Majority Merging: from Boolean Spaces to Affine Spaces

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Abstract. This paper is centered on the problem of merging (possibly conflicting) information coming from different sources. Though this problem has attracted much attention in propositional settings, propositional languages remain typically not expressive enough for a number of applications, especially when spatial information must be dealt with. In order to fill the gap, we consider a (limited) first-order logical setting, expressive enough for representing and reasoning about information modeled as half-spaces from metric affine spaces. In this setting, we define a family of distance-based majority merging operators which includes the propositional majority operator $\Delta^{d_H,\Sigma}$. We identify a subclass of interpretations of our representation language for which the result of the merging process can be computed and expressed as a formula.

1 INTRODUCTION

The problem of merging information coming from different sources arises in many applications, e.g. distributed knowledge systems. Due to the multiplicity of the sources providing information, combining them may lead to conflicts. However, one would need to get from a set of (possibly conflicting) belief/goal bases a single consistent belief/goal base representing a global view of the input set, taking into account every source as much as possible.

In the particular framework of propositional logic (PL), merging multiple belief/goal bases has been the point of interest of many works the last two decades [10, 11, 3, 7, 9, 8], a prominent reason being that many problems can be conveniently expressed using PL [4]. However, as PL only deals with true/false statements as units we often need a more expressive language to express information, in particular when the considered variables represent spatial entities, or when they range over an infinite, unbounded, or continuous domain. For instance, consider the following goal merging problem about a family wanting to purchase a new house. Suppose that the members of family compare accommodations on the basis of their type (flat or house), price and location. As one would expect, there is generally no type, price and location completely satisfying every person, though they need to find a compromise. PL is not sufficient to represent the available information, since two features in question (price and location) range over continuous domains, the price in a one-dimensional domain and the location in a two-dimensional domain. Yet each one of these domains is naturally associated with a proper *metric*, or *distance*. If half of the group would like to pay more than a certain price P while the other half would prefer a cheap house less than a price p with $p < P$, then it is natural to search for a house which price would range between p and P.

In this paper, we deal with the representation of such kind of information for which PL is not adequate. We define a class of merging operators that return a consistent set of information from a set of possibly conflicting sources. We consider quite a general setting, allowing each variable considered in the merging process to take its value out of a proper domain defined as unions and intersections of half-spaces from finite-dimensional metric affine spaces. The essence of our merging method then exploits the metrics associated to these domains. For this purpose, we take our inspiration from distance-based merging operators proposed in the PL framework [10, 8]. We propose a similar procedure for merging belief/goal bases expressed in a more general setting, which can be viewed as a fragment of First-Order Logic (FOL), monadic, typed, without symbol of quantification or function; furthermore we consider a particular class of interpretations. Since a common and natural choice to deal with conflicts among a group is to let the majority decide [10], we focus in this paper on the class of *majority merging operators*, which aims at computing a consistent result minimizing the global dissatisfaction of the input sources. The solid theoretical background on propositional majority merging operators [10, 8] and their aptitude to satisfy a significant set of rationality postulates motivates our choice. In particular we show that the class of merging operators we define includes the propositional majority merging operator $\Delta^{d_H,\Sigma}$. We identify a subclass of interpretations for which the result of the merging can be computed and expressed as a formula.

The rest of the paper is organized as follows: in the next section we provide some necessary mathematical background and we define the syntax and the semantics of our representation language. In Section 3 we define a class of merging operators. In Section 4 we point out how to compute the result of the merging under some restrictions. We conclude and present some perspectives for further work in Section 5. The results obtained are illustrated through the motivating example sketched above.

2 PRELIMINARIES

2.1 Background on linear spaces

For a given positive integer n, $\mathbb{R}^n$ denotes the n-dimensional real affine space. A *hyperplane* of $\mathbb{R}^n$ is a $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$. For instance, a line (resp. a plane) is a hyperplane of $\mathbb{R}^2$ (resp. of $\mathbb{R}^3$). Formally, a hyperplane h of $\mathbb{R}^n$ is characterized by a vector $(h_0, \ldots, h_n)$ of $\mathbb{R}^{n+1}$ and is defined by the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid h_1x_1 + \ldots + h_nx_n = h_0\}$. Let $k \in \{1, \ldots, n\}$. A hyperplane h of $\mathbb{R}^n$ is said to be *rectilinear* if it is parallel to $n-1$ axes of $\mathbb{R}^n$, formally h is defined by the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_k = h_0\}$ for some $k \in \{1, \ldots, n\}$. A hyperplane h of $\mathbb{R}^n$ is associated to two *closed half-spaces* $h^{\leq}$ and

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$h^{\geq}$, namely two subsets of $\mathbb{R}^n$ defined as $h^{\leq} = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid h_1x_1 + \ldots + h_nx_n \leq h_0\}$ and $h^{\geq} = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid h_1x_1 + \ldots + h_nx_n \geq h_0\}$. Thus a closed half-space associated to a hyperplane h of $\mathbb{R}^n$ is one of the two parts into which h divides $\mathbb{R}^n$. An *open half-space* of $\mathbb{R}^n$ is the complement of a closed half-space in $\mathbb{R}^n$. A half-space is *rectilinear* if it is associated to a rectilinear hyperplane. A *convex polyhedron* of $\mathbb{R}^n$ is the finite intersection of closed or open half-spaces of $\mathbb{R}^n$. A rectilinear convex polyhedron of $\mathbb{R}^n$, also called a *cuboid* of $\mathbb{R}^n$, is the finite intersection of closed or open rectilinear half-spaces of $\mathbb{R}^n$.

A *norm* N on $\mathbb{R}^n$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}$ satisfying the following properties, for every $e_1, e_2 \in \mathbb{R}^n$, for every $\lambda \in \mathbb{R}$:

$$\begin{cases} N(e_1) = 0 \text{ iff } e_1 \text{ is the zero vector (positive definiteness)}, \\ N(\lambda \cdot e_1) = |\lambda|.N(e_1) \text{ (homogeneity)}, \\ N(e_1 + e_2) \leq N(e_1) + N(e_2) \text{ (subadditivity)} \end{cases}$$

Given a norm N on $\mathbb{R}^n$, a *metric* (or distance) d on $\mathbb{R}^n$ is typically derived from N, namely, d is a mapping from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ defined for every $e_1, e_2 \in \mathbb{R}^n$ as $d(e_1, e_2) = N(e_1 + (-e_2))$. Let $E \subseteq \mathbb{R}^n$, $S \subseteq E$ and d be the metric induced by a norm on $\mathbb{R}^n$. S is called an *open set* of E iff $\forall e \in S, \exists r > 0$ such that $\{e' \in E \mid d(e, e') < r\} \subset S$. S is a *closed set* if its complement in E is an open set. The *closure* of S, denoted $\circ S$, is the smallest closed set containing S. S is *bounded* if $\exists \lambda > 0$ such that $\forall c_1, c_2 \in S, d(c_1, c_2) \leq \lambda$.

