Propositional Fragments for Knowledge Compilation and Quantified Boolean Formulae

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The QBF problem

- Canonical PSPACE-complete problem
- Can be used in many AI areas: planning, nonmonotonic reasoning, paraconsistent inference, abduction, etc

- High complexity, both in theory and in practice
- A possible solution: tractable classes
- Instances of those tractable classes hard for current QBF solvers (e.g. (renamable) Horn benchmarks)

Outline

QBF: formal definition

Definition *(QBF)*

A QBF Π is an expression of the form

$$Q_1 X_1 \dots Q_n X_n \Phi, \qquad (n \geq 0)$$

- $X_1 \dots X_n$ sets of propositional variables
- Φ a propositional formula on those variables
- $Q_i (0 \leq i \leq n)$ an existential $\exists$ or universal $\forall$ quantifier

Validity of a QBF

Existence of a winning strategy in a game against nature (∀)

Example

$$\forall\, x\ \exists\, y_1,\ y_2$$

$$[(y_1 \lor y_2) \land (\neg y_2 \lor x) \land$$

$$(\neg y_1 \lor \neg y_2) \land (y_2 \lor \neg x)]$$

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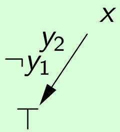

Validity of a QBF

Existence of a winning strategy in a game against nature ($\forall$)

Example

$$\forall\, x\ \exists\, y_1,\ y_2$$

$$[(y_1 \vee y_2) \wedge (\neg y_2 \vee x) \wedge$$

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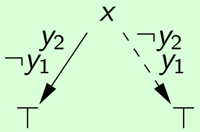
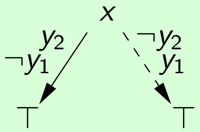


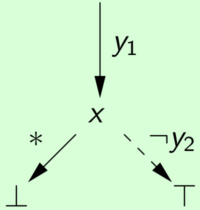
Validity of a QBF

Existence of a winning strategy in a game against nature ($\forall$)

Example

$$\forall\, x\, \exists\, y_1,\ y_2\ [(y_1 \vee y_2) \wedge (\neg y_2 \vee x) \wedge (\neg y_1 \vee \neg y_2) \wedge (y_2 \vee \neg x)] \quad \not\equiv \quad \exists\, y_1\, \forall\, x\, \exists\, y_2\ [(y_1 \vee y_2) \wedge (\neg y_2 \vee x) \wedge (\neg y_1 \vee \neg y_2) \wedge (y_2 \vee \neg x)]$$



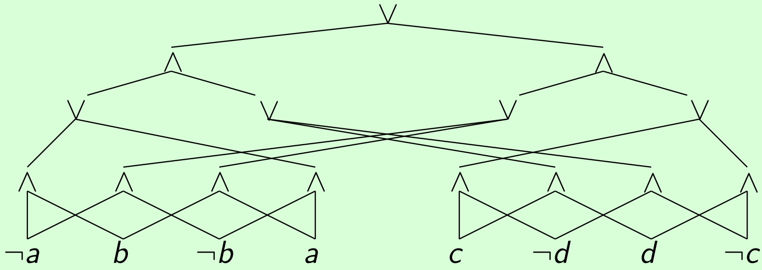


Definition (NNF [Darwiche 1999])

A formula in $\mathtt{NNF}_{PS}$ is a rooted DAG where:

- each leaf node is labeled with *true*, *false*, x or $\neg x$, $x \in PS$
- each internal node is labeled with $\land$ or $\lor$ and can have arbitrarily many children

Example

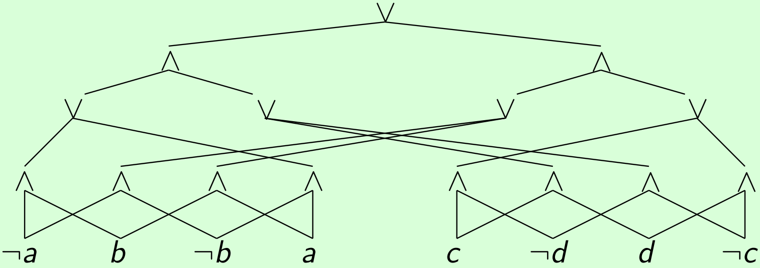


Properties [Darwiche 1999]

