Set-valued conditioning in a possibility theory setting

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Abstract. Possibilistic logic is a well-known framework for dealing with uncertainty and reasoning under inconsistent or prioritized knowledge bases. This paper deals with conditioning uncertain information where the weights associated with formulas are in the form of sets of uncertainty degrees. The first part of the paper studies set-valued possibility theory where we provide a characterization of set-valued possibilistic logic bases and set-valued possibility distributions by means of the concepts of compatible possibilistic logic bases and compatible possibility distributions respectively. The second part of the paper addresses conditioning set-valued possibility distributions. We first propose a set of three natural postulates for conditioning set-valued possibility distributions. We then show that any set-valued conditioning satisfying these three postulates is necessarily based on conditioning the set of compatible standard possibility distributions. The last part of the paper shows how one can efficiently compute set-valued conditioning over possibilistic knowledge bases.

1 INTRODUCTION

Possibilistic logic is a well-known framework for dealing with uncertainty, reasoning under inconsistent and prioritized knowledge bases and partial knowledge [25]. Many extensions have been proposed for possibilistic logic to deal for instance with imprecise certainty degrees [4, 5], symbolic certainty weights [6, 7], multi-agent beliefs [2], temporal and uncertain information [21], uncertain conditional events [10, 9, 11], generalized possibilistic logic [8, 18, 20], reasoning with justified beliefs [22], etc.

This paper proposes a new extension of possibilistic logic where the weights associated with formulas are in the form of sets of uncertainty degrees. Standard possibilistic logic expressions are propositional logic formulas associated with positive real degrees belonging to the unit interval $[0, 1]$. However, in practice it may be difficult for an agent to provide exact degrees associated with formulas of a knowledge base. This paper proposes an extension of standard possibility distributions and standard possibilistic bases where a set of possibility/certainty degrees may be associated with interpretations or formulas. A set of certainty degrees associated with a formula may represent the reliability levels of different sources that support the formula (see Example 1). Another important issue dealt with in this paper is the one of updating or conditioning a set-based knowledge base.

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Conditioning is an important task for updating the current uncertain information when a new sure piece of information is received. A conditioning operator is designed to satisfy some desirable properties such as giving priority to the new information and ensuring minimal change while transforming an initial distribution into a conditional one. This paper deals with conditioning in a possibility theory and possibilistic logic frameworks [8, 14, 18, 13]. Conditioning in standard (single-valued) possibility theory has been addressed in many works [24, 27, 17, 23, 16, 3]. There are two major definitions of possibility theory: $\min$-based (or qualitative) possibility theory and product-based (or quantitative) possibility theory. At the semantic level, these two theories share the same definitions, including the concepts of possibility distributions, necessity measures, possibility measures and the definition of normalization condition. However, they differ in the way they define possibilistic conditioning. This paper focuses on a so-called $\min$-based conditioning [24] (or qualitative-based conditioning) which is appropriate in situations where only the ordering between events is important. In this case, the unit interval $[0, 1]$ is viewed as an ordinal scale where only the minimum and the maximum operations are used for propagating and updating uncertainty degrees.

The first contribution of this paper concerns the definition of a set-valued possibility theory which generalizes both standard possibility theory and interval-based possibility theory [4]. The second contribution deals with conditioning in a set-valued possibility theory setting. We first propose three natural postulates for a set-valued conditioning. We show that any set-valued conditioning satisfying these postulates is necessarily based on applying $\min$-based conditioning on each compatible standard possibility distribution. We also provide the exact set of possibility degrees associated with $\min$-based conditioning a set-valued distribution. The last contribution concerns efficient and syntactic computations of conditioning set-valued knowledge bases.

The rest of this paper is organized as follows: Section 2 provides a brief refresher on the possibility theory and possibilistic logic settings. Section 3 presents set-valued possibility theory and set-valued possibilistic logic. In Section 4, we focus on set-valued conditioning while Section 5 provides a syntactic computing of set-valued conditioning. Section 6 provides concluding discussions.

2 BRIEF REMINDER ON POSSIBILITY THEORY

Possibility distributions: Possibility theory [29, 19] is a well-known uncertainty theory. It is based on the concept of possibility distribution π which associates every state ω of the world Ω (the universe of discourse) with a degree in the interval $[0, 1]$ expressing a partial knowledge over the world. In this paper, Ω denotes the set of propositional interpretations. $\omega \vDash \phi$ means that ω is a model of (or satisfies) ϕ in the sense of propositional logic. The degree $\pi(\omega)$ represents the degree of compatibility (or consistency) of the interpretation ω with the available knowledge. By convention, $\pi(\omega){=}1$ means that ω is fully consistent with the available knowledge, while $\pi(\omega){=}0$ means that ω is impossible. $\pi(\omega){>}\pi(\omega')$ simply means that ω is more compatible than ω'. A possibility distribution π is said to be normalized if there exists an interpretation ω such that $\pi(\omega){=}1$, it is said to be subnormalized otherwise.

As it is already mentioned in the introduction, possibility degrees are interpreted either i) *qualitatively* (in min-based possibility theory) where only the *ordering* of the values matters, or ii) *quantitatively* (in product-based possibility theory) where the possibilistic scale $[0, 1]$ is quantitative as in probability theory. Min-based or qualitative possibility theory refers to the possibilistic setting where only the ordering induced by possibility degrees is important. In this setting, only the $\max$ and $\min$ operators are used for the reasoning and updating tasks.

Min-based conditioning: In the standard possibilistic setting, conditioning comes down to updating a possibility distribution π encoding the current knowledge when a completely sure event called *evidence* or *observation*, denoted by $\phi{\subseteq}\Omega$ is received. This results in a conditional possibility distribution denoted by $\pi(.|\phi)$. There are many definitions of conditioning operators in the standard possibilistic setting [24, 27, 17, 23, 16].

Hisdal [24] proposed that a definition of a conditioning operator in the qualitative setting should satisfy the condition:

$$\forall\omega \vDash \phi, \pi(\omega) = \min(\pi(\omega|\phi), \Pi(\phi)).$$

Where $\Pi(\phi)$ denotes the possibility measure of an event ϕ, defined by:

$$\Pi(\phi) = \max\{\pi(\omega) : \omega{\in}\Omega,\ \omega{\vDash}\phi\}.$$

Dubois and Prade [15] proposed to select the largest conditional possibility distribution satisfying this condition, leading to the following conditioning operator.

