Selecting the most relevant elements from a ranking over sets

Sébastien Konieczny[1 [0000−0002−2590−1222]], Stefano Moretti[2 [0000−0003−3627−3257]], Ariane Ravier[2], and Paolo Viappiani[2 [0000−0002−7922−3877]]

[1] CRIL - CNRS - Université d'Artois - France. konieczny@cril.fr
[2] LAMSADE, CNRS, Université Paris-Dauphine, Université PSL, 75016, Paris, France.
{stefano.moretti,ariane.ravier,paolo.viappiani}@lamsade.dauphine.fr

Abstract. In this paper we study the problem of selecting the most relevant elements of a finite set N of elements from a ranking of non-empty subsets of N, that represents the performance of different coalitions. To solve this problem, we first introduce the notion of coalitional social choice function (*i.e.*, a map that associates each total preorder of the non-empty subsets of N with a subset of N). Then, we provide four basic properties that a coalitional social choice function should satisfy to select the most relevant elements. Finally, we prove that the unique coalitional social choice function that satisfies such properties is the one selecting the elements ranked in the highest position of the lexicographic excellence ranking, which is computed according to a social ranking solution from the literature.

Keywords: social ranking · power relation · coalitions · social choice function.

1 Introduction

The problem of defining a ranking of individual elements based on their contribution in establishing the position of groups or coalitions within a society has been recently introduced in the literature related to the notion of social ranking [6,8]. Taking as input a ranking of sets of objects like, for instance, a ranking over all possible research groups of a department, or over alternative combinations of attackers in a football team, or even a dichotomous order of winning or losing coalitions within a voting body, a social ranking generates a ranking of the individuals (researchers, football players, voters...) reflecting the overall relevance of individuals within the rank of coalitions [6,2,1].

In [6], for example, the individuals are ranked according to their frequency in the highest positions in the ranking of sets and a special social ranking function, called *lexicographic excellence* (*lex-cel*), has been identified as the unique one satisfying a set of appealing properties. A generalization of the lex-cel has been introduced in papers [2,5] considering the size of coalitions, in addition to their positions in the ranking of sets. For other notions of social ranking solutions see also the papers [5,3,9] (for a software implementation of social ranking solutions from the literature, the interested reader is referred to the R package [7]).

Sometimes, however, we are not interested in generating an entire ranking of the individual elements, but instead just want to select the most important ones. Consider for instance the problem of identifying the most influential scientists within an Academic Association (AA) based on the number and the quality of their publications. One of the major difficulties in comparing scientists is taking into account their contributions to multi-authored publications [4,12]. Several bibliometric indices exist to compare the impact of research activity of individual scientists and of research groups, and the choice of the appropriate index goes beyond the scope of this paper (see, for instance, the paper [11] for an in-depth analysis of the problem). Nevertheless, each group of scientists can be characterized by a record of jointly published papers (possibly with other authors who do not belong to the AA). So, in principle, a ranking of groups of scientists can be established according to a predefined bibliometric criterion for the comparison of the overall influence of research groups (see Example 1 for a toy situation illustrating some naive approaches aimed at ranking groups starting from records of papers published in journals). Starting from such a ranking of groups, which scientists can be identified as the most influential in a way that the scientists' ranking positions over different groups are considered?

In this paper we provide an answer to this question following a property-driven approach. We first single out a set of reasonable properties that a method selecting the set of most relevant elements (namely, a *coalitional social choice function*) from a ranking over groups should satisfy. Then we prove that the unique method satisfying those properties is the one which selects the best elements according to the ranking produced by the lex-cel.

As a first property, we consider the *All-Indifferent-All-Winners* axiom, which indicates that all the elements should be treated in the same way; so, if all groups are equally powerful and there is no reason to distinguish among the roles played by the elements, then all the elements should be selected. The second axiom we introduce is a *Monotonicity for Winners* property stating that if some elements are selected as the most relevant in a ranking of groups, and a new ranking over groups is produced improving the position of groups in the last equivalence class (but without affecting the comparison with groups in the other equivalence classes), then the most frequent elements over the improved groups should be considered as the most relevant ones also in the new ranking of groups. The third property, called the *Dominance* axiom, deals with the same kind of improvements described for the monotonicity axiom, but in this case it rules out the elements that are less represented among the improved groups. Finally, the *Independence for Losers from the Worst Set* axiom, prevents the elements that are excluded from being the most relevant ones in an original ranking of groups to become the most relevant in any other ranking of groups obtained from the original one by partitioning the last equivalence class. In other words, once the decision to exclude an element from the most relevant set is taken, the decision shall not be affected by changes involving the worst groups.