2.2 Syntax and semantics of $\mathcal{L}$

Given the above preliminaries, we now define the syntax and semantics of our representation language $\mathcal{L}$. The alphabet of $\mathcal{L}$ consists of a finite set of variables $\mathcal{V} = \{x^1, x^2, \ldots\}$, a finite set of unary predicate symbols $\mathcal{P}$, noted $X, Y, \ldots$, or X^i_j, i and j being two positive integers, the usual logical connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), the usual constant symbols $\top$ (true) and $\bot$ (false) and the punctuation symbols $'('$ and $')'$.

Let $\mathcal{T} = \{t_1, t_2, \ldots\}$ be a finite set of *types*. We assume that we are given a mapping τ from $\mathcal{V} \cup \mathcal{P}$ to $\mathcal{T}$, namely every symbol of variable or predicate has a type. An *atom* is of the form $(x^i \in X)$, with $x^i \in \mathcal{V}$, $X \in \mathcal{P}$ and $\tau(x^i) = \tau(X)$. A *literal* is an atom or its negation. The language $\mathcal{L}$ is inductively defined as follows: every atom is a formula, $\top$, $\bot$ are formulas and given two formulas α and β, $(\neg\alpha)$, $(\alpha \wedge \beta)$, $(\alpha \vee \beta)$ are also formulas.

We consider a particular class $\mathcal{H}$ of interpretations for $\mathcal{L}$. An interpretation $\mathcal{I}$ from $\mathcal{H}$ is defined as a pair $\langle \mathcal{D}_\mathcal{I}, \mathcal{M}_\mathcal{I} \rangle$, where:

- $\mathcal{D}_\mathcal{I}$ is a mapping which associates to every type t_j of $\mathcal{T}$ a tuple (n_j, H_j, D_j, d_j), where:
  - n_j is a positive integer,
  - H_j is a finite non-empty set of hyperplanes of $\mathbb{R}^{n_j}$,
  - $D_j \subseteq \mathbb{R}^{n^j}$ results from some finite unions, intersections and complements of closed half-spaces associated to hyperplanes of H_j,
  - d_j is a metric on D_j induced by a norm on $\mathbb{R}^{n_j}$.
- $\mathcal{M}_\mathcal{I}$ is mapping which associates to every predicate symbol $X \in \mathcal{P}$ of type t_j a closed half-space $\mathcal{M}_\mathcal{I}(X)$ associated to a hyperplane of H_j, i.e., $\mathcal{M}_\mathcal{I}(X) = h^{\leq}$ or $\mathcal{M}_\mathcal{I}(X) = h^{\geq}$ with $h \in H_j$. We require that for every D_j, there exists $X_1, \ldots, X_n \in \mathcal{P}$ such that D_j is equal to a finite combination of $\mathcal{M}_\mathcal{I}(X_1), \ldots, \mathcal{M}_\mathcal{I}(X_n)$ where allowed combinations are unions, intersections and complements.

We also define the class $\mathcal{H}_S$ of interpretations as being the subclass of $\mathcal{H}$ satisfying the two following conditions, for every type t_j:

- every H_j is a finite set of rectilinear hyperplanes of $\mathbb{R}^{n_j}$,
- for every predicate symbol X of type t_j such that $\mathcal{M}_\mathcal{I}(X) = h^{\leq}$ with $h \in H_j$, there exists a predicate symbol Y such that $\mathcal{M}_\mathcal{I}(Y) = h^{\geq}$, and vice-versa,
- every metric d_j is induced by the Manhattan norm N_M on $\mathbb{R}^{n_j}$ defined for every $e = (e_1, \ldots, e_{n_j}) \in \mathbb{R}^{n_j}$ as $N_M(e) = \sum\{|e_k| \mid k \in \{1, \ldots, n_j\}\}$.

For each variable $x^i \in \mathcal{V}$ of type $t_j \in \mathcal{T}$, D_j is called the *domain* of x^i, also noted $dom(x^i)$. Let $\mathcal{I}$ be an interpretation from $\mathcal{H}$. An *$\mathcal{I}$-assignment* ω (on $\mathcal{V}$) is a mapping which associates to each variable $x^i \in \mathcal{V}$ an element of its domain. The semantics of an atom of the form $(x^i \in X)$ for a given $\mathcal{I}$-assignment ω is defined as $[\![(x^i \in X)]\!](\mathcal{I})(\omega) =$ true if $\omega(x^i) \in \mathcal{M}_\mathcal{I}(X)$, false otherwise.

In the rest of the paper, an $\mathcal{I}$-assignment ω is also considered as the vector $(\omega(x^1), \ldots, \omega(x^{|\mathcal{V}|})) \in \prod\{dom(x^i) \mid x^i \in \mathcal{V}\}$.

$\mathcal{W}_\mathcal{I}$ denotes the set of all $\mathcal{I}$-assignments. An $\mathcal{I}$-assignment ω is an *$\mathcal{I}$-model* of a formula ϕ (denoted $\omega \models_\mathcal{I} \phi$) iff it makes the formula ϕ true. A formula is said to be *$\mathcal{I}$-consistent* if it admits an $\mathcal{I}$-model. The set of $\mathcal{I}$-models of a formula ϕ is denoted $Mod_\mathcal{I}(\phi)$. Two formulas ϕ and ψ are *$\mathcal{I}$-equivalent* (denoted $\phi \equiv \psi$) iff $Mod_\mathcal{I}(\phi) = Mod_\mathcal{I}(\psi)$.

Notice that using a formula ϕ of $\mathcal{L}$ in the context of an interpretation $\mathcal{I} \in \mathcal{H}$, for every variable x^i of type t_j and for every hyperplane $h \in H_j$, if there exists two predicate symbols X and Y such that $\mathcal{M}_\mathcal{I}(X) = h^{\leq}$ and $\mathcal{M}_\mathcal{I}(Y) = h^{\geq}$, then we can easily express the fact that x^i belongs to one of the closed half-spaces $h^{\leq}$ and $h^{\geq}$ as well as the open half-spaces associated to h, the hyperplane h itself or the whole space $\mathbb{R}^{n_j}$ by use of the logical connectives $\neg$, $\wedge$ and $\vee$.

A *cube* is a finite conjunction of literals. A formula is said to be in *disjunctive normal form* (DNF) if it is a disjunction of cubes (also viewed as a set of cubes). Any formula $\phi \in \mathcal{L}$ can be transformed into an $\mathcal{I}$-equivalent DNF formula denoted $DNF(\phi)$, in a finite number of steps (though exponential in the size of the formula).

