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2$. We consider two cases:
(1) $\mu_1 \wedge \mu_2 \wedge \bigwedge E$ is consistent. Then $\mu_1 \wedge \bigwedge E$ is consistent as well and we have $\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E) \equiv \mu_1 \wedge \mu_2 \wedge \bigwedge E$. Hence $\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E) \models \widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2$, and **(IC8)** is

satisfied.

(2) $\mu_1 \wedge \mu_2 \wedge \bigwedge E$ is inconsistent. In this situation, two cases are possible:

- $\{\omega \in [\mu_1 \wedge \mu_2] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is not empty.

  Then $[\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E)] = \{\omega \in [\mu_1 \wedge \mu_2] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$.

  Suppose that $\mu_1 \wedge \bigwedge E$ is consistent. Then $\widehat{\triangle^k_{\mu_1}}(E) \equiv \mu_1 \wedge \bigwedge E$. As $\mu_1 \wedge \mu_2 \wedge \bigwedge E$ is inconsistent, we deduce that $\widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2$ is also inconsistent, which contradicts the assumption. So $\mu_1 \wedge \bigwedge E$ is inconsistent.

  As $\{\omega \in [\mu_1 \wedge \mu_2] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\} \subseteq \{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$, we know that $\{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is not empty. Since $[\widehat{\triangle^k_{\mu_1}}(E)] = \{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$, we have:

  $$\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E) \models \widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2$$

  and **(IC8)** holds.

- $\{\omega \in [\mu_1 \wedge \mu_2] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\}$ is empty.

  Then $[\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E)] = [\mu_1 \wedge \mu_2]$. By assumption, $\widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2$ is consistent, so we can deduce that $\widehat{\triangle^k_{\mu_1}}(E) \not\equiv \mu_1 \wedge \bigwedge E$ and $[\widehat{\triangle^k_{\mu_1}}(E)] \neq \{\omega \in [\mu_1] \mid \#(\{K_i \in E \mid \omega \models K_i\}) \geq k\} = \emptyset$. This shows that $\widehat{\triangle^k_{\mu_1}}(E) \equiv \mu_1$.

  Obviously,

  $$\widehat{\triangle^k_{\mu_1 \wedge \mu_2}}(E) \models \widehat{\triangle^k_{\mu_1}}(E) \wedge \mu_2$$

  and **(IC8)** holds also in this case.

(Disj) Consider the following counter-example: $\mathcal{P} = \{a, b\}$, $E = \{K_1, K_2\}$ with $K_1 = \{a\}$, $K_2 = \{b\}$, $k = 2$, $\mu = \neg a$. We have $\widehat{\triangle^k_{\mu}}(E) \equiv \neg a$. Clearly, while $\bigvee E$ is consistent with μ, we do not have $\widehat{\triangle^k_{\mu}}(E) \models \bigvee E$.

(Maj) Consider the same counter-example as the one given in the same item of the proof of Proposition 6.

□

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