- Decomposability
- Determinism
- Smoothness
- Decision
- Ordering

Fragments of $\mathtt{NNF}_{PS}$: examples

Example

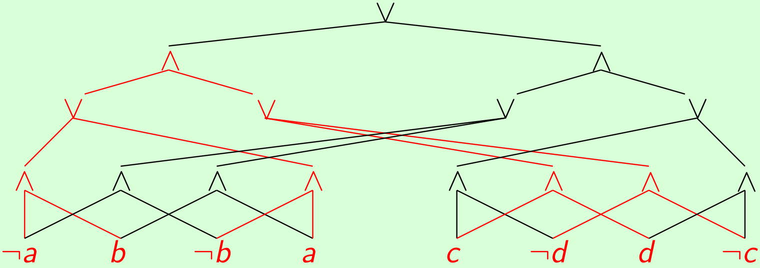


Fragments of $\mathtt{NNF}_{PS}$: examples

Decomposability: if $C_1, \ldots, C_n$ are the children of **and-node** C, then $Var(C_i) \cap Var(C_j) = \emptyset$ for $i \neq j$

Example

Decomposability

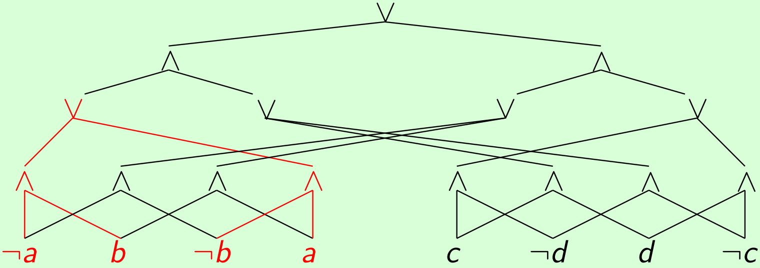


Fragments of $\mathtt{NNF}_{PS}$: examples

Determinism: if $C_1, \ldots, C_n$ are the children of **or-node** C, then $C_i \wedge C_j \models \mathit{false}$ for $i \neq j$

Example

Determinism

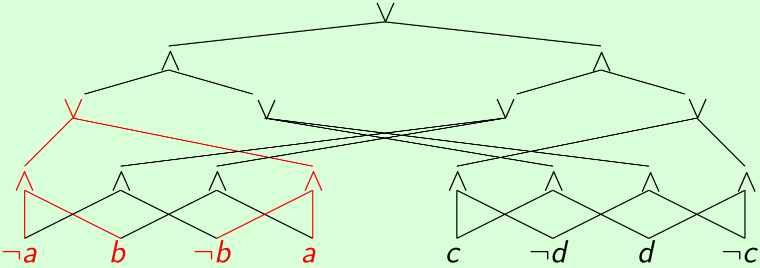


Fragments of $\mathtt{NNF}_{PS}$: examples

Smoothness: if $C_1, \ldots, C_n$ are the children of **or-node** C, then $Var(C_i) = Var(C_j)$