Example. We take the case drafted in the introduction to illustrate an interpretation $\mathcal{I}^*$ of $\mathcal{H}$. We consider the variables $\mathcal{V} = \{x^1, x^2, x^3\}$ where x^1 represents the kind of accomodation (house or flat), x^2 represents its price in thousands of euros (k€) ranging over the interval $[0, +\infty[$ and x^3 represent the location ranging over the plane $\mathbb{R}^2$. Since the variables range over different domains, we consider three types $\mathcal{T} = \{t_1, t_2, t_3\}$ such that for every $i \in \{1, 2, 3\}$, x^i has the type t_i. We consider a set of predicates $\mathcal{P}$ composed of 18 symbols denoted $X^1_1, \ldots, X^1_4, X^2_1, \ldots, X^2_5, X^3_1, \ldots, X^3_9$, such that for every $X^i_j \in \mathcal{P}$, X^i_j has the type t_i.

Assume that the family is composed of three members Helen, David and Marc. Figure 1 depicts some regions A, B and C of $\mathbb{R}^2$ considered by the family for the location of the accomodation, the half-spaces (i.e., half-planes) defining these regions, denoted $h^{\leq}_i$ and $h^{\geq}_i$, $i \in \{1, \ldots, 9\}$, themselves associated to 9 hyperplanes (i.e., lines); the presence of the dashed region will be explained in the next section. The equations of every hyperplane considered are defined as follows:

$$\begin{array}{lll} h_1 : 2x^3_1 - x^3_2 = -4, & h_2 : x^3_1 = 2, & h_3 : x^3_2 = 4, \\ h_4 : x^3_2 = 6, & h_5 : x^3_1 + x^3_2 = 11, & h_6 : 2x^3_1 + x^3_2 = 21, \\ h_7 : x^3_1 = 8, & h_8 : x^3_2 = 6, & h_9 : x^3_1 = 4. \end{array}$$

We ask each member of the family to express their wishes in terms of a formula. Helen prefers a flat between 90 and 100 k€ in the region A. David wants a flat below 80 k€ in the region B or a house

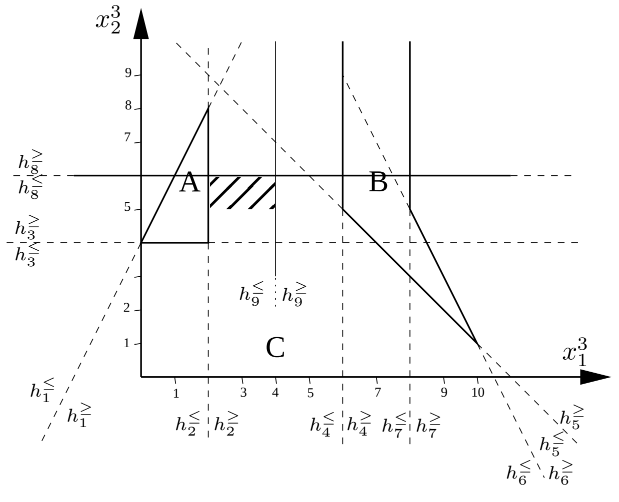

Figure 1. Nine lines $h_1, \ldots, h_9$ forming the set $\mathcal{H}_3$.

above 130 k€ in the region A. Marc expects an accomodation in the region C regardless of its price and of its type.

The interpretation $\mathcal{I}^* = \langle \mathcal{D}_{\mathcal{I}^*}, \mathcal{M}_{\mathcal{I}^*} \rangle$ on which we focus is defined as follows:

- x^1 is a one-dimensional binary variable. We set $n_1 = 1$, $\mathcal{H}_1 = \{0, 1\}$, $D_1 = \{0, 1\}$ and d_1 the restriction on D_1 of the usual distance over reals. 0 represents the flat and 1 the house. Notice that elements of $\mathcal{H}_1$ are points as hyperplanes of $\mathbb{R}^1$.
- x^2 is a one-dimensional variable ranging over the half-space $[0, +\infty[$. We set $n_2 = 1$, $\mathcal{H}_2 = \{0, 80, 90, 100, 130\}$, $D_2 = [0, +\infty[$ and d_2 the restriction on D_2 of the usual distance over reals.
- x^3 is a two-dimensional variable ranging over $\mathbb{R}^2$. We set $n_3 = 2$, $\mathcal{H}_3 = \{h_1, \ldots, h_9\}$, $D_3 = \mathbb{R}^2$ and d_3 the euclidean distance of the plane.
- $\mathcal{M}_{\mathcal{I}^*}$ is defined as follows:
  - predicate symbols of type t_1:

$$\begin{array}{ll} \mathcal{M}_{\mathcal{I}^*}(X_1^1) =]-\infty, 0], & \mathcal{M}_{\mathcal{I}^*}(X_2^1) = [0, +\infty[, \\ \mathcal{M}_{\mathcal{I}^*}(X_3^1) =]-\infty, 1], & \mathcal{M}_{\mathcal{I}^*}(X_4^1) = [1, +\infty[, \end{array}$$

  - predicate symbols of type t_2:

$$\begin{array}{ll} \mathcal{M}_{\mathcal{I}^*}(X_1^2) = [90, +\infty[, & \mathcal{M}_{\mathcal{I}^*}(X_2^2) =]-\infty, 100], \\ \mathcal{M}_{\mathcal{I}^*}(X_3^2) =]-\infty, 80], & \mathcal{M}_{\mathcal{I}^*}(X_4^2) = [130, +\infty[, \\ \mathcal{M}_{\mathcal{I}^*}(X_5^2) = [0, +\infty[, & \end{array}$$

  - predicate symbols of type t_3:

$$\begin{array}{lll} \mathcal{M}_{\mathcal{I}^*}(X_1^3) = h_1^{\geq}, & \mathcal{M}_{\mathcal{I}^*}(X_2^3) = h_2^{\leq}, & \mathcal{M}_{\mathcal{I}^*}(X_3^3) = h_3^{\geq}, \\ \mathcal{M}_{\mathcal{I}^*}(X_4^3) = h_4^{\geq}, & \mathcal{M}_{\mathcal{I}^*}(X_5^3) = h_5^{\geq}, & \mathcal{M}_{\mathcal{I}^*}(X_6^3) = h_6^{\leq}, \\ \mathcal{M}_{\mathcal{I}^*}(X_7^3) = h_7^{\leq}, & \mathcal{M}_{\mathcal{I}^*}(X_8^3) = h_8^{\leq}, & \mathcal{M}_{\mathcal{I}^*}(X_9^3) = h_9^{\leq}. \end{array}$$

  Notice that for every D_j there exists a formula ϕ such that $Mod_{\mathcal{I}^*}(\phi) = D_j$. Indeed, for instance, $Mod_{\mathcal{I}^*}(((x^1 \in X_1^1) \wedge (x^1 \in X_2^1)) \vee ((x^1 \in X_3^1) \wedge (x^1 \in X_4^1))) = \{0, 1\} = D_1$.

