On Iterated Contraction: Syntactic Characterization, Representation Theorem and Limitations of the Levi Identity

Sébastien Konieczny[1(✉)] and Ramón Pino Pérez[2]

[1] CRIL - CNRS, Université d'Artois, Lens, France
konieczny@cril.fr
[2] Facultad de Ciencias, Universidad de Los Andes, Mérida, Venezuela
pino@ula.ve

Abstract. In this paper we study iterated contraction in the epistemic state framework, offering a counterpart of the work of Darwiche and Pearl for iterated revision. We provide pure syntactical postulates for iterated contraction, that is, the postulates are expressed only in terms of the contraction operator. We establish a representation theorem for these operators. Our results allow to highlight the relationships between iterated contraction and iterated revision. In particular we show that iterated revision operators form a larger class than that of iterated contraction operators. As a consequence of this, in the epistemic state framework, the Levi identity has limitations; namely, it doesn't allow to define all iterated revision operators.

1 Introduction

Belief change theory [1,7–9,11,12] aims at modelling the evolution of the logical beliefs of an agent according to new inputs the agent receives.

The two main classes of operators are revision operators, which allow to correct some wrong beliefs of the agent, and contraction operators, which allow to remove some pieces of beliefs from the beliefs of the agent.

Contraction and revision, though being different processes, are closely linked. Two *identities* allow to define contraction from revision and vice-versa. One can define a revision operator from a contraction operator by the *Levi identity*, which states that, in order to define a revision by α, one can first perform a contraction by $\neg\alpha$ and then an expansion[1] by α [15]. Conversely, one can define a contraction operator from a revision operator by using the *Harper identity*: what is true after contraction by α is what is true in the current state and in the result of the revision by $\neg\alpha$ [10]. To give the formal definition of these identities, let us denote by K a theory (a deductively closed set of logical sentences), and let α be a formula. Let us denote $\star$ a revision operator, $\div$ a contraction operator, and $\oplus$ the expansion:

[1] See [8] for exact definition, but one can safely identify expansion with conjunction/union in most cases.

S. Moral et al. (Eds.): SUM 2017, LNAI 10564, pp. 348–362, 2017.
DOI: 10.1007/978-3-319-67582-4_25

Levi identity	$K \star \alpha = (K \div \neg\alpha) \oplus \alpha$
Harper identity	$K \div \alpha = K \cap (K \star \neg\alpha)$

The connection obtained through these identities is very strong, since one obtains in fact a bijection between the set of revision operators and the set of contraction operators [8]. So, in the AGM framework these two classes of operators are two sides of the same coin, and one can study either revision or contraction, depending on which operator is chosen as more basic/natural.

Although intrinsically a dynamic process, initial works on belief change only address the (static) one-step change [1,8,11], and were not able to cope with iterated change.

After many unsuccessful attempts, a solution for modelling iterated revision was provided by Darwiche and Pearl [6]. They provide additional postulates to govern iterated change. These additional constraints for iteration cannot use simple logical theories for belief representation, as used in the one-step case. One has to shift to a more powerful representation, epistemic states, which allow to encode the revision strategy of the agent, and allow to ensure dynamic coherence of changes.

It is interesting to note that the work of Darwiche and Pearl was dedicated to iterated revision. One could expect that a characterization of iterated contraction could be obtained safely from generalizations of the identities. In fact this is not the case. First, until now there is no proposal for postulates nor representation theorem for iterated contraction. There were some works on iterated contraction that we will discuss in the related work section, but no real counterpart of the work of Darwiche and Pearl for contraction existed so far. This is what we propose in this paper.

The class of iterated contraction operators obtained is very interesting in different respects. It allows to obtain a better understanding of belief change theory. Actually, some of the consequences of our representation theorem are quite surprising.

First, when comparing the two representation theorems (the one for iterated contraction and the one for iterated revision), it is clear that they share the same class of (faithful) assignments. This means that the difference between iterated contraction and iterated revision is not a matter of nature, but a matter of degree (revision being a bigger change than contraction, not a different kind of change). This also means that there is a deep relationship between iterated contraction and iterated revision via the representation theorems and the assignments. One could think that the generalization of the Levi and Harper identities could be easily attained thanks to these theorems. Nevertheless, it is not the case.