Definition 1 (min-based conditioning)**.** *Let π be a possibility distribution, $\phi \subseteq \Omega$ be a sure event.* min*-based conditioning of π by ϕ, simply denoted by $\pi(.|_m\phi)$, is defined as:*

$$\forall\omega{\in}\Omega, \pi(\omega|_m\phi) = \begin{cases} 1 & \textit{if } \pi(\omega){=}\Pi(\phi) \textit{ and } \omega{\in}\phi; \\ \pi(\omega) & \textit{if } \pi(\omega){<}\ \Pi(\phi) \textit{ and } \omega{\in}\phi; \\ 0 & \textit{otherwise.} \end{cases} \quad (1)$$

When $\Pi(\phi){=}0$, then by convention $\forall\omega{\in}\Omega$, $\pi(\omega|_m\phi){=}1$.

Possibilistic knowledge bases: A possibilistic formula is a pair (φ, α) where φ is a propositional logic formula and $\alpha{\in}[0, 1]$ is a certainty degree associated with φ. The higher the certainty degree α is, the more important is the formula φ. A possibilistic base $K{=}\{(\varphi_i, \alpha_i), i = 1, ..., n\}$ is simply a set of possibilistic formulas.

A possibilistic knowledge base is a well-known compact representations of a possibility distribution. Given a possibilistic base K, we can generate a unique possibility distribution where interpretations ω satisfying all propositional formulas in K have the highest possible degree $\pi(\omega){=}1$ (since they are fully consistent), whereas the others are pre-ordered with respect to the highest formulas they falsify. More formally:

Definition 2. *Let K be a possibilistic knowledge base. Then, the corresponding possibility distribution π_K is given by: $\forall\omega{\in}\Omega$,*

$$\pi_K(\omega) = \begin{cases} 1 & \textit{if } \forall(\varphi, \alpha){\in}K,\ \omega \vDash \varphi \\ 1{-}max\{\alpha_i : (\varphi_i, \alpha_i){\in}K, \omega \nvDash \varphi_i\} & \textit{otherwise.} \end{cases} \quad (2)$$

The following lemma will be helpful for establishing proofs of some propositions. It states that 'zero-weighted' formulas can be added or removed from possibilistic knowledge bases without changing theirs distributions.

Lemma 1. *Let K be a possibilistic knowledge base K such that $(\delta, 0) \in K$. Let $K'{=}K \setminus \{(\delta, 0)\}$. Then $\forall\omega \in \Omega$, $\pi_K(\omega){=}\pi_{K'}(\omega)$.*

This lemma can be easily shown since if a formula δ has a certainty degree equal to 0, then if there is an interpretation ω that falsifies only the formula δ then, according to Definition 2, the possibility degree associated to ω will be $1{-}0{=}1$.

An important notion that plays a central role in the inference process and conditioning is the one of α-cut. Let α be a positive real number. An α-cut is a set of propositional formulas defined by $K_{\geq\alpha}{=}\{\varphi : (\varphi, \beta){\in}K$ and $\beta{\geq}\alpha\}$.

The concept of α-cut can be used to provide the syntactic counterpart of conditioning a possibilistic knowledge base with a propositional formula:

Definition 3. *Let K be a possibilistic knowledge and ϕ be a sure piece of information. The result of conditioning K by ϕ, denoted K_ϕ is defined as follows:*

$$K_\phi = \{(\phi, 1)\} \cup \{(\varphi, \alpha) : (\varphi, \alpha) \in K \textit{ and } K_{\geq\alpha} \wedge \phi \textit{ is consistent.}\}$$

Namely, K_ϕ is obtained by considering ϕ with a certainty degree '1', plus weighted formulas (φ, α) of K such that their α-cut is consistent with ϕ. It can be checked that:

$$\forall\omega \in \Omega,\ \pi_{K_\phi}(\omega) = \pi_K(\omega|_m\phi),$$

where π_K and π_{K_ϕ} are given using Definition 2 and $\pi_K(.|_m\phi)$ is obtained using Definition 1.

Next section generalizes standard possibility theory and possibilistic logic into a set-valued setting.

3 SET-VALUED POSSIBILITY THEORY AND SET-VALUED POSSIBILISTIC LOGIC

Let us first start with a short example to motivate our extension.

Example 1. *Suppose we are interested in the amenities and facilities of a hotel in Paris to organize a conference. For this, we posted a question on a specialized Internet platform. To simplify, the question was about the presence of a large conference room in the hotel (represented by the variable c) and if the hotel has a great restaurant (represented by a the variable r) to host the gala dinner. We also asked people to specify how certain of the answers they are, using a unit scale $[0, 1]$. Assume that we got three answers of three people: p_1 is a former hotel employee, the second, p_2, is an employee of the Paris tourism office and the third, $p3$, is a client of the hotel. The certainty levels of these people with respect to different scenarios*[3] *are summarized as follows:*

Table 1. Example of multiple sources information

	p_1	p_2	p_3
cr	1	1	1
$\neg cr$	1	1	1
$c\neg r$	.3	.2	.4
$\neg c\neg r$	.4	.4	.4

In this example, the confidence degrees provided by the responders can be viewed as possibility degrees. Now, suppose that we got hundreds or thousands of answers or suppose that there is a large number of variables, then it will be interesting to find a compact way to encode the obtained answers and more importantly to reason with them (answer any request of interest and update the available information when new sure information is obtained). Set-valued possibility theory is especially tailored to this type of information.

Let us now introduce the concept of set-valued possibility distribution.

3.1 Set-valued possibility distributions

In the set-valued possibilistic setting, the available knowledge is encoded by a set-valued possibility distribution $S\pi$ where each state ω is associated with a finite set $S\pi(\omega)$ of possible values of possibility degrees $\pi(\omega)$.

If S is a set, then we denote by $\overline{S}$ and $\underline{S}$ the maximum and minimum values of S respectively. When all S's associated with interpretations (or formulas) are singletons (meaning that $\overline{S} = \underline{S}$), we refer to standard distributions (resp. standard possibilistic bases). Here, $\underline{S\pi}(\omega)$ (resp. $\overline{S\pi}(\omega)$) denotes the minimum (resp. maximum) of the possibility degrees of ω.