The roadmap of the paper is as follows. In the next section we introduce some preliminary definitions and a motivational example for the computation of the lex-cel. In Section 3, we formally introduce the four axioms and we illustrate them along the lines of the example introduced in Section 2. Section 4 is devoted to the axiomatic characterization of the lex-cel coalitional social choice function using the axioms introduced in Section 3, and to establishing their logical independence. Section 5 concludes with some comments and directions for future research.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set of *elements* (individuals, items, etc.) and let $\mathcal{P}(N)$ be the set of the non-empty subsets of N (also called *coalitions* or *groups*). A binary relation $R \subseteq N \times N$ is said to be: *reflexive*, if for each $i \in N$, iRi; *transitive*, if for each $i, j, k \in N$, iRj and $jRk \Rightarrow iRk$; *total*, if for each $i, j \in N$, iRj or jRi; *antisymmetric*, if for each $i, j \in N$, iRj and $jRi \Rightarrow i = j$. A *total preorder* (also called a *ranking*) is a reflexive, transitive and total binary relation. A *total order* is a reflexive, transitive, total and antisymmetric binary relation. $\boldsymbol{\mathcal{R}}(N)$ denotes the set of rankings (or total preorders) on a given set N.

A total preorder $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ is called a *power relation*. Given $S, T \in \mathcal{P}(N)$, $S \succeq T$ means that "S is at least as powerful as T with respect to the power relation $\succeq$". We denote by $\sim$ the symmetric part of $\succeq$ (*i.e.* $S \sim T$ if $S \succeq T$ and $T \succeq S$) and by $\succ$ its asymmetric part (*i.e.* $S \succeq T$ and not $T \succeq S$). So, for each pair of subsets $S, T \in \mathcal{P}(N)$, $S \succ T$ means that S is *strictly more powerful* than T, whereas $S \sim T$ means that S and T are *are equally powerful*.

Let $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ be a power relation of the form $S_1 \succeq S_2 \succeq \cdots \succeq S_{|\mathcal{P}(N)|}$. The *quotient order* of $\succeq$ is denoted as $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$ in which the subsets S_j are grouped in the *equivalence classes* Σ_k generated by the symmetric part of $\succeq$. This means that all the sets in Σ_1 are equally powerful to S_1 and are strictly better than the sets in Σ_2 and so on.

Given a power relation $\succeq$ and its associated quotient order $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$, we denote by $i_k = |\{S \in \Sigma_k : i \in S\}|$ the number of sets in Σ_k containing i for $k = 1, \ldots, l$. Now, let $\theta^{\succeq}(i)$ be the l-dimensional vector $\theta^{\succeq}(i) = (i_1, \ldots, i_l)$ associated to $\succeq$. Consider the lexicographic order $\geq_L$ among vectors $\mathbf{i}$ and $\mathbf{j}$: $\mathbf{i} \geq_L \mathbf{j}$ if either $\mathbf{i} = \mathbf{j}$ or there exists t such that $i_t > j_t$ and $i_r = j_r$ for all $r \in \{1, \ldots, t-1\}$.

Let $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$. The *lexicographic excellence (lex-cel)* [6] is the binary relation $R_{le}^{\succeq}$ such that for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ and all $i, j \in N$:

$$i \; R_{le}^{\succeq} \; j \iff \theta^{\succeq}(i) \quad \geq_L \quad \theta^{\succeq}(j).$$

(in the remaining $I_{le}^{\succeq}$ and $P_{le}^{\succeq}$ stand for the symmetric part and the asymmetric part of $R_{le}(\succeq)$, respectively.)

We illustrate the computation of the lex-cel via the following toy example.

Example 1. Consider an AA formed by three scientists $N = \{1, 2, 3\}$ and the problem of identifying the most influential scientist(s) within the AA based on the number of papers published in journals ranked according to the SCImago Journal Rank (SJR) and awarding the publication in the highest position of the SJR. Suppose that the three scientists never worked all together on a research project, so there is no paper co-authored by the three scientists together. Instead, some published papers exist which were co-authored by pairs of authors. Moreover, while scientists 1 and 2 are more experienced and they published some papers without the help of the others, 3 is a young researcher and he never published a paper alone. Journals related to the AA discipline are grouped into four quartiles Q_1, Q_2, Q_3 and Q_4, where Q_1 is the first quartile formed by the top 25% of journals in the SJR list, Q_2 is the second quartile (from 25% to 50%), Q_3 the third one (from 50% to 75%) and Q_4 the last one (from 75% to 100%). The number of papers co-authored by each group of scientists is summarized in Table 1.

Table 1. Number of published papers co-authored by groups of scientists in each SJR quartile.