The formulas encoding the information provided by Helen, David and Marc are respectively:

- $\phi_1 = (x^1 \in X_1^1) \wedge (x^1 \in X_2^1) \wedge (x^2 \in X_1^2) \wedge (x^2 \in X_2^2) \wedge (x^3 \in X_1^3) \wedge (x^3 \in X_2^3) \wedge (x^3 \in X_3^3)$,
- $\phi_2 = ((x^1 \in X_1^1) \wedge (x^1 \in X_2^1) \wedge (x^2 \in X_3^2) \wedge (x^3 \in X_4^3) \wedge (x^3 \in X_5^3) \wedge ((x^3 \in X_6^3) \vee (x^3 \in X_7^3))) \vee ((x^1 \in X_3^1) \wedge (x^1 \in X_4^1) \wedge (x^2 \in X_4^2) \wedge (x^3 \in X_1^3) \wedge (x^3 \in X_2^3) \wedge (x^3 \in X_3^3))$,
- $\phi_3 = (x^3 \in X_8^3)$.

3 THE MERGING PROCESS

In this section $\mathcal{I}$ is any interpretation from $\mathcal{H}$. Recall that $\mathcal{W}_{\mathcal{I}}$ denotes the set of all $\mathcal{I}$-assignments. A *belief/goal base* is a finite set of formulas $\phi_1, \phi_2, \ldots$ from $\mathcal{L}$, also viewed as the formula that is conjunction of its elements. A *profile* is a finite multiset $\mathcal{K} = \{K_1, \ldots, K_m\}$ of $\mathcal{I}$-consistent belief/goal bases. A merging operator Δ is a mapping which associates to a profile $\mathcal{K}$ and a $\mathcal{I}$-consistent formula IC representing integrity constraints a subset $\Delta_{IC}(\mathcal{K})$ of $\mathcal{W}_{\mathcal{I}}$.

As in the PL case [8] we define a distance-based merging method of a profile via a three-step process: we first define the distance d between two $\mathcal{I}$-assignments $\omega, \omega' \in \mathcal{W}_{\mathcal{I}}$ as follows:

$$d(\omega, \omega') = \sum \{d_j(\omega(x^i), \omega'(x^i)) \mid x^i \in V, \tau(x^i) = t_j\}.$$

Given an $\mathcal{I}$-assignment ω, its "local distance" to a belief/goal base K is defined as follows: $d_K(\omega) = \inf\{d(\omega, \omega') \mid \omega' \models_{\mathcal{I}} K\}$. Then we use an aggregation function to compute the "global distance" between ω and the profile $\mathcal{K}$. As argued in the introduction, we focus in this paper on the aggregation function $\sum$ which supports the majority point of view of the belief/goal bases [10]. The global distance between ω and $\mathcal{K}$ is then denoted $d_{\mathcal{K}}(\omega) = \sum\{d_{K_k}(\omega) \mid K_k \in \mathcal{K}\}$. Lastly, the result of the merging satisfies $\Delta_{IC}(\mathcal{K}) = \{\omega \in \circ Mod_{\mathcal{I}}(IC) \mid d_{\mathcal{K}}(\omega) \text{ is minimal}\}$.

In PL the set $\Delta_{IC}(\mathcal{K})$ is obviously a non-empty set satisfying IC. Indeed every formula of PL admits a finite set of models over its set of variables, thus there exists at least one model ω of IC for which the distance $d_{\mathcal{K}}(\omega)$ reaches a minimum. However, in our class $\mathcal{H}$ of interpretations of $\mathcal{L}$, $Mod_{\mathcal{I}}(IC)$ is usually an infinite set and it can be the case that $\Delta_{IC}(\mathcal{K})$ is empty or that some $\mathcal{I}$-assignment ω in it is not an $\mathcal{I}$-model of IC. To circumvent these problems, we consider $\mathcal{I}$-assignments of $\circ Mod_{\mathcal{I}}(IC)$ as candidates. This choice ensures $\Delta_{IC}(\mathcal{K})$ to be a non-empty set when $Mod_{\mathcal{I}}(IC)$ is a bounded set. Furthermore, it does not question the natural distance requirement w.r.t. IC: each $\mathcal{I}$-assignment of $\circ Mod_{\mathcal{I}}(IC)$ is at distance 0 of IC w.r.t. $d_{\mathcal{K}}$. The following proposition holds:

Proposition 1. *(1) Assume $Mod_{\mathcal{I}}(IC)$ is a bounded non-empty set. Then $\Delta_{IC}(\mathcal{K})$ is a closed and bounded non-empty set.*
(2) Assume in addition that $Mod_{\mathcal{I}}(IC)$ is a closed set. Then $\Delta_{IC}(\mathcal{K})$ is a subset of $Mod_{\mathcal{I}}(IC)$.

Proof. (1) By definition the set $\circ Mod_{\mathcal{I}}(IC)$ is a closed subset of $\mathcal{W}_{\mathcal{I}}$ and it is bounded since $Mod_{\mathcal{I}}(IC)$ is bounded. Let us prove that $\Delta_{IC}(\mathcal{K})$ is a non-empty set. A norm on $\mathbb{R}^n$ is a continuous function. Since for every $x^i \in \mathcal{V}$ of type t_j, d_j is a distance induced by a norm and since $d_{\mathcal{K}}$ results from operations on d_j preserving continuity, $d_{\mathcal{K}}$ is a continuous function on $\mathcal{W}_{\mathcal{I}}$. Since $\circ Mod_{\mathcal{I}}(IC)$ is a closed and bounded set, the minimum of $d_{\mathcal{K}}$ on $\circ Mod_{\mathcal{I}}(IC)$ is reached in at least one point (from the Weierstrass extreme value theorem). This means that $\Delta_{IC}(\mathcal{K})$ is a non-empty subset of $\circ Mod_{\mathcal{I}}(IC)$, it is also closed since it is the inverse image of a closed set (a singleton) by a continuous function.
(2) If $Mod_{\mathcal{I}}(IC)$ is a closed set, it is equal to $\circ Mod_{\mathcal{I}}(IC)$. Therefore, $\Delta_{IC}(\mathcal{K})$ is a subset of $Mod_{\mathcal{I}}(IC)$ by definition. □

Example (continued). Let $\mathcal{K} = \{K_1, K_2, K_3\}$ with for every $K_k \in \mathcal{K}$, $K_k = \{\phi_k\}$. Let $IC = (x^3 \in X_9^3)$, i.e., the integrity constraints only bear on the location of the accomodation: available accomodations must be contained in the half-plane $h_9^{\leq}$. Let D be the closed region of $\mathbb{R}^2$ which is dashed in Figure 1. Then $\Delta_{IC}(\mathcal{K}) = \{\omega \in \mathcal{W}_{\mathcal{I}} \mid \omega(x^1) = 0, \omega(x^2) \in [80, 90], \omega(x^3) \in D\}$.