Example

Smoothness



Fragments of $\mathtt{NNF}_{PS}$: definitions

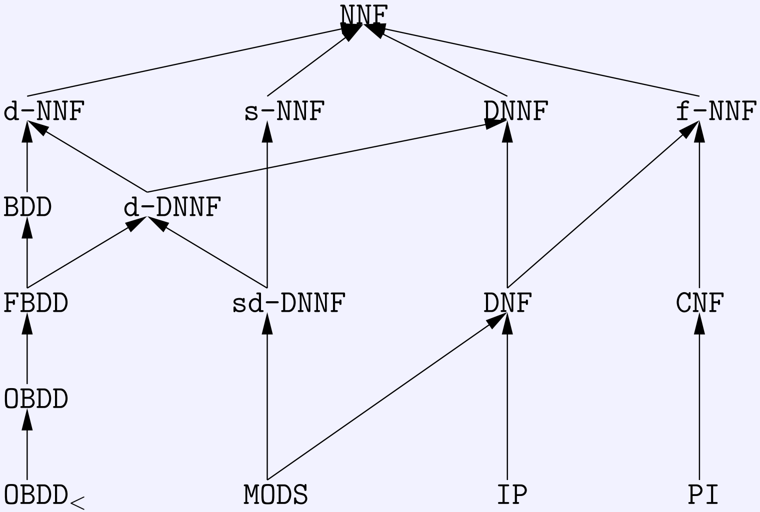
Definition *(Propositional fragments [Darwiche & Marquis 2001])*

- $\mathtt{DNNF}$: $\mathtt{NNF}_{PS}$ + decomposability.
- $\mathtt{d\text{-}DNNF}$: $\mathtt{NNF}_{PS}$ + decomposability and determinism.
- $\mathtt{FBDD}$: $\mathtt{NNF}_{PS}$ + decomposability and decision.
- $\mathtt{OBDD}_{<}$: $\mathtt{NNF}_{PS}$ + decomposability, decision and ordering.
- $\mathtt{MODS}$: $\mathtt{DNF} \cap \mathtt{d\text{-}DNNF}$ + smoothness.

Complexity results for QBF

Fragment	Complexity
$PROP_{PS}$ (general case)	PSPACE-c
CNF	PSPACE-c
DNF	PSPACE-c
d-DNNF	PSPACE-c
DNNF	PSPACE-c
FBDD	PSPACE-c
$\texttt{OBDD}_<$	PSPACE-c
$\texttt{OBDD}_<$ (compatible prefix)	$\in$ P
PI	PSPACE-c
IP	PSPACE-c
MODS	$\in$ P

Inclusion of fragments [Darwiche & Marquis 2001]



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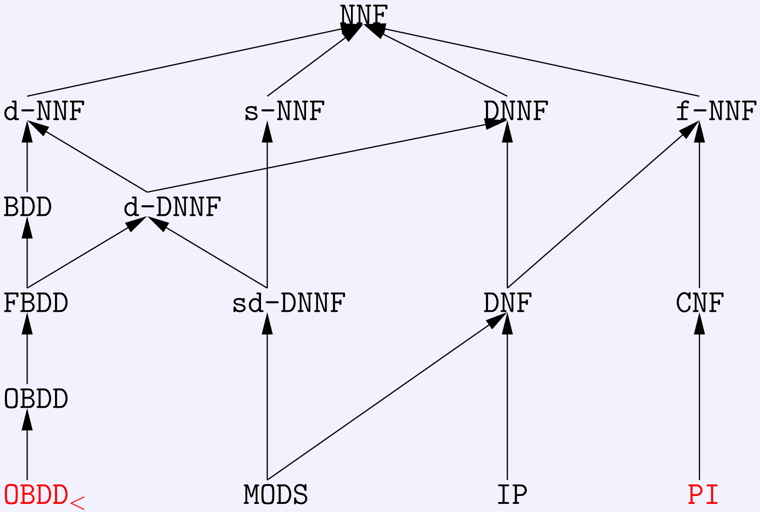