Groups	Q_1	Q_2	Q_3	Q_4
$\{1,2,3\}$	0	0	0	0
$\{1,2\}$	5	0	0	15
$\{1,3\}$	0	10	5	0
$\{2,3\}$	5	0	5	0
$\{1\}$	5	0	0	15
$\{2\}$	0	0	10	5
$\{3\}$	0	0	0	0

To reward the publications in the highest-ranked journals in the SJR, one may argue that the performance of the above groups may be ranked according to the lexicographic comparison (from the first quartile to the fourth one) of the rows of Table 1. As a result, we obtain the following power relation $\succeq$ (in the following, to avoid cumbersome notation, commas and parentheses for sets are omitted, e.g., we use 123 instead of $\{1, 2, 3\}$):

$$23 \succ 1 \sim 12 \succ 13 \succ 2 \succ 3 \sim 123.$$

So, the quotient order $\succ$ of $\succeq$ is

$$\Sigma_1 = \{23\} \succ \Sigma_2 = \{1, 12\} \succ \Sigma_3 = \{13\} \succ \Sigma_4 = \{2\} \succ \Sigma_5 = \{3, 123\}.$$

Assume that the lex-cel is adopted to rank the scientists according to their influence over the ranking of groups. To compute the lex-cel on $\succeq$, we need to lexicographically compare the vectors $\theta^{\succeq}(i)$, $i \in N$, defined earlier in this section. So,

$$\theta^{\succeq}(2) = (1, 1, 0, 1, 1) >_L \theta^{\succeq}(3) = (1, 0, 1, 0, 2) >_L \theta^{\succeq}(1) = (0, 2, 1, 0, 1)$$

and therefore the lex-cel $R^{\succeq}_{le}$ gives a total order over N such that $2P^{\succeq}_{le}3P^{\succeq}_{le}1$.

Notice that alternative approaches, less oriented to rewarding the highest positions in the SJR, might be adopted. Therefore, a different power relation could be generated from Table 1. For instance, the use of the total number of publications as bibliometric index would produce the power relation $\sqsupseteq$

$$1 \sim 12 \sqsupset 2 \sim 13 \sqsupset 23 \sqsupset 3 \sim 123.$$

It is easy to verify that the lex-cel in this case gives the ranking $1P^{\sqsupseteq}_{le}2P^{\sqsupseteq}_{le}3$.

3 Properties for coalitional social choice functions

We define the notion of *coalitional social choice function* (cscf) as a map

$$B : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$$

that associates to each power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ a non-empty subset $B(\succeq)$ $\in \mathcal{P}(N)$ which is interpreted as the set of *most relevant elements* or *winners* in $\succeq$. We now introduce some properties for a cscf.

The first axiom states a principle of neutrality for elements: if all coalitions are indifferent then there is no way to distinguish a major contribution of any element and all the elements should be considered winners.

Axiom 1 (All-Indifferent-All-Winners (AIAW)) Consider a power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ such that for all $S, T \in \mathcal{P}(N)$:

$$S \sim T. \tag{1}$$

Then a cscf B satisfies the property *All-Indifferent-All-Winners* if it holds that $B(\succeq) = N$.

Example 2. Consider again an AA with $N = \{1, 2, 3\}$, as in Example 1, but suppose now that all non-empty subsets of the set N of scientists have published precisely the same number of papers in each SJR quartile. Based on such information, all coalitions may be considered equally powerful. So, we have a power relation $\succeq$ such that.

$$1 \sim 2 \sim 3 \sim 12 \sim 13 \sim 23 \sim 123.$$

A cscf B satisfying Axiom 1 selects as winners $B(\succeq) = \{1, 2, 3\}$.

The second axiom is a particular monotonicity condition for winners: improving the position of some coalitions in the worst equivalence class, but keeping the same relation among coalitions in the other equivalence classes, should not affect the status of most represented winners (over the improved coalitions). In the following, given a family of coalitions $\Sigma \subseteq \mathcal{P}(N)$ and any element $i \in N$, we use the notation i_Σ to denote the number of coalitions in Σ to which i belongs, *i.e.* $i_\Sigma = |\{S \in \Sigma : i \in S\}|$.

Axiom 2 (Monotonicity for Winners (MW)) Consider two power relations $\succeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ and their respective quotient orders $\succ$ and $\sqsupset$ such that:

- $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$,
- $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset \Sigma \sqsupset \Sigma_l \setminus \Sigma$,
- with $\Sigma \subseteq \Sigma_l$.