In PL every variable $x^i \in \mathcal{V}$ is binary, i.e., of type t_1 according to our running example. Hence, for two given assignments ω, ω' of PL we simply have $d_1(\omega(x^i), \omega'(x^i)) = 0$ if $\omega(x^i) = \omega'(x^i)$, 1 otherwise; this implies that the distance between two assignments ω, ω' defined as $d(\omega, \omega') = \sum\{d_1(\omega(x^i), \omega'(x^i)) \mid x^i \in \mathcal{V}\}$ corresponds to the Hamming distance d_H between assignments of PL. This shows that in the restricted PL setting, $\Delta_{IC}(\mathcal{K})$ corresponds to the propositional majority merging operator $\Delta_{IC}^{d_H,\Sigma}(\mathcal{K})$ [10, 8].

Notice also that in PL the metric is the same for every variable. Yet in our framework the variables can range over different domains and be associated to different metrics. Due to the incommensurability of these metrics, our majority merging operator does not garantee to give the same importance to every variable. If one would like to overcome this problem, one would need to add an upstream step of normalization of the different metrics considered.

Finally, in PL the set of all possible assignments (propositional worlds in this case) is finite, hence the result of the merging process can be computed in a finite number of steps and be represented as a formula of PL. However, in our case $\mathcal{W}_{\mathcal{I}}$ is not finite so that $\Delta_{IC}(\mathcal{K})$ cannot always be expressed as a formula of $\mathcal{L}$ (see the example above). We intend in the sequel to overcome this problem.

In the rest of the paper, IC_* will denote a cube and $\mathcal{K}_*$ a profile in which every belief/goal base is a cube. We first point out two preliminary results. The first one states that computing $\Delta_{IC_*}(\mathcal{K}_*)$ comes down to computing componentwise some $\Delta^i_{IC_*}(\mathcal{K}_*) \subseteq dom(x^i)$ for every $x^i \in \mathcal{V}$. The second result exploits the first one and provides a generic method to compute $\Delta_{IC}(\mathcal{K})$ in a piecewise fashion, when IC and formulas of $\mathcal{K}$ are given in DNF (which can be assumed without loss of expressiveness).

Let x^i a variable of type t_j. Given a cube ϕ, the *projection* of ϕ on x^i, denoted ϕ^i, is the conjunction of literals appearing in ϕ bearing on x^i; $S^i(\phi)$ denotes the set $\bigcap\{\mathcal{M}_{\mathcal{I}}(X) \mid (x^i \in X)$ is a literal of $\phi^i\} \cap \bigcap\{dom(x^i) \setminus \mathcal{M}_{\mathcal{I}}(X) \mid \neg(x^i \in X)$ is a literal of $\phi^i\}$. $\Delta^i_{IC_*}(\mathcal{K}_*)$ is defined in a three-step process. The i-local distance d^i_S between an element e of $dom(x^i)$ and a subset S of $dom(x^i)$ is defined as $d^i_S(e) = \inf\{d_j(e, e') \mid e' \in S\}$. The i-global distance $d^i_{\mathcal{K}_*}$ between an element e of $dom(x^i)$ and the multiset $\{S^i(K_k) \mid K_k \in \mathcal{K}_*\}$ is defined as $d^i_{\mathcal{K}_*}(e) = \sum\{d^i_{S^i(K_k)}(e) \mid K_k \in \mathcal{K}_*\}$. The subset $\Delta^i_{IC_*}(\mathcal{K}_*)$ of $dom(x^i)$ is then defined as $\Delta^i_{IC_*}(\mathcal{K}_*) = \{e \in \circ S^i(IC_*) \mid d^i_{\mathcal{K}_*}(e)$ is minimal$\}$. The following proposition holds:

Proposition 2. $\Delta_{IC_*}(\mathcal{K}_*) = \prod\{\Delta^i_{IC_*}(\mathcal{K}_*) \mid x^i \in \mathcal{V}\}$.

Proof. Let $\omega \in \mathcal{W}_{\mathcal{I}}$ and $K_k \in \mathcal{K}_*$. By definition, the local distance between ω and K_k is $d_{K_k}(\omega) = \inf\{\sum\{d_j(\omega(x^i), \omega'(x^i)) \mid x^i \in \mathcal{V}, \tau(x^i) = t_j\} \mid \omega' \models_{\mathcal{I}} K_k\}$. Yet since K_k is a cube, $Mod_{\mathcal{I}}(K_k) = \prod\{S^i(K_k) \mid x^i \in \mathcal{V}\}$. Hence,

$$\begin{aligned} d_{K_k}(\omega) &= \textstyle\sum\{\inf\{d_j(\omega(x^i), e') \mid e' \in S^i(K_k)\} \mid x^i \in \mathcal{V}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \tau(x^i) = t_j\} \\ &= \textstyle\sum\{d^i_{S^i(K_k)}(\omega(x^i)) \mid x^i \in \mathcal{V}, \tau(x^i) = t_j\}. \end{aligned}$$

Then we have:

$$\begin{aligned} d_{\mathcal{K}_*}(\omega) &= \textstyle\sum\{\sum\{d^i_{S^i(K_k)}(\omega(x^i)) \mid x^i \in \mathcal{V}\} \mid K_k \in \mathcal{K}_*\} \\ &= \textstyle\sum\{\sum\{d^i_{S^i(K_k)}(\omega(x^i)) \mid K_k \in \mathcal{K}_*\} \mid x^i \in \mathcal{V}\} \\ &= \textstyle\sum\{d^i_{\mathcal{K}_*}(\omega(x^i)) \mid x^i \in \mathcal{V}\}. \end{aligned}$$

For every $\omega_\Delta \in \mathcal{W}_{\mathcal{I}}$, $\omega_\Delta \in \Delta_{IC_*}(\mathcal{K}_*)$ iff $d_{\mathcal{K}_*}(\omega_\Delta) = \min\{\sum\{d^i_{\mathcal{K}_*}(\omega(x^i)) \mid x^i \in \mathcal{V}\} \mid \omega \in \circ Mod_{\mathcal{I}}(IC_*)\}$. Yet since IC_* is a cube, $\circ Mod_{\mathcal{I}}(IC_*) = \prod\{\circ S^i(IC_*) \mid x^i \in \mathcal{V}\}$.

Hence, $\omega_\Delta \in \Delta_{IC_*}(\mathcal{K}_*)$ iff $d_{\mathcal{K}_*}(\omega_\Delta) = \sum\{\min\{d^i_{\mathcal{K}_*}(e) \mid e \in \circ S^i(IC_*)\} \mid x^i \in \mathcal{V}\}$. This means that $\Delta_{IC_*}(\mathcal{K}_*) = \prod\{\Delta^i_{IC_*}(\mathcal{K}_*) \mid x^i \in \mathcal{V}\}$. □

Proposition 2 shows that computing $\Delta_{IC_*}(\mathcal{K}_*)$ comes down to computing componentwise a set $\Delta^i_{IC_*}(\mathcal{K}_*)$ for every $x^i \in \mathcal{V}$ since the former results from the Cartesian product of the latters. This property holds for our class of majority merging operators since the aggregation function $\sum$ commutes with itself (see the proof of Proposition 2). Contrastingly, arbitration operators [11, 7] using e.g. MAX as the aggregation function would not satisfy this property, since MAX does not commute with $\sum$.