Clearly, set-valued possibility theory is also an extension of interval-based possibility theory [4], where the set is denoted as an interval of possible values. Therefore, we now consider sets of degrees and we define a set-valued possibility distribution as follows:

Definition 4 (Set-valued possibility distribution)**.** *A set-valued possibility distribution $S\pi$ is a mapping $S\pi : \Omega \to \mathcal{S}$ from the universe of discourse Ω to the set $\mathcal{S}$ of all sub-sets included in the interval $[0, 1]$, with the normalization property requiring that $\max_{\omega\in\Omega} \overline{S\pi}(\omega)=1$.*

The information corresponding to Example 1 could be compactly encoded as follows:

Example 2. *(***Example 1 cont'd.***) Let us represent the available knowledge from Example 1 as a set-valued possibility distribution given in Table 2.*

Table 2. Set-valued distribution corresponding to the multiple source information of Table 1.

	$S\pi$
cr	$\{1\}$
$\neg cr$	$\{1\}$
$c\neg r$	$\{.2, .3, .4\}$
$\neg c\neg r$	$\{.4\}$

As in an interval-based possibility theory [4], we also interpret a set-valued possibility distribution as a family of compatible standard possibility distributions defined by:

Definition 5. *Let $S\pi$ be a set-valued possibility distribution. A normalized possibility distribution π is said to be compatible with $S\pi$ if and only if $\forall\omega \in \Omega,\ \pi(\omega) \in S\pi(\omega)$.*

As shown in Example 3, compatible distributions are not unique. We denote by $\mathcal{C}(S\pi)$ the set of all possibility distributions compatible with $S\pi$.

Example 3. *Let $S\pi$ be a set-valued possibility distribution described in the Table 3.*

Then following Definition 5, the possibility distributions π_1 and π_2 (from Table 3) are compatible with $S\pi$.

However, π_3 is not compatible with $S\pi$ since $\pi_3(cr)=.4 \notin S\pi(cr)=\{1\}$.

Table 3. Example of set-valued possibility distribution $S\pi$, compatible possibility distributions π_1 and π_2 and a non compatible one π_3.

$\omega\in\Omega$	$S\pi$
cr	$\{1\}$
$\neg cr$	$\{1\}$
$c\neg r$	$\{.2, .3, .4\}$
$\neg c\neg r$	$\{.4\}$

$\omega\in\Omega$	π_1	π_2	π_3
cr	1	1	.4
$\neg cr$	1	1	1
$c\neg r$	.3	.4	.2
$\neg c\neg r$	.4	.4	.4

Let us now see how to generalize standard possibilistic logic into a set-valued setting.

3.2 Set-valued possibilistic logic

Contrary to standard possibilistic logic where the uncertainty is described with single values, set-valued possibilistic logic uses sets.

The syntactic representation of set-valued possibilistic logic generalizes the notion of a possibilistic base to a set-valued possibilistic knowledge base as follows:

Definition 6. *A set-valued possibilistic knowledge base, denoted by SK, is a set of propositional formulas associated with sets:*

$$SK = \{(\varphi, S), \varphi \in \mathcal{L} \textit{ and } S \textit{ is a set of degrees in } [0, 1]\}$$

[3] In this example, the scenario cr means that the hotel has a conference room and has a great restaurant while the scenario $c\neg r$ means that the hotel has a conference room but does not have a great restaurant .

In Definition 6, $\varphi \in \mathcal{L}$ denotes again a formula of a propositional language $\mathcal{L}$.

A set-valued possibilistic base SK can be viewed as a family of standard possibilistic bases called compatible bases. More formally:

Definition 7 (Compatible possibilistic base)**.** *A possibilistic base K is said to be compatible with a set-valued possibilistic base SK if and only if K is obtained from SK by replacing each set-valued formula (φ, S) by a standard possibilistic formula (φ, α) with $\alpha \in S$.*

In other words, each compatible possibilistic base is such that $K = \{(\varphi, \alpha) : (\varphi, S) \in SK$ and $\alpha \in S\}$.

We also denote by $\mathcal{C}(SK)$ the finite set of all compatible possibilistic bases associated with a set-valued possibilistic base SK.

Example 4. *In the following, we will use this set-valued possibilistic knowledge base to illustrate our propositions. Let SK be a set-valued possibilistic knowledge base such that:*

$$SK = \{(\neg c \vee r, \{.4, .7, .8\}), (r, \{.6\})\}.$$

An example of a compatible possibilistic knowledge base is:

$$K = \{(\neg c \vee r, .4), (r, .6)\}.$$

As in standard possibilistic logic, a set-valued knowledge base SK is also a compact representation of a set-valued possibility distribution $S\pi_{SK}$.

3.3 From set-valued possibilistic bases to set-valued possibility distributions

Let us go one step further with the contribution on how to compute the set-valued possibility distribution from a set-valued base.

Let $SK = \{(\varphi_i, S_i)$: i=1, ..., $n\}$ be a set-valued possibilistic knowledge base. A natural way to define a set-valued possibility distribution, associated with SK and denoted by $S\pi_{SK}$, is to consider all standard possibility distributions associated with each compatible knowledge base. Namely:

Definition 8. *Let SK be a set-valued possibilistic knowledge base. The set-valued possibility distribution $S\pi_{SK}$ associated with SK is defined by:*

$$\forall \omega \in \Omega,\ S\pi_{SK}(\omega) = \{\pi_K(\omega) : K \in \mathcal{C}(SK)\}.$$

Recall that $\mathcal{C}(SK)$ is the set of compatible knowledge bases (given in Definition 7) and π_K is given by Definition 2. Similar to the single valued possibilistic logic setting, we can get rid of some formulas of a set-valued knowledge base without any information loss. More precisely, we can ignore any formula of SK attached with only one certainty degree equal to zero, as stated in the following lemma.

Lemma 2. *Let SK be a set-valued possibilistic base such that $(\delta, \{0\}) \in SK$. Let $SK' = SK \setminus \{(\delta, \{0\})\}$. Then $\forall \omega \in \Omega$, $S\pi_{SK}(\omega) = S\pi_{SK'}(\omega)$.*

Lemma 2 is again useful for establishing proofs of some propositions. The idea behind this lemma stands in the definition of compatible bases and Lemma 1. Indeed, in the case where SK is such that $(\delta, \{0\}) \in SK$, then in every compatible base K, we have $(\delta, 0) \in K$, therefore, as stated in Lemma 1, the weighted formula $(\delta, 0)$ can be ignored from K without changing its associated distributions, and this can be generalized to the set-valued formula $(\delta, \{0\})$.