Take a cscf B and let $T \subseteq B(\succeq)$ be the set of most represented winners over Σ, *i.e.*

$$T = \{i \in B(\succeq) : i_\Sigma \geq j_\Sigma \ \forall j \in B(\succeq)\}. \tag{2}$$

We say that B satisfies the *Monotonicity for Winners* property if it holds that

$$T \subseteq B(\sqsupseteq).$$

Example 3. Consider again the power relation $\succeq$ of Example 2. Now, suppose that, after an update of the publication records, it turns out the number of joint-papers published by scientists 2 and 3 together (and without author 1) improves with respect to SJR. As a consequence, a new power relation $\sqsupseteq$ is considered where coalition 23 is the most powerful coalition:

$$23 \sqsupset 1 \sim 2 \sim 3 \sim 12 \sim 13 \sim 123.$$

According to relation (2), the set of most represented winners is $T = \{2, 3\}$. So, a cscf B satisfying Axiom 2 is such that $2, 3 \in B(\sqsupseteq)$. Notice that, based on Axiom 2, we cannot affirm whether element 1 belongs to $B(\sqsupseteq)$ or not.

The third axiom states that after making an improvement to coalitions in the worst equivalence class (as in the previous axiom), the winners that are now strictly less represented than other winners over the improved coalitions (*i.e.*, they are dominated by other winners) become losers (no longer winners).

Axiom 3 (Dominance (D)) Consider two power relations $\succeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ and their respective quotient orders $\succ$ and $\sqsupset$ such that:

- $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$,
- $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset \Sigma \sqsupset \Sigma_l \setminus \Sigma$,
- with $\Sigma \subseteq \Sigma_l$.

Take a cscf B and let $L \subseteq B(\succeq)$ be the set of winners that are strictly less represented than other winners over Σ, *i.e.*

$$L = \{j \in B(\succeq) : \exists i \in B(\succeq) \text{ with } i_\Sigma > j_\Sigma\}. \tag{3}$$

We say that B satisfies the *Dominance* property if it holds that

$$B(\sqsupseteq) \subseteq N \setminus L.$$

Remark 1. Notice that the set L in Axiom 3 is the complement in the set of winners of the set T in Axiom 2, so $T \cup L = B(\succeq)$ and $T \cap L = \emptyset$.

Example 4. Consider again the power relation $\succeq$ of Example 2 and the power relation $\sqsupseteq$ of Example 3. According to Remark 1, we have $L = N \setminus \{2, 3\} = \{1\}$. So, a cscf B satisfying Axiom 3 is such that scientist 1 does not belong to the set $B(\sqsupseteq)$ of most important elements.

Finally, the last axiom states that once an element becomes a loser it remains a loser over all possible power relation obtained fractioning the worst equivalence class of the quotient order.

Axiom 4 (Independence for Losers from the Worst Set (ILWS))
We say that a cscf B satisfies the property of *Independence for Losers from the Worst Set* if for any power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ with the associated quotient order $\succ$ such that

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$$

and $i \in N$ such that

$$i \notin B(\succeq),$$

then for any partition $T_1, \ldots, T_m$ of Σ_l and for any power relation $\sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ with the associated quotient order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset T_1 \sqsupset \cdots \sqsupset T_m,$$

it holds that

$$i \notin B(\sqsupseteq).$$

Example 5. Consider $\sqsupseteq$ of Example 3 and its quotient order $\sqsupset$ such that $\Sigma_1 = \{23\}$ and $\Sigma_2 = \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \setminus \{23\}$. Now, suppose that after an update in the publications records of groups of scientists, we obtain a new power relation $\sqsupseteq'$ obtained from $\sqsupseteq$ via a partition of Σ_2 and such that

$$23 \sqsupset' 1 \sim' 12 \sqsupset' 13 \sqsupset' 2 \sqsupset' 3 \sim' 123.$$

Notice that $\sqsupset'$ coincides with $\succeq$ of Example 1. Consider a cscf B satisfying Axiom 4 and suppose $1 \notin B(\sqsupseteq)$ (like in Example 4). A direct consequence of Axiom 4 is that 1 does not even belong to $B(\sqsupseteq')$.

4 The lex-cel coalitional social choice function

A particular cscf denoted as B_{le} is based on the lex-cel and it associates to a each power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ the set of elements in the highest positions in the ranking $R^{\succeq}_{le}$ provided by the lex-cel.

Definition 1 (Lex-cel cscf). Let $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$. The *lex-cel cscf* is the map $B_{le} : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$ such that for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$:

$$B_{le}(\succeq) = \{i \in N : i \; R^{\succeq}_{le} \; j \;\; \forall j \in N\}.$$

Example 6. Consider the power relation $\succeq$ of Example 1 and the lex-cel ranking computed over $\succeq$ in the same example. Then, $B_{le}(\succeq) = \{2\}$.

To axiomatically characterize the lex-cel cscf we first need to introduce the following lemma.