In [10] the authors proposed a syntactic characterization of the result of the merging process for propositional majority merging operators. They assumed that every propositional belief/goal base of the profile to be merged is given in DNF. In the following we generalize their approach. For this purpose, we assume now that IC and every belief/goal base of $\mathcal{K}$ are formulas of $\mathcal{L}$ given in DNF. We exploit Proposition 2 and provide a generic algorithm to compute $\Delta_{IC}(\mathcal{K})$ in a piecewise fashion.

Proposition 3.

$$\begin{aligned} \Delta_{IC}(\mathcal{K}) = \textstyle\bigcup\{&\Delta_{IC_*}(\mathcal{K}_*) \mid \\ &IC_* \in IC,\ \mathcal{K}_* \in K_1 \times \ldots \times K_m, \\ &d_{\mathcal{K}}(\omega) \textit{ is minimal, for some } \omega \in \Delta_{IC_*}(\mathcal{K}_*)\}. \end{aligned}$$

Proof. Let R be the set on the right side of the equality.
$\subseteq$: let $\omega \in \Delta_{IC}(\mathcal{K})$. Let $\omega_1 \models_{\mathcal{I}} K_1, \ldots, \omega_m \models_{\mathcal{I}} K_m$ such that $d_{\mathcal{K}}(\omega) = \sum\{d(\omega, \omega_k) \mid K_k \in \mathcal{K}\}$. There exists a cube $IC_* \in IC$ such that $\omega \models_{\mathcal{I}} IC_*$ and for every $K_k \in \mathcal{K}$ there exists a cube $T \in K_k$ such that $\omega_k \models_{\mathcal{I}} T$. Thus $\omega \in R$.
$\supseteq$: let $\omega \in \Delta_{IC_*}(\mathcal{K}_*)$, with $IC_* \in IC$ and $\mathcal{K}_* \in K_1 \times \ldots \times K_m$ such that $d_{\mathcal{K}}(\omega)$ is minimal. Assuming that there exists $\omega' \in \Delta_{IC}(\mathcal{K})$ with $d_{\mathcal{K}}(\omega') < d_{\mathcal{K}}(\omega)$ would contradict the minimality of $d_{\mathcal{K}}(\omega)$. Therefore $\omega \in \Delta_{IC}(\mathcal{K})$. □

Taking advantage of Propositions 2 and 3, we get Algorithm 1 for computing $\Delta_{IC}(\mathcal{K})$:

Proposition 4. *Assume we are given an algorithm to compute $\Delta^i_{IC_*}(\mathcal{K}_*)$ for every $x^i \in \mathcal{V}$, and let $f(IC_*, \mathcal{K}_*)$ be its time complexity. Let $\alpha = \max\{|IC|, \max\{|K_k| \mid K_k \in \mathcal{K}\}\}$. Then the space and time complexities of Algorithm 1 are respectively in $\mathcal{O}(|\mathcal{V}|\alpha^m)$ and $\mathcal{O}(|\mathcal{V}|\alpha^m f(IC_*, \mathcal{K}_*))$.*

Proof. Obvious. □

Example (continued). ϕ_1, ϕ_3 and IC are cubes and thus are already in DNF. A DNF formula $\mathcal{I}$-equivalent to ϕ_2 is $\phi'_2 = \{T_1^{\phi'_2}, T_2^{\phi'_2}, T_3^{\phi'_2}\}$ where:

- $T_1^{\phi'_2} = (x^1 \in X^1_1) \wedge (x^1 \in X^1_2) \wedge (x^2 \in X^2_3) \wedge (x^3 \in X^3_4) \wedge (x^3 \in X^3_5) \wedge (x^3 \in X^3_6)$,
- $T_2^{\phi'_2} = (x^1 \in X^1_1) \wedge (x^1 \in X^1_2) \wedge (x^2 \in X^2_3) \wedge (x^3 \in X^3_4) \wedge (x^3 \in X^3_5) \wedge (x^3 \in X^3_7)$,
- $T_3^{\phi'_2} = (x^1 \in X^1_3) \wedge (x^1 \in X^1_4) \wedge (x^2 \in X^2_4) \wedge (x^3 \in X^3_1) \wedge (x^3 \in X^3_2) \wedge (x^3 \in X^3_3)$.

Algorithm 1: Computing $\Delta_{IC}(\mathcal{K})$

Input : a profile $\mathcal{K}$ and a formula IC
Output: $\Delta_{IC}(\mathcal{K})$

1 **begin**
2 $\quad \Delta_{IC}(\mathcal{K}) = \emptyset$;
3 $\quad d_{min} = +\infty$;
4 $\quad$ **forall** $IC_* \in IC$, $\mathcal{K}_* \in K_1 \times \ldots \times K_m$ **do**
5 $\quad\quad$ **forall** $x^i \in \mathcal{V}$ **do**
6 $\quad\quad\quad$ Compute $\Delta^i_{IC_*}(\mathcal{K}_*)$;
7 $\quad\quad\quad$ Pick up any $\omega^i \in \Delta^i_{IC_*}(\mathcal{K}_*)$;
8 $\quad\quad$ **end**
9 $\quad\quad$ **if** $\sum\{d_{\mathcal{K}_*}(\omega^i) \mid x^i \in \mathcal{V}\} \leq d_{min}$ **then**
10 $\quad\quad\quad$ **if** $\sum\{d_{\mathcal{K}_*}(\omega^i) \mid x^i \in \mathcal{V}\} < d_{min}$ **then**
11 $\quad\quad\quad\quad \Delta_{IC}(\mathcal{K}) = \emptyset$;
12 $\quad\quad\quad \Delta_{IC}(\mathcal{K}) = \Delta_{IC}(\mathcal{K}) \cup \prod\{\Delta^i_{IC_*}(\mathcal{K}_*) \mid x^i \in V\}$;
13 $\quad\quad$ **end**
14 $\quad$ **end**
15 $\quad$ **return** $\Delta_{IC}(\mathcal{K})$;
16 **end**

Since $|IC \times K_1 \times K_2 \times K_3| = 3$, the main loop of Algorithm 1 (lines 4 to 14) is performed three times. The definition of $\Delta_{IC}(\mathcal{K})$ previously reported in this paper is given by $\Delta_{IC}(\mathcal{K}_*) \cup \Delta_{IC}(\mathcal{K}'_*)$ with $\mathcal{K}_* = \{K_1, T_1^{\phi'_2}, K_3\}$ and $\mathcal{K}'_* = \{K_1, T_2^{\phi'_2}, K_3\}$.

The algorithm proposed in [10] for computing $\Delta^{d_H,\Sigma}$ is a special case of Algorithm 1. When all variables of $\mathcal{V}$ are Boolean ones, computing $\Delta^i_{IC_*}(\mathcal{K}_*)$ simply consists in electing using strict majority the truth value(s) of the literals bearing on x^i appearing in $\mathcal{K}_*$.