More surprisingly, we prove that there are more iterated revision operators than contraction operators (oppositely to the bijection obtained in the classical - AGM - framework). As a consequence, one can not expect to have generalizations of the Levi identity for the iterated case: some iterated revision operators are out of reach from iterated contraction ones. This seems to suggest that iterated belief revision operators are more basic than iterated belief contraction ones.

In the next Section we will provide the formal preliminaries for this paper. In Sect. 3 we give the logical postulates for modelling iterated contraction. In Sect. 4 we provide a representation theorem for these contractions. In Sect. 5 we study the links between iterated contraction and iterated revision. In Sect. 6 we discuss some related works and we conclude in Sect. 7.

2 Preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the standard connectives. Let $\mathcal{L}^*$ denote the set of consistent formulae of $\mathcal{L}$.

An interpretation ω is a total function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is denoted by $\mathcal{W}$. An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[\![\alpha]\!]$ denotes the set of models of the formula α, i.e., $[\![\alpha]\!] = \{\omega \in \mathcal{W} \mid \omega \models \alpha\}$. $\vdash$ denotes implication between formulae, i.e. $\alpha \vdash \beta$ means $[\![\alpha]\!] \subseteq [\![\beta]\!]$.

$\leq$ denotes a pre-order on $\mathcal{W}$ (i.e., a reflexive and transitive relation), and $<$ denotes the associated strict order defined by $\omega < \omega'$ if and only if $\omega \leq \omega'$ and $\omega' \not\leq \omega$. A pre-order is *total* if for all $\omega, \omega' \in \mathcal{W}$, $\omega \leq \omega'$ or $\omega' \leq \omega$. If $A \subseteq \mathcal{W}$, then the set of minimal elements of A with respect to a total pre-order $\leq$, denoted by $\min(A, \leq)$, is defined by $\min(A, \leq) = \{\omega \in A \mid \nexists \omega' \in A \text{ such that } \omega' < \omega\}$.

We will use epistemic states to represent the beliefs of the agent, as usual in iterated belief revision [6]. An epistemic state Ψ represents the current beliefs of the agent, but also additional conditional information guiding the revision process (usually represented by a pre-order on interpretations, a set of conditionals, a sequence of formulae, etc.). Let $\mathcal{E}$ denote the set of all epistemic states. A projection function $B : \mathcal{E} \longrightarrow \mathcal{L}^*$ associates to each epistemic state Ψ a consistent formula $B(\Psi)$, that represents the current beliefs of the agent in the epistemic state Ψ. We will call models of the epistemic state the models of its beliefs, i.e. $[\![\Psi]\!] = [\![B(\Psi)]\!]$.

A concrete and very useful representation of epistemic states are total pre-orders over interpretations. In this representation, if $\Psi =\leq$, $B(\Psi)$ is a propositional formula which satisfies $[\![B(\Psi)]\!] = \min(\mathcal{W}, \leq)$. We call this concrete representation of epistemic states the *canonical representation*.

For simplicity purpose we will only consider in this paper consistent epistemic states and consistent new information. Thus, we consider change operators as functions $\circ$ mapping an epistemic state and a consistent formula into a new epistemic state, i.e. in symbols, $\circ : \mathcal{E} \times \mathcal{L}^* \longrightarrow \mathcal{E}$. The image of a pair (Ψ, α) under $\circ$ will be denoted by $\Psi \circ \alpha$.

Let us now recall Darwiche and Pearl proposal for iterated revision [6]. Darwiche and Pearl modified the list of KM postulates [11] to work in the more general framework of epistemic states:

(R*1) $B(\Psi \star \alpha) \vdash \alpha$
(R*2) If $B(\Psi) \wedge \alpha \nvdash \bot$ then $B(\Psi \star \alpha) \equiv \varphi \wedge \alpha$
(R*3) If $\alpha \nvdash \bot$ then $B(\Psi \star \alpha) \nvdash \bot$