Lemma 1. *Consider two power relations $\succeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ and their respective quotient orders $\succ$ and $\sqsupset$ such that:*

- $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$,
- $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset \Sigma \sqsupset \Sigma_l \setminus \Sigma$,
- *with* $\Sigma \subseteq \Sigma_l$.

Then, $B_{le}(\sqsupseteq) = T = \{i \in B_{le}(\succeq) : i_\Sigma \geq j_\Sigma \ \forall j \in B_{le}(\succeq)\}$.

Proof. First, notice that for each $i \in N$

$$\boldsymbol{\theta}^{\sqsupseteq}(i) = (i_1, \ldots, i_{l-1}, i_\Sigma, i_l - i_\Sigma). \tag{4}$$

where $i_k = |\{S \in \Sigma_k : i \in S\}|$ is the number of sets in Σ_k containing i for $k = 1, \ldots, l$. We prove that, for each $i \in T$, we have $i \in B_{le}(\sqsupseteq)$ or, equivalently by the definitions of lex-cel, that

$$\boldsymbol{\theta}^{\sqsupseteq}(i) \ \geq_L \ \boldsymbol{\theta}^{\sqsupseteq}(j) \ \forall j \in N.$$

We distinguish three cases:

i) $j \in T$: since $i, j \in T$, we have that $i_\Sigma = j_\Sigma$; moreover, $i, j \in B_{le}(\succeq)$ implies that $\boldsymbol{\theta}^{\succeq}(i) = \boldsymbol{\theta}^{\succeq}(j))$. Then, by relation (4), $\boldsymbol{\theta}^{\sqsupseteq}(i) = \boldsymbol{\theta}^{\sqsupseteq}(j)$;
ii) $j \in B_{le}(\succeq) \setminus T$: since $i, j \in B_{le}(\succeq)$, we have that $i_k = j_k$ for all $k = 1, \ldots, l$; moreover, $j \in B_{le}(\succeq) \setminus T$ implies that $i_\Sigma > j_\Sigma$. Then, by relation (4), $\boldsymbol{\theta}^{\sqsupseteq}(i) >_L \boldsymbol{\theta}^{\sqsupseteq}(j)$
iii) $j \in N \setminus B_{le}(\succeq)$: since $i \in B_{le}(\succeq)$ but $j \notin B_{le}(\succeq)$, it must exist $k = 1, \ldots, l-1$ such that $i_k > j_k$. Then, by relation (4), $\boldsymbol{\theta}^{\sqsupseteq}(i) >_L \boldsymbol{\theta}^{\sqsupseteq}(j)$.

So, $T \subseteq B_{le}(\sqsupseteq)$. Moreover, by the the above points *ii*) and *iii*) for each $j \in N \setminus T$ we have that $\boldsymbol{\theta}^{\sqsupseteq}(i) >_L \boldsymbol{\theta}^{\sqsupseteq}(j)$ for all $i \in T$. So $B_{le}(\sqsupseteq) \subseteq T$, which concludes the proof.

We are now ready to introduce the main result of this section.

Theorem 5. *The cscf B_{le} is the unique cscf fulfilling Axioms 1, 2, 3 and 4.*

Proof. We first prove that the cscf B_{le} satisfies the four axioms.

To see that B_{le} satisfies Axiom 1, consider a power relation $\succeq \in \mathcal{R}(\mathcal{P}(N))$ such that $S \sim T$ for all $S, T \in \mathcal{P}(N)$. We have that $\theta^{\succeq}(i) = (2^{n-1})$ for any $i \in N$ since any element i belongs to precisely 2^{n-1} coalitions in the same equivalence class $\Sigma_1 = \mathcal{P}(N)$ of the quotient order $\succ$.

The fact that B_{le} satisfies Axiom 2 immediately follows from Lemma 1, for $B(\sqsupseteq) = T$ for any $\succeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(N))$ with the properties considered in Axiom 2.

The fact that B_{le} satisfies Axiom 3 immediately follows from Lemma 1 and Remark 1, for $B(\sqsupseteq) = T = B(\succeq) \setminus L \subseteq N \setminus L$ for any $\succeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ with the properties considered in Axiom 3.

Finally, to see that B_{le} satisfies Axiom 4, consider any power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ with quotient order $\succ$ such that

$$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$$

and $i \in N$ such that $i \notin B(\succeq)$. For any partition $T_1, \dots, T_m$ of Σ_l and for any power relation $\sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ with quotient order $\sqsupset$ such that

$$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset T_1 \sqsupset \cdots \sqsupset T_m,$$

we have that

$$\boldsymbol{\theta}^{\sqsupseteq}(i) = (i_1, \dots, i_{l-1}, i_{T_1}, \dots, i_{T_m}), \tag{5}$$

where $i_{T_p} = |\{S \in T_p : i \in S\}|$ is the number of sets in T_p containing i for $p = 1, \dots, m$.