Let us now characterize $S\pi_{SK}$. The following proposition provides the conditions under which the highest possibility degree '1' belongs to $S\pi_{SK}(\omega)$:

Proposition 1. *Let SK be a set-valued possibilistic knowledge base. Let ω be an interpretation. Then:*

$$1 \in S\pi_{SK}(\omega) \text{ iff } \omega \vDash \bigwedge\{\varphi : (\varphi, S) \in SK \text{ and } \underline{S} > 0\}$$

Namely, $1 \in S\pi_{SK}(\omega)$ if and only if ω satisfies all formulas having a strictly positive certainty degree.

Proof. Recall that $1 \in S\pi_{SK}(\omega)$ means that there exists a compatible possibilistic base $K \in \mathcal{C}(SK)$ such that $\pi_K(\omega) = 1$. Now, formulas of K having a certainty degree equal to '0' can be removed, thanks to Lemma 1, without changing π_K. The fact that $\pi_K(\omega) = 1$ implies that ω is a model of $\{\varphi : (\varphi, \alpha) \in K, \alpha > 0\}$. This also means that ω is also a model of $\{\varphi, (\varphi, S) \in SK,\ \underline{S} > 0\}$.

Let us now show the converse. Assume that ω is a model of $\{\varphi, (\varphi, S) \in SK,\ \underline{S} > 0\}$. Let K be a compatible possibilistic knowledge base obtained from SK by replacing each set-valued S by its lower bound $\underline{S}$. Clearly, $\{\varphi : (\varphi, \underline{S}) \in K\}$ is satisfied by ω. Hence, $1 \in S\pi_{SK}(\omega)$. □

Example 5. *(***Example 4 cont'd***) Let us continue with the knowledge base from Example 4. Recall that*

$$SK = \{(\neg c \vee r, \{.4, .7, .8\}), (r, \{.6\})\}$$

Following Proposition 1, interpretations cr and $\neg cr$ will have among their possibility degrees the degree 1 (namely $1 \in S\pi_{SK}(cr)$ and $1 \in S\pi_{SK}(\neg cr)$) since these interpretations are models of all the formulas of SK attached only to strictly positive degrees.

We now study under which conditions a possibility degree $(1-\alpha)$ belongs to $S\pi_{SK}(\omega)$, with $\alpha \in [0, 1]$. Clearly, if $(1-\alpha) \in S\pi(\omega)$ then there exists a compatible base K such that $\pi_K(\omega) = 1-\alpha$. Hence, there exists $(\varphi, \alpha) \in K$ such that $\omega \nvDash \varphi$. Then there exists $(\varphi, S) \in SK$ such that $\omega \nvDash \varphi$ and $\alpha \in S$.

To determine the possible values of $S\pi_{SK}(\omega)$, it is enough to browse all certainty degrees associated with formulas of SK falsified by ω and check whether their inverse will belong or not to $S\pi_{SK}(\omega)$.

This is precisely specified by the following proposition:

Proposition 2. *Let ω be an interpretation. Let $A = \bigcup\{S : (\varphi, S) \in SK,\ \omega \nvDash \varphi\}$. Let $a \in A \cup \{0\}$. Then,*

$$(1-a) \in S\pi_{SK}(\omega) \text{ iff } \omega \vDash \{\varphi : (\varphi, S) \in SK,\ \underline{S} > a\}$$

Proof. Proposition 2 recovers Proposition 1 in case where a=0. Hence, we only focus on the case a>0. To see the proof, assume that a>0 and $(1-a)\in S\pi_{SK}(\omega)$. This means that there exists a compatible possibilistic knowledge base $K \in \mathcal{C}(SK)$, such that $\pi_K(\omega)$=1$-a$.

This means that $\{\varphi : (\varphi, b), b > a\}$ is consistent and satisfied by ω. Since $\{\varphi : (\varphi, S), \underline{S} > a\} \subseteq \{\varphi : (\varphi, b), b > a\}$, this also means that $\{\varphi : (\varphi, S),\ \underline{S} > a\}$ is consistent and satisfied by ω.

Let us show the converse. Assume that $\omega \models \{\varphi : (\varphi, S),\ \underline{S}>a\} \wedge \omega$. Clearly, if A=$\emptyset$ (namely, a=0) or A=$\{0\}$ then whatever is the compatible base K, ω will satisfy each formula in K, hence $\pi_K(\omega)$=1, and $(1 - a) \in S\pi_{SK}(\omega)$. Assume that $a \in A$ and $a > 0$. Let (φ_1, S_1) be a formula of SK such that $a \in S_1$ and $\omega \not\models \varphi_1$. Let K be a compatible base defined by:

$$K = \{(\varphi, \underline{S}) : (\varphi, S) \in SK,\ \varphi \neq \varphi_1\} \cup \{(\varphi_1, a)\}.$$

Namely, K is obtained from SK by replacing S by $\underline{S}$ for each formula in SK, except for φ_1 where a is used instead of $\underline{S}$. It is easy to see that K is compatible with SK, namely $K \in \mathcal{C}(SK)$. It is also easy to see that $\pi_K(\omega) = 1 - a$, since $\{\varphi : (\varphi, b) \in K, b > a\}$ is satisfied by ω, $\{\varphi : (\varphi, b) \in K, b > a\} \cup \{(\varphi_1, a)\}$ is falsified by ω. Therefore $(1 - a) \in S\pi_{SK}(\omega)$. □

Let us continue our example, and illustrate Proposition 2.