4 COMPUTING $\Delta^i_{IC_*}(\mathcal{K}_*)$ UNDER $\mathcal{H}_S$

In this section we consider interpretations of $\mathcal{H}_S$ for $\mathcal{L}$ and show that in this context $\Delta^i_{IC_*}(\mathcal{K}_*)$ can be expressed as a formula of $\mathcal{L}$. In our running example, the considered interpretation $\mathcal{I}^*$ does not belong to $\mathcal{H}_S$. So we need to switch to a particular interpretation $\mathcal{I}_S \in \mathcal{H}_S$.

Example (continued). Figure 2 depicts three regions A', B' and C' considered by the family for the location of the accomodation. Each one of these regions is a cuboid of $\mathbb{R}^2$ (i.e., a rectangle), thus all the lines defining them are rectilinear hyperplanes as required, i.e., every line of $\mathcal{H}_3$ is parallel to the axis x_1^3 or x_2^3.

Their equations are defined as follows:

$$\begin{array}{lll} h_1 : x_1^3 = 0, & h_2 : x_1^3 = 2, & h_3 : x_2^3 = 4, \\ h_4 : x_1^3 = 6, & h_5 : x_2^3 = 1, & h_6 : x_1^3 = 10, \\ h_7 : x_2^3 = 8, & h_8 : x_2^3 = 6, & h_9 : x_1^3 = 4. \end{array}$$

$\mathcal{I}_S$ requires that for every predicate symbol X of type t_j such that $\mathcal{M}_{\mathcal{I}_S}(X) = h^{\leq}$ with $h \in H_j$, there exists a predicate symbol Y such that $\mathcal{M}_{\mathcal{I}_S}(Y) = h^{\geq}$, and vice-versa. Following this condition, we consider 18 predicate symbols $X_1^3, \ldots, X_{18}^3$ of type t_3. $\mathcal{M}_{\mathcal{I}_S}$ is defined as follows, for every predicate symbol of type t_3, for every $j \in \{1, \ldots, 9\}$:

$$\mathcal{M}_{\mathcal{I}_S}(X^3_{2j-1}) = h_j^{\leq} \text{ and } \mathcal{M}_{\mathcal{I}_S}(X^3_{2j}) = h_j^{\geq}.$$

Let $\mathcal{K} = \{K_1, K_2, K_3\}$. Let IC_* be a cube of IC and for every $K_k \in \mathcal{K}$, let T_k be a cube of K_k such that by projecting formulas T_1, T_2, T_3, IC_* on x^3, we get respectively four cubes $T_1^3, T_2^3, T_3^3, IC_*^3$ bearing on x^3, defined as

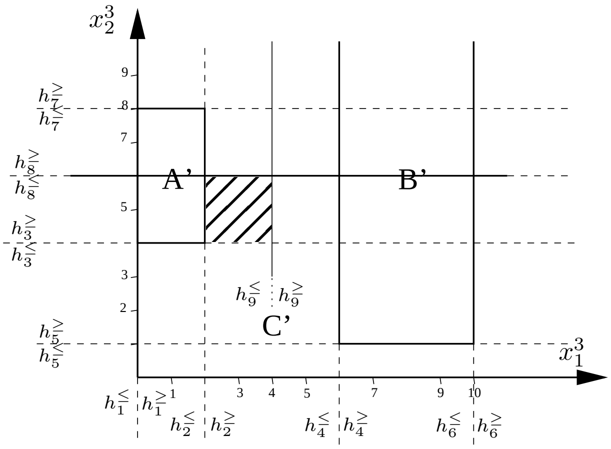

Figure 2. Nine rectilinear lines $h_1, \ldots, h_9$ forming the set $\mathcal{H}_3$.

- $IC_*^3 = (x^3 \in X_{17}^3)$,
- $T_1^3 = (x^3 \in X_2^3) \wedge (x^3 \in X_6^3) \wedge (x^3 \in X_3^3) \wedge (x^3 \in X_{13}^3)$,
- $T_2^3 = (x^3 \in X_8^3) \wedge (x^3 \in X_{10}^3) \wedge (x^3 \in X_{11}^3)$,
- $T_3^3 = (x^3 \in X_{15}^3)$.

Notice that for every $k \in \{1, 2, 3\}$, T_k^3 is a cube, hence every set $S^3(T_k^3)$ represents a cuboid of $\mathbb{R}^2$ (i.e., a rectangle). In fact, the sets $S^3(T_1^3)$, $S^3(T_2^3)$ and $S^3(T_3^3)$ respectively correspond to the regions A', B' and C'. Let $\mathcal{K}_* = \{T_1, T_2, T_3\}$. Then $\Delta^3_{IC_*}(\mathcal{K}_*)$ is the closed set of points of $\circ S^3(IC_*)$ the "closest" ones to regions A', B' and C' (i.e., the dashed region in Figure 2).

Let $\mathcal{I} \in \mathcal{H}_S$ and x^i a variable of $\mathcal{V}$ of type t_j. For every $K_k \in \mathcal{K}_*$, recall that K_k^i denotes the conjunction of literals appearing in K_k bearing on x^i. For every $l \in \{1, \ldots, n_j\}$, let $K_k^i(x_l^i)$ be the conjunction of literals of K_k^i defined on predicate symbols X where $\mathcal{M}_{\mathcal{I}}(X)$ is a half-space which the associated hyperplane is orthogonal to the axis x_l^i (namely, the only axis which is not parallel to the hyperplane associated to $\mathcal{M}_{\mathcal{I}}(X)$). In our running example, the lines h_1 and h_2 are orthogonals to the axis x_1^3 but not h_3 and h_7, hence $K_1^3(x_1^3) = (x^3 \in X_2^3) \wedge (x^3 \in X_3^3)$ and $K_1^3(x_2^3) = (x^3 \in X_6^3) \wedge (x^3 \in X_{13}^3)$.

Algorithm 2 allows us to compute $\Delta^i_{IC_*}(\mathcal{K}_*)$ for a given $x^i \in \mathcal{V}$ of type t_j. The key of this algorithm is that in $\mathcal{H}_S$, $\circ S^i(IC_*)$ is a cuboid, as well as $S^i(K_k)$ for every $K_k \in \mathcal{K}_*$. Hence, for computing $\Delta^i_{IC_*}(\mathcal{K}_*)$, it is enough, for each axis x_l^i, to project the cuboids on x_l^i, then to "merge" locally the set of intervals resulting from the projections on x_l^i, and lastly to return the Cartesian product of all the results. Notice that this decomposition on each axis is only feasible when the metric d_j is induced by the Manhattan norm, as it is required by $\mathcal{H}_S$. For the sake of clarity, we use the following notation in Algorithm 2. Let $\mathbb{R}_\infty = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\alpha, \beta \in \mathbb{R}_\infty$, (α, β) denotes the interval in which each bound is either open or closed and $[\alpha, \beta]$ denotes a (possibly unbounded) closed interval; for a given finite set $E \subseteq \mathbb{R}_\infty$, if there exists $\gamma \in E$ such that $\gamma = -\infty$, then $\min\{\gamma \mid \gamma \in E\} = -\infty$, and if there exists $\gamma \in E$ such that $\gamma = +\infty$, then $\max\{\gamma \mid \gamma \in E\} = +\infty$.