(R*4) If $\Psi_1 = \Psi_2$ and $\alpha_1 \equiv \alpha_2$ then $B(\Psi_1 \star \alpha_1) \equiv B(\Psi_2 \star \alpha_2)$
(R*5) $B(\Psi \star \alpha) \wedge \psi \vdash B(\Psi \star (\alpha \wedge \psi))$
(R*6) If $B(\Psi \star \alpha) \wedge \psi \nvdash \bot$ then $B(\Psi \star (\alpha \wedge \psi)) \vdash B(\Psi \star \alpha) \wedge \psi$

For the most part, the DP list is obtained from the KM list by replacing each φ by $B(\Psi)$ and each $\varphi \star \alpha$ by $B(\Psi \star \alpha)$. The only exception to this is (R*4), which is stronger than its simple translation.

In addition to this set of basic postulates, Darwiche and Pearl proposed a set of postulates devoted to iteration:

(DP1) If $\alpha \vdash \mu$ then $B((\Psi \star \mu) \star \alpha) \equiv B(\Psi \star \alpha)$
(DP2) If $\alpha \vdash \neg\mu$ then $B((\Psi \star \mu) \star \alpha) \equiv B(\Psi \star \alpha)$
(DP3) If $B((\Psi \star \alpha) \vdash \mu$ then $B((\Psi \star \mu) \star \alpha) \vdash \mu$
(DP4) If $B((\Psi \star \alpha) \nvdash \neg\mu$ then $B((\Psi \star \mu) \star \alpha) \nvdash \neg\mu$

And then they give a representation theorem in terms of pre-orders on interpretations:

Definition 1. *A faithful assignment is a mapping that associates to any epistemic state Ψ a total pre-order $\leq_\Psi$ on interpretations such that:*

1. *If $\omega \models B(\Psi)$ and $\omega' \models B(\Psi)$, then $\omega \simeq_\Psi \omega'$*
2. *If $\omega \models B(\Psi)$ and $\omega' \not\models B(\Psi)$, then $\omega <_\Psi \omega'$*
3. *If $\Psi_1 = \Psi_2$, then $\leq_{\Psi_1} = \leq_{\Psi_2}$*

Theorem 1 ([6]). *An operator $\star$ satisfies (R*1)–(R*6) if and only if there is a faithful assignment that maps each epistemic state Ψ to a total pre-order on interpretations $\leq_\Psi$ such that:*

$$[\![\Psi \star \mu]\!] = \min([\![\mu]\!], \leq_\Psi)$$

Theorem 2 ([6]). *Let $\star$ be a revision operator that satisfies (R*1)–(R*6). This operator satisfies (DP1)–(DP4) if and only if this operator and its faithful assignment satisfies:*

(CR1) *If $\omega \models \mu$ and $\omega' \models \mu$, then $\omega \leq_\Psi \omega' \Leftrightarrow \omega \leq_{\Psi\star\mu} \omega'$*
(CR2) *If $\omega \models \neg\mu$ and $\omega' \models \neg\mu$, then $\omega \leq_\Psi \omega' \Leftrightarrow \omega \leq_{\Psi\star\mu} \omega'$*
(CR3) *If $\omega \models \mu$ and $\omega' \models \neg\mu$, then $\omega <_\Psi \omega' \Rightarrow \omega <_{\Psi\star\mu} \omega'$*
(CR4) *If $\omega \models \mu$ and $\omega' \models \neg\mu$, then $\omega \leq_\Psi \omega' \Rightarrow \omega \leq_{\Psi\star\mu} \omega'$*

The first aim of this paper is to provide a similar direct characterization of iterated contraction. Then we will study the links between iterated revision and iterated contraction operators.