Example 6. *(**Example 4 cont'd**) We need to check which degrees belong to $S\pi_{SK}(\omega)$. For each interpretation, we first compute $A = \bigcup\{S : (\varphi, S) \in SK,\ \omega \not\models \varphi\}$. For instance, let us consider ω=$c\neg r$ then A=$\{.4, .7, .8, .6\}$. Now, let us analyse each value a of A∪$\{0\}$,*

- *For a=0, $c\neg r \not\models \{\neg c \vee r, r\}$, then $1 \notin S\pi_{SK}(c\neg r)$;*
- *For a=.4, $c\neg r \not\models \{r\}$, then $.6 \notin S\pi_{SK}(c\neg r)$;*
- *For a=.7, $\emptyset \wedge c\neg r$ is consistent, then $.3\in S\pi_{SK}(c\neg r)$;*
- *For a=.8, $\emptyset \wedge c\neg r$ is consistent, then $.2\in S\pi_{SK}(c\neg r)$*
- *Finally, for a=.6, $\emptyset \wedge c\neg r$ is consistent, then $.4\in S\pi_{SK}(c\neg r)$.*

Then we can conclude that $S\pi_{SK}(c\neg r)$=$\{.2, .3, .4\}$.

Let us take another interpretation, for instance ω=$\neg c\neg r$. Then $A = \{.6\}$ and for each $a\in A\cup\{0\}$,

- *For a=0, $\neg c\neg r \not\models \{\neg c \vee r, r\}$, then $1 \notin S\pi_{SK}(\neg c\neg r)$;*
- *And for a=.6, $\emptyset \wedge \neg c\neg r$ is consistent, then $.4 \in S\pi_{SK}(\neg c\neg r)$.*

We can conclude that $S\pi_{SK}(\neg c\neg r)$=$\{.4\}$.
The whole distribution is exactly the one given in Example 2.

Let us now deal with the issue of conditioning a set-valued possibilistic base. The following section extends min-based conditioning to set-valued possibility distributions.

4 CONDITIONING SET-VALUED POSSIBILISTIC INFORMATION

Before providing our extension of min-based conditioning to the set-valued setting, let us first focus on the natural properties that a set-valued conditioning operator should fulfill.

4.1 Three natural requirements for the set-valued conditioning

The first natural requirement (called recovering standard conditioning) is that in the *degenerate* case, namely when each set $S\pi(\omega)$ contains exactly one single degree $\pi(\omega)$, the result of the new conditioning procedure should coincide with the result $\pi(.|_m\phi)$ of the original conditioning procedure (Definition 1). For each possibility distribution π, by $\{\pi(\omega)\}$ we denote its set-valued representation, i.e., a set-valued possibility distribution for which, for every $\omega\in\Omega$, we have $S\pi(\omega)$=$\{\pi(\omega)\}$. In these terms, the above requirement takes the following form:

S1. If for every $\omega\in\Omega$, we have $S\pi(\omega)$=$\{\pi(\omega)\}$, then $S\pi(\omega|\phi)$=$\{\pi(\omega|_m\phi)\}$ for all ω and ϕ.

The second requirement (called specificity) is related to the fact that we do not know the precise values $S\pi(\omega)$ since we only have partial information about them. In principle, if we can get some additional information about these values, then this would lead, in general, to narrower sets (indeed, the cardinality of a set captures the ignorance regarding the exact value of $\pi(\omega)$). Let us define the concepts of specificity between set-valued possibility distribution:

Definition 9. *Let $S\pi$ and $S\pi'$ be two set-valued possibility distributions. Then $S\pi$ is said to be more specific than $S\pi'$, denoted $S\pi\subseteq S\pi'$, if $S\pi(\omega)\subseteq S\pi'(\omega)$ holds for all $\omega\in\Omega$.*

S2. If $S\pi(\omega)\subseteq S\pi'(\omega)$ for all ω, then $S\pi(\omega|\phi)\subseteq S\pi'(\omega|\phi)$ for all ω.

Of course, these two postulates are not sufficient. For example, we can take $S\pi(.|\phi)$=$\{\pi(.|_m\phi)\}$ for degenerate set-valued possibility distributions and $S\pi(\omega|\phi)$=$[0, 1]$ for any other set-valued distribution $S\pi$. To avoid such extensions, it is reasonable to impose the following minimality condition:

S3. There does not exist a conditioning operation '$|_1$' that satisfies both properties **S1**–**S2** and for which:

- $S\pi(\omega|_1\phi) \subseteq S\pi(\omega|\phi)$ for all $S\pi$, ω, and ϕ,
- $S\pi(\omega|_1\phi) \neq S\pi(\omega|\phi)$ for some $S\pi$, ω, and ϕ.

S3 is called minimality condition. The following theorem provides one of our main results where we show that there is only one set-valued conditioning satisfying **S1-S3** and where the set conditional possibility degree $S\pi(\omega|\phi)$ is defined as the closure of the set of all $\pi(.|_m\phi)$, where π is compatible with $S\pi$.

Theorem 1. *There exists exactly one set-valued conditioning, also denoted by $S\pi(.|\phi)$ for sake of simplicity, that satisfies the properties* **S1**–**S3**, *and which is defined by: $\forall\omega \in \Omega$,*

$$S\pi(\omega|\phi) = \{\pi(\omega|_m\phi) : \pi \in \mathcal{C}(S\pi)\} \qquad (3)$$

where $|_m$ is the min*-based conditioning given in Definition 1.*

Proof. 1°. Let us denote the corresponding set-based conditioning by $S\pi(.|\phi)$. We need to prove:

- that this closure $S\pi(.|\phi)$ satisfies the properties **S1**–**S3**, and

- that every operation $S\pi(.|_1\phi)$ that satisfies the properties **S1**–**S3** coincides with the set-conditioning $S\pi(.|\phi)$.

2°. One can easily see that the operation $S\pi(.|\phi)$ satisfies the properties **S1**–**S2**.

3°. Let us now prove that if an operation $S\pi(.|_1\phi)$ satisfies the properties **S1**–**S2**, then for every $S\pi$ and ϕ, we have $S\pi(.|\phi) \subseteq S\pi(.|_1\phi)$.

Then, for every distribution $\pi \in \mathcal{C}(S\pi)$, we have $\{\pi\} \subseteq S\pi$ and thus, due to the postulate **S2**, we have $\{\pi\}(.|_1\phi) \subseteq S\pi(.|\phi)$. By the property **S1**, we have $\{\pi\}(\omega|_1\phi) = \{\pi(\omega|_m\phi)\}$. Thus, the above inclusion means that $\pi(.|_m\phi) \in S\pi(.|_1\phi)$.