Proposition 5. *Let $x^i \in \mathcal{V}$ of type t_j.*
(1) Algorithm 2 computes $\Delta^i_{IC_*}(\mathcal{K}_*)$ in $\mathcal{O}(n_j|\mathcal{V}|)$ time.
(2) There exists a formula ϕ of $\mathcal{L}$ such that $S^i(\phi) = \Delta^i_{IC_*}(\mathcal{K}_*)$.

We omit the proof of this proposition for space reasons.

Algorithms 1 and 2 allow us to compute $\Delta_{IC}(\mathcal{K})$ when for every

Function: Match

Input : two closed intervals $[IC_*^-, IC_*^+]$ and $[S^-, S^+]$
Output: a closed interval $R \subseteq [IC_*^-, IC_*^+]$
begin
  if $S^- > IC_*^+$ **then return** $\{IC_*^+\}$;
  else if $S^+ < IC_*^-$ **then return** $\{IC_*^-\}$;
  else return $[IC_*^-, IC_*^+] \cap [S^-, S^+]$;
end

Algorithm 2: Computing $\Delta^i_{IC_*}(\mathcal{K}_*)$

Input : a profile $\mathcal{K}_*$, a formula IC_* and a variable x^i of type t_j
Output: $\Delta^i_{IC_*}(\mathcal{K}_*)$
begin
  forall $l \in \{1, \ldots, n_j\}$ **do**
    forall $K_k \in \mathcal{K}_*$ **do** $(I_k^-, I_k^+) = S^i(K_k^i(x_l^i))$;
    $IntSet = \{[I_k^-, I_k^+] \mid K_k \in \mathcal{K}_*\}$;
    $I^+_{min} = \min\{I_k^- \mid [I_k^-, I_k^+] \in IntSet\}$;
    $I^-_{max} = \max\{I_k^+ \mid [I_k^-, I_k^+] \in IntSet\}$;
    while $Int \neq \emptyset$ **and** $\bigcap\{I \mid I \in IntSet\} = \emptyset$ **do**
      $I^+_{min} = \min\{I_k^+ \mid [I_k^-, I_k^+] \in IntSet\}$;
      $I^-_{max} = \max\{I_k^- \mid [I_k^-, I_k^+] \in IntSet\}$;
      Remove an interval $[I_k^-, I^+_{min}]$ from $IntSet$;
      Remove an interval $[I^-_{max}, I_k^+]$ from $IntSet$;
    end
    $(IC_*^-, IC_*^+) = S^i(IC_*^i(x_l^i))$;
    if $IntSet = \emptyset$ **then**
      $I_l =$ Match $([IC_*^-, IC_*^+], [I^+_{min}, I^-_{max}])$;
    else
      $I_l =$ Match $([IC_*^-, IC_*^+],$
        $[I^+_{min}, I^-_{max}] \cap \bigcap\{I \mid I \in IntSet\})$;
    end
    return $\Delta^i_{IC_*}(\mathcal{K}_*) = \prod\{I_l \mid l \in \{1, \ldots, n_j\}\}$;
  end
end

$x^i \in \mathcal{V}$, $\tau(x^i) = t_j$, *every considered region of D_j can be represented as the finite union of cuboids*. In the general case (i.e., when considering an interpretation of $\mathcal{H}$), every region of D_j represents the finite union of (non-rectilinear) convex polyhedra. Interestingly, an approximation of the union of a finite set $\mathcal{P}$ of (non-rectilinear) convex polyhedra by the union of a finite set $\mathcal{P}'$ of cuboids can be computed by considering a regular mesh for discretizing the space and the union of a finite set $\mathcal{P}'$ of cuboids covering $\bigcup\{P \mid P \in \mathcal{P}\}$ w.r.t. the mesh (see e.g. [1]). Such an approximation can be chosen as close as desired from the initial region by choosing an appropriated size for the mesh. Of course, the number of cuboids of the resulting set $\mathcal{P}'$ corresponds to the size of the formula representing the region $\bigcup\{P \mid P \in \mathcal{P}'\}$. Yet the size of the formulas (the maximal number of cubes in them) is the prominent parameter in the complexity of Algorithm 1. Hence, the number of cuboids of the resulting set $\mathcal{P}'$ needs to be minimized. For this purpose, many polynomial-time algorithms can be exploited (see [6]).

5 CONCLUSION AND PERSPECTIVES

In this paper we have proposed a framework for merging (possibly conflicting) information from different sources. We have considered a (limited) first-order logical representation language $\mathcal{L}$, expressive enough for representing information modeled as polyhedra of $\mathbb{R}^n$ for a specific class $\mathcal{H}$ of interpretations. We have defined a class Δ of distance-based majority merging operators. We have shown that Δ includes the propositional merging operator $\Delta^{d_H,\Sigma}$. By generalizing a previous approach [10], we have also provided a generic method for computing the result of the merging process in a piecewise fashion. We have identified a subclass $\mathcal{H}_S$ of $\mathcal{H}$ allowing us to model cuboids of $\mathbb{R}^n$, and pointed out (and evaluated) an algorithm to compute the result of the merging process and express it as a formula.

We have noticed that any region of $\mathbb{R}^n$ can be approximated as close as desired by means of the union of some cuboids of $\mathbb{R}^n$. This allows us to represent polyhedra of $\mathbb{R}^n$ using formulas of $\mathcal{L}$ under $\mathcal{H}_S$. Nevertheless, when high, the expected quality of the approximation can lead to increase significantly the size of the induced formulas. Furthermore, while the class $\mathcal{H}$ considers any metric induced by a norm on $\mathbb{R}^n$, $\mathcal{H}_S$ only deals with the metric induced by the Manhattan norm. As a perspective we plan to study some alternatives to the approach pointed out here for approximating the result of the merging process under $\mathcal{H}$. For this purpose, we will import the solid theoretical background on convex optimization [2] into our framework in order to compute an element "close" enough to the result of the merging process w.r.t. some arbitrary thresholds. We have already proved that for every variable x^i, the i-global distance is a *convex* function. This property allows us to approximate efficiently under an arbitrary threshold a minimum of the i-global distance by using a projected subgradient method of minimization [12]. However, as far as we know, such thresholds are not directly related to the distance, hence it is not possible to check how far the element is from the result of the merging process. This problem has been addressed in the literature and some thresholds sensitive to this distance have been proposed [5]. The theoretical and practical study of their integration in our framework is let as an issue for further research. Another perspective concerns the rationality postulates issue. A characterization of propositional majority merging operators using such postulates has been given in [9]. We plan to investigate how such postulates can be extended to our setting.

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