3 Iterated Contraction

Let us first give the basic postulates for contraction of epistemic states. We use the contraction postulates given for propositional logic formulas in [4], that are equivalent to the original AGM ones [1], that we only adapt for epistemic states:

(C1) $B(\Psi) \vdash B(\Psi - \alpha)$
(C2) If $B(\Psi) \nvdash \alpha$, then $B(\Psi - \alpha) \vdash B(\Psi)$
(C3) If $B(\Psi - \alpha) \vdash \alpha$, then $\vdash \alpha$
(C4) $B(\Psi - \alpha) \wedge \alpha \vdash B(\Psi)$
(C5) If $\alpha_1 \equiv \alpha_2$ then $B(\Psi - \alpha_1) \equiv B(\Psi - \alpha_2)$
(C6) $B(\Psi - (\alpha \wedge \beta)) \vdash B(\Psi - \alpha) \vee B(\Psi - \beta)$
(C7) If $B(\Psi - (\alpha \wedge \beta)) \nvdash \alpha$, then $B(\Psi - \alpha) \vdash B(\Psi - (\alpha \wedge \beta))$

(C1) states that contraction can just remove some information, so the beliefs of the posterior epistemic state are weaker than the beliefs of the prior one. (C2) says that if the epistemic state does not imply the formula by which one wants to contract, then the posterior epistemic state will have the same beliefs as the prior one. (C3) is the success postulate, it states that the only case where contraction fails to remove a formula from the beliefs of the agent is when this formula is a tautology. (C4) is the recovery postulate, it states that if we do the contraction by a formula followed by a conjunction by this formula, then we will recover the initial beliefs. This ensures that no unnecessary information is discarded during the contraction. (C5) is the irrelevance of syntax postulate, that says that the syntax does not have any impact on the result of the contraction. (C6) says that the contraction by a conjunction implies the disjunction of the contractions by the conjuncts. (C7) says that if α is not removed during the contraction by the conjunction by $\alpha \wedge \beta$, then the contraction by α implies the contraction by the conjunction.

Now let us introduce the postulates for iterated contraction:

(C8) If $\neg\alpha \vdash \gamma$ then $B(\Psi - (\alpha \vee \beta)) \vdash B(\Psi - \alpha) \Leftrightarrow B(\Psi - \gamma - (\alpha \vee \beta)) \vdash B(\Psi - \gamma - \alpha)$
(C9) If $\gamma \vdash \alpha$ then $B(\Psi - (\alpha \vee \beta)) \vdash B(\Psi - \alpha) \Leftrightarrow B(\Psi - \gamma - (\alpha \vee \beta)) \vdash B(\Psi - \gamma - \alpha)$
(C10) If $\neg\beta \vdash \gamma$ then $B(\Psi - \gamma - (\alpha \vee \beta)) \vdash B(\Psi - \gamma - \alpha) \Rightarrow B(\Psi - (\alpha \vee \beta)) \vdash B(\Psi - \alpha)$
(C11) If $\gamma \vdash \beta$ then $B(\Psi - \gamma - (\alpha \vee \beta)) \vdash B(\Psi - \gamma - \alpha) \Rightarrow B(\Psi - (\alpha \vee \beta)) \vdash B(\Psi - \alpha)$

(C8) expresses the fact that if a contraction by a disjunction implies the contraction by one of the disjuncts, then it will be the same if we first contract by a formula that is a consequence of the negation of that disjunct. (C9) captures the fact that if a contraction by a disjunction implies the contraction by one of the disjuncts, then it will be the same if we first contract by a formula that implies this disjunct. (C10) expresses the fact that if a contraction by a disjunction implies the contraction by one of the disjunct after a contraction by a formula that is a consequence of the negation of the other disjunct, then it was already the case before this contraction. (C11) captures the fact that if a contraction by a disjunction implies the contraction by one of the disjunct after a contraction by a formula that implies the other disjunct, then it was already the case before this contraction.

The operators satisfying (C1)–(C7) will be called contraction operators. The operators satisfying (C1)–(C11) will be called iterated contraction operators.

4 Representation Theorem

Let us now provide a representation theorem for iterated contraction operators in terms of faithful assignments. These assignments associate to each epistemic state a total pre-order on interpretations, this total pre-order representing the relative plausibility of each interpretation, and the current beliefs of the agent being the most plausible ones.

And let us now state the basic theorem for contraction postulates in the epistemic state framework.