The set $S\pi(\omega|_1\phi)$ therefore contains all the values $\pi(\omega|_m\phi)$ corresponding to all possible $\pi \in \mathcal{C}(S\pi)$:

$$\{\pi(\omega|_m\phi) : \pi \in \mathcal{C}(S\pi)\} \subseteq S\pi(\omega|_1\phi).$$

Thus, we conclude that $S\pi(\omega|\phi) \subseteq S\pi(\omega|_1\phi)$ for all ω.

The statement is proven.

4°. We can now prove that $S\pi(.|\phi)$ also satisfies the property **S3**.

Indeed, if there is some other operation $|_1$ that satisfies **S1** and **S2**, and for which $S\pi(\omega|_1\phi) \subseteq S\pi(\omega|\phi)$ for all ω, then, since we have already proven the opposite inclusion in Part 3 of this proof, we conclude that $S\pi(\omega|_1\phi) = S\pi(\omega|\phi)$ for all ω, so indeed no narrower conditioning operation is possible.

5°. To complete the proof, let us show that if some $S\pi(.|_1\phi)$ satisfies the properties **S1**–**S3**, then it coincides with $S\pi(.|\phi)$.

Indeed, by Part 3 of this proof, we have $S\pi(\omega|\phi) \subseteq S\pi(\omega|_1\phi)$ for all ω. If we had $S\pi(\omega|\phi) \neq S\pi(\omega|_1\phi)$ for some ω and ϕ, this would contradict the minimality property **S3**. Thus, indeed, $S\pi(.|\phi) = S\pi(.|_1\phi)$. Uniqueness is proven, and so is for the theorem. □

4.2 Analyzing set-based conditioning

Now, we can go one step beyond Theorem 1 and provide the exact contents of the conditioned set $S\pi(.|_m\phi)$. Let us first start with the following lemma which delimits the set of possible values associated with models of ϕ after the conditioning operation.

Lemma 3. *Let $S\pi$ be a set-valued possibility distribution. Let $\phi \subseteq \Omega$. Then $\forall \omega \in \Omega$,*

- *If $\omega \nvDash \phi$, $S\pi(\omega|\phi) = \{0\}$,*
- *And if $\omega \vDash \phi$, $S\pi(\omega|\phi) \subseteq S\pi(\omega) \cup \{1\}$.*

The proof of this lemma is immediate. Indeed, if π is a standard possibility distribution, then by definition $\pi(\omega|_m\phi)$ is either equal to $\pi(\omega)$ or to 1 for models of ϕ. Hence, the only admissible values for $S\pi(\omega|\phi)$ are those in $S\pi(\omega)$ and the value 1. For counter-models of ϕ (namely, $\omega \nvDash \phi$), then clearly $S\pi(\omega|\phi) = \{0\}$ since $\pi(\omega|_m\phi) = 0$ for each compatible distributions π.

Given this lemma, we need to answer two questions:

- Under which conditions does the fully possibility degree 1 belong to $S\pi(\omega|\phi)$?
- Under which conditions will a given possibility degree $a \in S\pi(\omega)$ still belong to $S\pi(\omega|\phi)$?

The answer to these questions is given in the following proposition:

Proposition 3. *Let $S\pi$ be a set-valued possibility distribution. Let $\phi \subseteq \Omega$.*

i) $1 \in S\pi(\omega|\phi)$ iff $\forall \omega' \neq \omega$, $\overline{S\pi}(\omega) \geq \underline{S\pi}(\omega')$.
ii) Let $a \in S\pi(\omega)$ (with $a \neq 1$). Then $a \in S\pi(\omega|\phi)$ iff $\exists \omega' \neq \omega$, $\overline{S\pi}(\omega') > a$.

Proof. For item (i) assume that $1 \in S\pi(\omega|\phi)$. This means that there exists a compatible distribution π of $S\pi$ such that $\pi(\omega|_m\phi) = 1$. This also means that $\forall \omega' \neq \omega$, $\pi(\omega) \geq \pi(\omega')$. Since, $\overline{S\pi}(\omega) \geq \pi(\omega)$, and $\pi(\omega') \geq \underline{S\pi}(\omega')$, hence we have $\forall \omega' \neq \omega$, $\overline{S\pi}(\omega) \geq \underline{S\pi}(\omega')$. For the converse, assume that $\forall \omega'$, $\overline{S\pi}(\omega) \geq \underline{S\pi}(\omega')$. Let π be a compatible distribution such that $\pi(\omega) = \overline{S\pi}(\omega)$ and $\forall \omega' \neq \omega$, $\pi(\omega') = \underline{S\pi}(\omega)$. Clearly, $\forall \omega' \neq \omega$, $\pi(\omega) > \pi(\omega')$. Hence $\pi(\omega|_m\phi) = 1$ and $1 \in S\pi(\omega|\phi)$.

For item (ii), let $a \in S\pi(\omega)$ where $a \neq 1$. Assume that $\exists \omega' \neq \omega$, such that $\overline{S\pi}(\omega') > a$. Consider a compatible distribution π where $\pi(\omega') = \overline{S\pi}(\omega')$ and $\pi(\omega) = a$. Then clearly, $\pi(\omega_m|\phi) = a \in S\pi(\omega|\phi)$. For the converse, assume that $a \in S\pi(\omega|\phi)$ and $a \neq 1$. This means that there exists a compatible distribution π such that $\pi(\omega|_m\phi) = a < 1$. Hence, $\exists \omega'$, $\pi(\omega) = a < \pi(\omega')$. Since $\pi(\omega') \leq \overline{S\pi}(\omega')$ this means that $\overline{S\pi}(\omega') > a$.

□

Example 7. *In this example, we deal with conditioning a set-valued possibility distribution. Therefore, let us continue Example 2 and assume that the manager of the hotel tells us that the restaurant of the hotel has closed down definitively a few weeks ago. Then we need to condition with the new piece of information $\phi = \neg r$. Let us run the conditioning operation step by step. For every interpretation model of ϕ,*

- *For $\omega = c\neg r$,*
  i) since, with $\omega' = \neg c \neg r$, $.4 \geq .4$, then $1 \in S\pi(c\neg r|\neg r)$;
  ii) For $a = .2$, since, $\overline{S\pi}(\neg c \neg r) = .4 > .2$, then $.2 \in S\pi(c\neg r|\neg r)$.
  For $a = .3$, since, $\overline{S\pi}(\neg c \neg r) = .4 > .2$, then $.3 \in S\pi(c\neg r|\neg r)$.
  For $a = .4$, since, $\overline{S\pi}(\neg c \neg r) = .4 \not> .4$, then $.4 \notin S\pi(c\neg r|\neg r)$.
- *For the interpretation $\omega = \neg c \neg r$, we follow the same computation steps.*
- *For counter-models of $\neg r$, we have $S\pi(\omega|\phi) = \{0\}$.*

Given the distribution in Table 2, we sum up the result of conditioning this distribution in Table 4.