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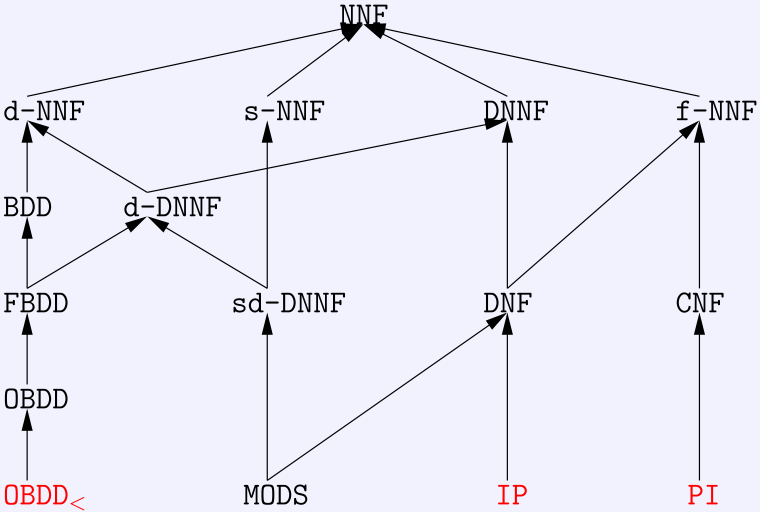

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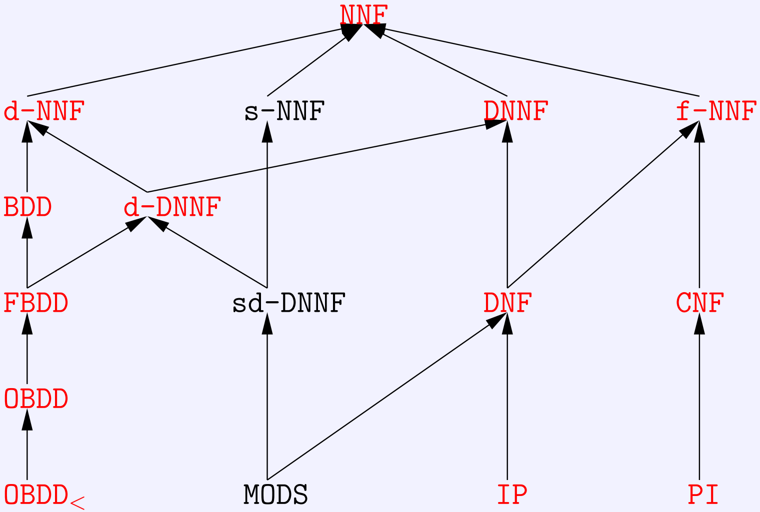

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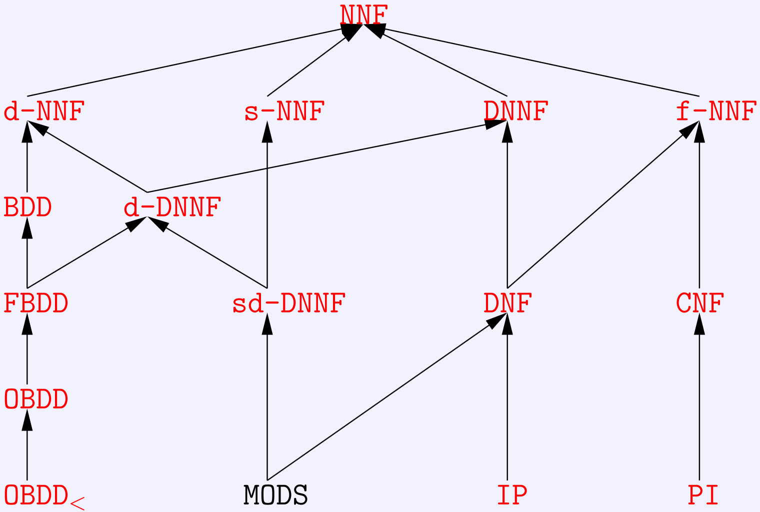