Since $i \notin B(\succeq)$, by Definition 1 there must exist $j \neq i$ such that $\boldsymbol{\theta}^{\succeq}(j) >_L \boldsymbol{\theta}^{\succeq}(i)$. So, for the sum $\sum_{k=1}^{l} i_k = 2^{n-1}$ for all $i \in N$, there must exist $s \in \{1, \dots, l-1\}$ such that $i_k = j_k$ for all $k \in \{1, \dots, s-1\}$ and $i_s < j_s$. Consequently, by relation (5), $\boldsymbol{\theta}^{\sqsupseteq}(j) >_L \boldsymbol{\theta}^{\sqsupseteq}(i)$; so $i \notin B_{le}(\sqsupseteq)$.

We now prove that if a cscf B satisfies the four axioms then $B = B_{le}$.

Consider a cscf B that satisfies Axioms 1, 2, 3 and 4. We want to prove that for any power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ (and its quotient order $\succ$ such that $\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_l$) it holds that $B(\succeq) = B_{le}(\succeq)$. The proof is by induction to the number l of equivalence classes in the quotient order $\succ$.

If $l = 1$ (therefore the power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ is such that $S \sim T$ for all $S, T \in \mathcal{P}(N)$), by Axiom 1 and the first part of this proof, we have that $B(\succeq) = N = B_{le}(\succeq)$.

Now, let $l \geq 1$ and suppose that the assertion $B(\succeq) = B_{le}(\succeq)$ has been proven for any power relation $\succeq$ such that the quotient order $\succ$ contains precisely l equivalence classes.

Let $\sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ be such that the quotient order $\sqsupset$ is $\Sigma_1 \sqsupset \cdots \sqsupset \Sigma_{l+1}$. Consider a new power relation $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ such that the quotient order $\succ$ is $\Sigma_1 \succ \cdots \succ \Sigma_l \cup \Sigma_{l+1}$, containing precisely l equivalence classes. By Lemma 1 with Σ_l in the role of Σ, we have that

$$B_{le}(\sqsupseteq) = T = \{i \in B_{le}(\succeq) : i_l \geq j_l \ \forall j \in B_{le}(\succeq)\}. \tag{6}$$

Application of the induction hypothesis on $\succeq$ yields $B(\succeq) = B_{le}(\succeq)$.

Now, by Axioms 2 and 3 on B, with Σ_l in the role of Σ, it follows that

$$T \subseteq B(\sqsupseteq) \subseteq N \setminus L, \tag{7}$$

where $L = \{j \in B(\succeq) : \exists i \in B(\succeq) \text{ with } i_l \geq j_l\}$. Then, by Axiom 4, with partition $T_1 = \Sigma_l, T_2 = \Sigma_{l+1}$ of the last equivalence class $\Sigma_l \cup \Sigma_{l+1}$ in $\succ$, we have

$$B(\sqsupseteq) \subseteq B_{le}(\succeq) = L \cup T. \tag{8}$$

Finally, by relations (7) and (8) we have that $T \subseteq B(\sqsupseteq) \subseteq T$. So, by relation (6), we have proven that $B(\sqsupseteq) = T = B_{le}(\sqsupseteq)$.

We now show that Axioms 1, 2, 3 and 4 are logically independent, which means that they are necessary for the axiomatic characterization of the lex-cel cscf provided in Theorem 5 .

Proposition 1. *Axioms 1, 2, 3 and 4 are logically independent.*

Proof. We want to prove that axioms 1, 2, 3 and 4 are necessary in order to uniquely characterize the cscf B_{le}. Therefore, we show that for any combination of three out of the four axioms, a cscf $B : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$, with $N = \{1, \dots, n\}$, that satisfies such three axioms does not necessarily satisfies the fourth one.

Axiom 1 is not satisfied:

Let $i \in N$ and consider a cscf $B^1 : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$ such that $B^1(\succeq) = \{i\}$ for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$.

- Clearly B^1 does not satisfy Axiom 1 if $|N| \geq 2$.
- B^1 trivially satisfies Axiom 2, since $T = \{i\} = B^1(\sqsupseteq)$ for all power relations $\succeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ as in Axiom 2.
- B^1 trivially satisfies Axiom 3, since $L = \emptyset$ and $B^1(\sqsupseteq) = \{i\} \subseteq N$ for all power relations $\succeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ as in Axiom 3.
- B^1 also satisfies Axiom 4, for all $j \in N \setminus \{i\}$, $j \notin B^1(\succeq)$ and $j \notin B^1(\sqsupseteq)$ for all power relations $\succeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ as in Axiom 4.