Table 4. Set-valued distribution $S\pi$ of Example 2 conditioned by $\phi = \neg r$.

	$S\pi(.\|\phi)$
cr	$\{0\}$
$\neg cr$	$\{0\}$
$c\neg r$	$\{.2, .3, 1\}$
$\neg c \neg r$	$\{1\}$

5 SYNTACTIC COUNTERPART OF SET-VALUED CONDITIONING

Let us first consider again conditioning a standard possibilistic knowledge base K and rewrite the result of conditioning K. Recall that $K_{\geq a}$=$\{\varphi : (\varphi, \alpha) \in K$ and $\alpha \geq a\}$ be a set of propositional formulas from K having a weight greater or equal to a. Then, the result of conditioning K by ϕ, denoted by K_ϕ, given by Definition 3 can be rewritten as:

$$\begin{array}{rcl} K_\phi & = & \{(\phi, 1)\} \\ & \cup & \{(\varphi, \alpha) : (\varphi, \alpha) \in K_{\geq \alpha} \wedge \phi \text{ is consistent }\} \\ & \cup & \{(\varphi, 0) : (\varphi, \alpha) \in K_{\geq \alpha} \wedge \phi \text{ is inconsistent }\}. \end{array}$$

The only difference with Definition 3 is that '0' weighted formulas have been added. This has no influence thanks to Lemma 1. Namely, K_ϕ is obtained from K by adding ϕ with a fully certainty degree and ignore some formulas from K. By ignoring some formulas, we mean the certainty degrees of these formulas are set to '0'.

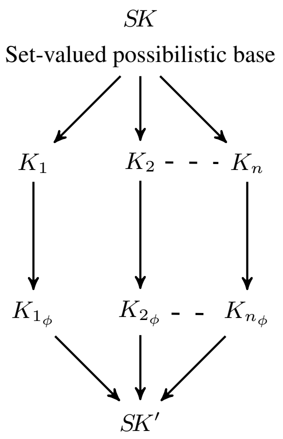

Figure 1. Compatible-based conditioning

The aim of this section is to provide syntactic computation of set-valued conditioning when set-valued possibility distributions are compactly represented by set-valued possibilistic knowledge bases. As illustrated in Figure 1, the input is an initial set-valued knowledge base SK and a formula ϕ. The output is a new set-valued knowledge base SK' that results from conditioning the set of all compatible bases of SK with ϕ. This new set-valued knowledge base SK' is obtained by considering the set of all compatible possibilistic knowledge bases, $K_i \in \mathcal{C}(SK)$. More precisely, it is done in three steps:

- First, from SK we generate the set of compatible bases $K_1, K_2, ..., K_n$
- then, we condition each compatible base K_i with ϕ. The result is K_{i_ϕ} and obtained using Definition 3.
- Lastly, we define SK' by associating with each formula φ of SK the set of degrees present in at least one conditioned K_{i_ϕ}.

Namely: $SK' = \{(\varphi, S) : S = \bigcup\{\alpha_k : (\varphi, \alpha_k) \in K_\phi, K \in \mathcal{C}(SK)\}\}$.

Hence, a naive algorithm for computing SK' is given.

Algorithm 1 Naive computation of SK'

Input: SK: a set-valued knowledge base
  ϕ: a propositional formula
Output: SK': the result of conditioning SK with ϕ
  $SK' \longleftarrow \{(\phi, 1)\}$
  foreach $(\gamma, S) \in SK$ **do**
    $S' \longleftarrow \emptyset$
    foreach K compatible with SK **do**
      Compute K_ϕ
      $S' \longleftarrow S' \cup \{\alpha : (\gamma, \alpha) \in K_\phi\}$
    end foreach
    $SK' \longleftarrow SK' \cup \{(\gamma, S')\}$
  end foreach
  return SK'

Clearly, this algorithm is not satisfactory since the number of compatible bases may be exponential.

Our aim is then to equivalently compute SK' without exploiting the set of all compatible possibilistic knowledge bases.

It is easy to show that $\forall\omega\in\Omega$, $\pi_{K'}(\omega)$=$\pi_K(\omega|\phi)$. Now, in the set-valued setting, conditioning SK comes down first to apply standard conditioning on each compatible base then gathering all certainty degrees. Clearly, SK' is obtained from SK by ignoring some weight. The conditions under which a weight should be ignored is given by the following proposition:

Proposition 4. *Let SK be a set-valued knowledge base, ϕ be a propositional formula. Let $(\gamma, S) \in SK$ and $a \in S$. Let S' be the new set associated with γ in SK'. Then:*

$a \in S'$ iff $\phi \wedge \{\varphi : (\varphi, S) \in SK, \underline{S} \geq a\} \wedge \gamma$ is consistent.

Proof. The proof is as follows. Assume that $a \in S'$. This means that there exists a compatible base K such that $(\gamma, a) \in K'$. Since $\{\varphi : (\varphi, \alpha) \in K'\}$ is consistent, and $(\gamma, a) \in K'$ and $(\phi, 1) \in K'$ then trivially $\phi \wedge \gamma \wedge \{\varphi : (\varphi, b) \in K'\}$ is consistent. Hence, $\phi \wedge \gamma \wedge \{\varphi : (\varphi, b) \in K', b \geq a\}$ is consistent and $\phi \wedge \gamma \wedge \{\varphi : (\varphi, S) \in SK, \underline{S} \geq a\}$ is consistent.