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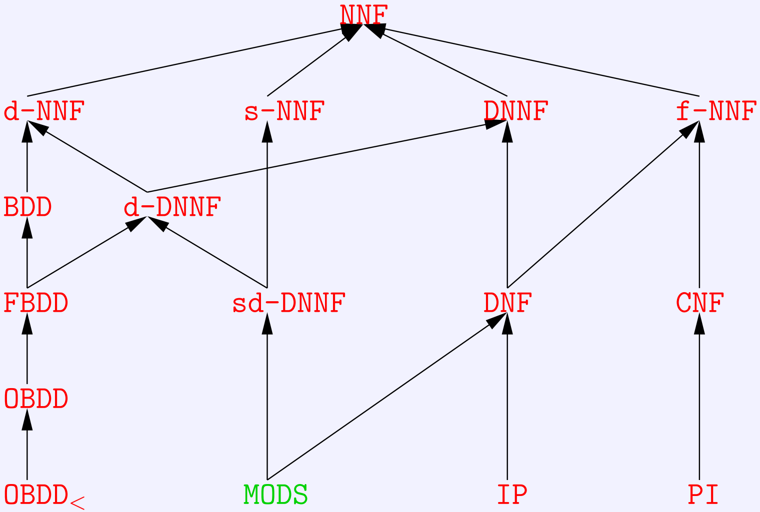


$\texttt{OBDD}_<$ with compatible prefix: a polynomial case

- Prefix compatible: $<$ extension of the variable ordering induced by the prefix of the QBF

- Eliminating quantifiers from the innermost to the outermost
  - Eliminating existential quantifiers
  - Eliminating universal quantifiers ($\forall x \equiv \neg \exists x \neg$)
  - Reduce the $\texttt{OBDD}_<$ at each elimination step

- Remark: Negation in constant time in $\texttt{OBDD}_<$

$\texttt{OBDD}_<$ with compatible prefix: a polynomial case

$$\Sigma = \exists x \, \forall y \, \exists z \, \phi$$

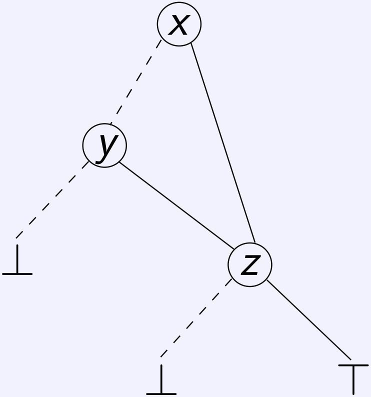
$$\phi \equiv (x \vee y) \wedge z$$

$\texttt{OBDD}_<$ with compatible prefix: a polynomial case

$$\Sigma = \exists x \; \forall y \; \exists z \; \phi$$

$$\phi \equiv (x \lor y) \land z$$

$\phi =$



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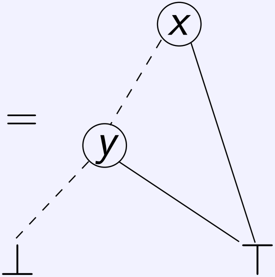

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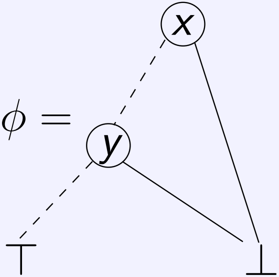

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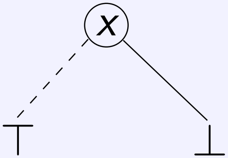

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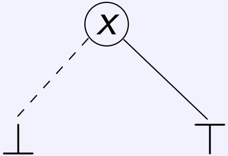

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$$\Sigma = \exists x\ \forall y\ \exists z\ \phi$$

$$\phi \equiv (x \vee y) \wedge z$$

$$\exists x \forall y \exists z\ \phi = \top$$

$\Rightarrow \Sigma$ is valid

Conclusion

- Presentation of propositional fragments
- Numerous intractable fragments
- Some tractability results (under very restrictive conditions)

Perspectives

- Study of other (polynomial) complete or incomplete fragments
- Integrate specialized algorithms into our solver
- A sharper analysis of the prefix in order to improve our solver

Any questions ?