Axiom 2 is not satisfied:

For any $S \in \mathcal{P}(N)$, let $b(S) = \min\{i \in S\}$ and consider a cscf $B^2 : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$ such that for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$

$$B^2(\succeq) = \begin{cases} N, & \text{if } \succeq = \succeq^0, \\ \{b(B_{le}(\succeq))\}, & \text{otherwise.} \end{cases} \tag{9}$$

where $\succeq^0 \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ is a power relation with quotient order $\succ^0$ having a unique equivalence class $\Sigma_1 = \mathcal{P}(N)$ (all coalitions are indifferent in $\succeq^0$).

- B^2 does not satisfy Axiom 2: take for instance the power relations $\succeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ such that the quotient order $\succ$ is

  $$\Sigma_1 = \mathcal{P}(N)$$

  and the quotient order $\sqsupset$ is

  $$\{12\} \sqsupset \mathcal{P}(N) \setminus \{12\}.$$

  Taking $\Sigma = \{12\}$, with the notations of Axiom 2, we have $T = \{1, 2\}$, but $B^2_{le}(\sqsupseteq) = \{1\}$, which violates the axiom.

- Clearly B^2 satisfies Axiom 1.
- Since B_{le} satisfies Axiom 3, the same holds for B^2, as the set L in Axiom 3 remains precisely the same for both cscfs.
- Finally, since all elements that do not belong to $B_{le}(\succeq)$, $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$, do not neither belong to $B^2(\succeq)$, it follows that B^2 also satisfies Axiom 4.

Axiom 3 is not satisfied

Consider a cscf $B^3 : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$ such that $B^3(\succeq) = N$ for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$.

- B^3 does not satisfy Axiom 3: take for instance two power relations $\succeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ such that the quotient order $\succ$ is

$$\Sigma_1 = \mathcal{P}(N)$$

  and the quotient order $\sqsupset$ is

$$\{1, 12\} \sqsupset \mathcal{P}(N) \setminus \{12, 1\}.$$

  Taking $\Sigma = \{1, 12\}$, with the notation of Axiom 3 we have $L = \{2\}$, but $B^3_{le}(\sqsupseteq) = N \nsubseteq N \setminus \{2\}$, which violates the axiom.
- On the other hand, B^3 clearly satisfies Axiom 1.
- B^3 trivially satisfies Axiom 2, as we have $T \subseteq N$.
- B^3 also satisfies Axiom 4, as the set of elements not in $B^3(\succeq)$ is empty, for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$.

Axiom 4 is not satisfied:

To define the last solution $B^4 : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$, we first need to introduce the notion of dual lex-cel (see [6]). Consider the lexicographic* order $\geq_{L^*}$ among vectors $\mathbf{i}$ and $\mathbf{j}$: $\mathbf{i} \geq_{L^*} \mathbf{j}$ if either $\mathbf{i} = \mathbf{j}$ or there exists t such that $i_t < j_t$ and $i_r = j_r$ for all $r \in \{t+1, \ldots, l\}$. The *dual lex-cel* is the binary relation $R^{\succeq}_{d}$ such that for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ and all $i, j \in N$:

$$i \; R^{\succeq}_{d} \; j \Longleftrightarrow \theta^{\succeq}(i) \quad \geq_{L^*} \quad \theta^{\succeq}(j).$$

Consider the *dual lex-cel* cscf $B^4 : \boldsymbol{\mathcal{R}}(\mathcal{P}(N)) \to \mathcal{P}(N)$ such that for all $\succeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$:

$$B^4(\succeq) = \{i \in N : i \; R^{\succeq}_{d} \; j \;\forall j \in N\}.$$

- B^4 does not satisfy Axiom 4: take for instance two power relations $\succeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$ such that the quotient order $\succ$ is

$$\{1, 12\} \succ \mathcal{P}(N) \setminus \{1, 12\}$$

  and the quotient order $\sqsupset$ is

$$\{1, 12\} \sqsupset \{2, 23\} \sqsupset 13 \sqsupset \mathcal{P}(N) \setminus \{1, 12, 2, 23, 13\}.$$

Notice that the conditions of Axiom 4 for $\succ$ and $\sqsupset$ apply with the partition of the worst equivalence class in $\succ$ into the three disjoint sets $T_1 = \{2, 23\}$, $T_2 = \{13\}$ and $T_3 = \mathcal{P}(N) \setminus (T_1 \cup T_2)$.
However, it easy to check that $1P_d(\succeq)i$ for all $i \in N \setminus \{1\}$, so, $2 \notin B^4(\succeq)$, whereas $2P_d(\sqsupseteq)j$ for all $j \in N \setminus \{2\}$ and so, $2 \in B^4(\sqsupseteq)$.
- We leave to the reader to verify that B^4 satisfies Axioms 1, 2 and 3 by following the same steps as in the proof of Theorem 5 where we proved that B_{le} satisfies those axioms.