Now, assume that $\phi \wedge \gamma \wedge \{\varphi : (\varphi, S) \in SK, \underline{S} \geq a\}$ is consistent. Let K be a compatible base, where each (φ, S) such that $\varphi \neq \gamma$ is replaced by $(\varphi, \underline{S})$ and (γ, S) is replaced by (γ, a). Clearly, K is a compatible. Besides, $(\gamma, a) \in K'$ since $K_{\geq a} \wedge \phi$ is consistent. Hence, $a \in S'$. □

Based on the above propositions, we propose an algorithm (Algorithm 2) to compute the result of conditioning SK with ϕ. It consists in browsing all the degrees of SK and checking whether each degree should be replaced by 0 or not.

In Algorithm 2, the costly task is checking consistency of the statement marked by (#). Hence, the complexity of computing SK' is $O(|SK| * n * SAT)$ where n is the number of different certainty levels in SK (namely, $n = |\bigcup\{S : (\varphi, S) \in SK\}|$). This is stated in the following proposition.

Proposition 5. *Let SK be a set-valued possibilistic knowledge base and ϕ be the new evidence. Let SK' be a set-valued possibilistic knowledge base computed using Algorithm 2. Then computing SK_ϕ is in $O(|SK| * n * SAT)$ where SAT is a satisfiability test of a set propositional clauses and n is the number of different weights in SK.*

Algorithm 2 Syntactic set-valued conditioning

Input: SK: a set-valued knowledge base
  ϕ: a propositional formula
Output: SK': the result of conditioning SK with ϕ

```
SK' ⟵ {(φ, 1)}
foreach (γ, S) ∈ SK do
  S' ⟵ ∅
  foreach a ∈ S do
    if (#) φ ∧ γ ∧ {φ : (φ, S) ∈ SK, S̲ ≥ a} is consistent then
      S' ⟵ S' ∪ {a}
    else
      S' ⟵ S' ∪ {0}
    end if
    SK' ⟵ SK' ∪ {(γ, S')}
  end foreach
end foreach
return SK'
```

Example 8. *Let us illustrate Algorithm 2. To do so, we continue Example 4 where* $SK = \{(\neg c \vee r, \{.4, .7, .8\}), (r, \{.6\})\}$ *and with the new information* $\phi = \neg r$. *For each pair* (φ, S),

- *First let us take* $(\neg c \vee r, \{.4, .7, .8\})$ *then:*
  - *For* $a = .4$, $\{r, \neg c \vee r\} \wedge \{\neg r\} \wedge \{\neg c \vee r\}$ *is not consistent then,* $0 \in S'$*;*
  - *For* $a = .7$, $\emptyset \wedge \{\neg r\} \wedge \{\neg c \vee r\}$ *is consistent then,* $.7 \in S'$*;*
  - *We use the same reasoning for* a=.8*, then,* $.8 \in S'$.
- *Now for the second pair* $(r, \{.6\})\}$ *we have:*
  - *For* a=.6*,* $\{r\} \wedge \{\neg r\} \wedge \{r\}$ *is not consistent so* $0 \in S'$*;*

The new base is SK'=$\{(\neg r, \{1\}), (\neg c \vee r, \{0, .7, .8\}), (r, \{0\})\}$. *Thanks to Lemma 2, we can exclude the pair* $(r, \{0\})$*, this is our new base:* SK'=$\{(\neg r, \{1\}), (\neg c \vee r, \{0, .7, .8\})\}$. *The corresponding set-valued possibility distribution according Definition 8 is given in Table 5.*

Table 5. Set-valued distribution corresponding to set-valued knowledge base SK'.

	$S\pi_{SK'}$
cr	$\{0\}$
$\neg cr$	$\{0\}$
$c\neg r$	$\{.2, .3, 1\}$
$\neg c\neg r$	$\{1\}$

6 RELATED WORKS AND DISCUSSIONS

This paper dealt with representing and reasoning with qualitative information in a possibilistic setting and it provided three main contributions:

- The first one is a new extension of possibilistic logic called set-valued possibilistic logic particularly suited for reasoning with qualitative and multiple source information. We provided a natural semantics in terms of compatible possibilistic bases and compatible possibility distributions.
- The second main contribution deals with a generalization of the well-known min-based or qualitative conditioning to the new set-valued setting. The paper proposes three natural postulates ensuring that any set-valued conditioning satisfying these three postulates is necessarily based on the set of compatible standard possibility distributions.
- The third main contribution concerns the syntactic characterization of set-valued conditioning. Efficient procedures are proposed to compute the exact set-valued possibility distributions and their syntactic counterparts. Interestingly enough, the proposed setting generalizes standard possibilistic and conditioning does not require extra computational cost with respect to the standard single valued possibilistic setting. We provide an algorithm which does not generate explicitly the set of all compatible possibilistic knowledge bases.

Many extensions have been proposed to generalize possibilistic logic. The closest one to set-valued possibilistic logic, proposed in this paper, is interval-based possibilistic logic [4, 11, 5]. The two settings view a knowledge base (resp. possibility distribution) as a family of compatible bases bases (resp. distributions). Of course, intervals are particular sets. However, in [5] conditioning operator deals only with quantitative interpretation of possibility theory [5] while set-valued possibilistic logic deals with qualitative possibility theory. Besides, the rational postulates given in [5] does not characterise the uniqueness of conditioning operator while in this paper, this three postulates **S1**, **S2**, and **S3** guarantee the uniqueness of the conditioning operation.

Among the other extensions, symbolic possibilistic logic [6, 7] deals with a special type of uncertainty where the available uncertain information is in the form of partial knowledge on the relative certainty degrees (symbolic weights) associated with formulas. In [2], a multiple agent extension of possibilistic logic is proposed. This extension associates sets of agents to sets of possibilistic logic formulas and aims to reason on the individual and mutual beliefs of the agents. Note that no form of conditioning the whole knowledge is proposed for this setting.

Note that the idea of compatible-based conditioning in the interval-based possibilistic setting is somehow similar to conditioning in credal sets [1, 26] and credal networks [12] where the concept of convex set refers to the set of compatible probability distributions composing the credal set. Regarding the computational cost, conditioning in credal sets is done on the set of extreme points (edges of the polytope representing the credal set) but their number can reach $N!$ where N is the number of interpretations [28]. In this paper, our set-valued conditioning operator has a complexity close to the one of standard possibilistic knowledge base.

Clearly, many of the qualitative extensions of possibilistic logic mentioned in this section could benefit from our conditioning operators as far as they can be encoded as set-valued possibilistic bases. This will be our main track for future works.

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