We have therefore proven that the cscf B_{le} is the unique solution satisfying the four proposed axioms and that these four axioms are logically independent. So we obtained a full characterization of this method.

As already noticed in [6], the dual lex-cel R_d (cf. to its definition in the proof of Proposition 1) penalizes elements appearing many times in the worst coalitions of a power relation. As a consequence, the dual lex-cel cscf B^4 shows a similar behaviour for the selection of the most relevant elements, as illustrated in the following example.

Example 7. Consider again the power relation $\succeq$ of Example 1. It is easy to verify that the dual lex-cel ranking $R_d^{\succeq}$ is such that $1P_d^{\succeq}2P_d^{\succeq}3$. So, the corresponding dual lex-cel cscf B^4 yields the most relevant element $B^4(\succeq) = \{1\}$. In contrast with the lex-cel cscf, which rewards the excellence of element 2 as a member of the best coalitions (specifically, coalitions 23 and 12) according to the power relation $\succeq$, the dual lex-cel csfc punishes the presence of element 2 in the worst coalitions (specifically, coalitions 123 and 2) according to $\succeq$.

5 Conclusion and Future Work

There are many situations where sets of elements are compared; for instance, in sports, in job performance evaluations, etc. A common task in these situations is to reason about the elements in order to assess the performance; typically one might want to compare elements based on their performance when part of different teams. An approach that has been recently proposed is the *lexicographic excellence (lex-cel)* method [6], that provides a full ranking of the elements given the full ranking of all possible coalitions (called power relation). The peculiar characteristic of the lex-cel is its qualitative nature and the fact that is a relatively simple rule to use.

In this work we addressed the problem of determining the most relevant element(s) given in input the ranking over coalitions. We define a coalitional social choice function (cscf) as a mapping from a power relation to a (typically small) subset of "winners". We adopted an axiomatic approach and stated four axioms that we believe constitute reasonable behaviors of the desired cscf. We then showed that the cscf which returns the first element(s) in the ranking obtained by lex-cel is the only cscf compatible with the four axioms. So we proposed the first, as far as we know, method for determining the most relevant elements given a ranking of coalitions, as well as its corresponding characterization in terms of four intuitive axioms. Although there are similarities between some of the axioms introduced in this paper and those for social rankings introduced in [6], we would like to point out some important differences suggesting a novel interpretation of the lex-cel ranking when it is applied to select the most relevant elements from a power relation. First, our axiomatic characterization of the lex-cel cscf does not use any property directly related to the *Coalitional Anonymity* axiom proposed in [6], which is a strong property requiring that the relative ranking of two individual elements should exclusively depend on the relative positions of groups containing just one of them and disregarding the structure or the size of those groups. Second, our axioms highlight the fundamental mechanism driving the selection of the most relevant elements, which is essentially based on changes (i.e., improvements or partitioning) in the structure of the worst equivalence class of a power relation (in [6], only the *Independence from the Worst Set* axiom concerned the worst equivalence class). In conclusion, we believe that the set of (easy to understand) axioms used in this paper reveals in an explicit way the crucial role of the worst equivalence class in determining the most relevant elements according to the lex-cel method.

Concerning future works, we believe that an important direction is to deal with situations where there is uncertainty about the preference order over the coalitions. Preference uncertainty can arise because of a lack of information on how certain coalitions compare, or because of the user's reluctance to rank an exponential number of coalitions (due to cognitive cost or time constraints).

Indeed, the main obstacle to the adoption of lex-cel lies in the exponential size of its input (the exponential number of elements in a power relation). Therefore, we think it is worth studying the application of the lex-cel (and as well of similar methods) when a partial order of the coalitions is given as an input. This partial order can be interpreted in the sense of "strict uncertainty" [10]: we can assume that there exists a "true" complete ranking $\succeq^* \in \boldsymbol{\mathcal{R}}(\mathcal{P}(N))$, but that it is unknown to us and only a reduction of the ranking is given. We can then design methods to reason about all possible completion of the partial ranking given as input, and view these methods as coalitional social choice functions taking *partial* orders as input. Considering the running example dealing with the determination of the most influential scientists with an AA, this could lead to simplification of the process of comparison of collaborations: for instance, it could be possible to have m sub-committees tasked with ranking certain subsets of $\mathcal{P}(N)$, *i.e.* the work of comparing all potential collaborations is divided between these m committees. From their conclusions, the application of a cscf equipped to work with missing information will be used to determine the ranking over the researchers' influence. In this way, it will be possible to deal with real-life situations, where a complete order is seldom given.

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