Theorem 3. *An operator $-$ satisfies the postulates **(C1)–(C7)** if and only if there exists a faithful assignment that associates to each epistemic state Ψ a total pre-order $\leq_\Psi$ on interpretations such that*

$$[\![\Psi - \alpha]\!] = [\![\Psi]\!] \cup \min([\![\neg\alpha]\!], \leq_\Psi)$$

We will say that the faithful assignment given by the previous theorem *represents* the operator $-$. The proof of this theorem follows the same lines as the proof for the representation theorem in the propositional case [4]. For space reason it will not be included here. We concentrate our effort in proving a representation theorem for iterated contraction operators.

Let us now state the representation theorem for iterated contraction.

Theorem 4. *Let $-$ be a contraction operator that satisfies **(C1)–(C7)**. This operator satisfies **(C8)–(C11)** if and only if this operator and its faithful assignment satisfies:*

4. If $\omega, \omega' \in [\![\gamma]\!]$ then $\omega \leq_\Psi \omega' \Leftrightarrow \omega \leq_{\Psi-\gamma} \omega'$
5. If $\omega, \omega' \in [\![\neg\gamma]\!]$ then $\omega \leq_\Psi \omega' \Leftrightarrow \omega \leq_{\Psi-\gamma} \omega'$
6. If $\omega \in [\![\neg\gamma]\!]$ and $\omega' \in [\![\gamma]\!]$ then $\omega <_\Psi \omega' \Rightarrow \omega <_{\Psi-\gamma} \omega'$
7. If $\omega \in [\![\neg\gamma]\!]$ and $\omega' \in [\![\gamma]\!]$ then $\omega \leq_\Psi \omega' \Rightarrow \omega \leq_{\Psi-\gamma} \omega'$

We will call a faithful assignment that satisfies properties 4 *to* 7 *an iterated faithful assignment.*

Condition 4 captures the fact that the plausibility between the models of γ is exactly the same before the contraction and after the contraction by γ. Condition 5 captures the fact that the plausibility between the models of $\neg\gamma$ is exactly the same before the contraction and after the contraction by γ. Conditions 4 and 5 will be called *rigidity conditions* (called also ordered preservation conditions in [18]). Condition 6 captures the fact that if a model of $\neg\gamma$ is strictly more plausible than a model of γ before the contraction by γ then it will be the case after the contraction by γ. Condition 7 captures the fact that the plausibility of the models of $\neg\gamma$ does not decrease with respect the models of γ after contraction by γ. More precisely, if a model of $\neg\gamma$ is at least as plausible as a model of γ before the contraction it will be the case after the contraction by γ. Conditions 6 and 7 will be called *non-worsening conditions*.

Please note that the iterated faithful assignments are directly related to the ones for iterated revision (cf. Theorem 2), there are only some inversions in conditions 6 and 7 (compared to CR3 and CR4), that are due to the fact that one can see a contraction by $\neg\alpha$ as a softer change (improvement [14]) than a revision by α (see discussion at the beginning of Sect. 6).

For space reasons we only give a sketch of the proof of Theorem 4. An iterated contraction operator is indeed a contraction operator. Thus, we know by Theorem 3 that there is a faithful assignment representing it. Thus the proof of Theorem 4 will consist in proving that under the assumption of the basic contraction postulates, the iteration postulates entail conditions 4–7 for the faithful assignment representing the operator. And reciprocally, that if the iterated faithful assignment represents the operator the postulates of the iteration are satisfied. Thus, from now on, in this section we suppose that $-$ is a contraction operator and $\Psi \mapsto \leq_\Psi$ is the faithful assignment representing it, that is the equation in Theorem 3 holds. Actually, we prove the following facts which are enough to conclude:

(i) The assignment satisfies condition 4 if and only if postulate (C8) holds.
(ii) The assignment satisfies condition 5 if and only if postulate (C9) holds.
(iii) Suppose the contraction operator satisfies (C8). Then the assignment satisfies condition 6 if and only if postulate (C10) holds.
(iv) The assignment satisfies condition 7 if and only if postulate (C11) holds.

In order to give a flavor of the whole proof we give the proof of the Fact (iii) (which is perhaps a little more complicated than the other three facts). First we have the following observations:

Observation 1. *Suppose that the assignment satisfies condition 4, then for any μ such that $\mu \vdash \gamma$ we have* $\min([\![\mu]\!], \leq_\Psi) = \min([\![\mu]\!], \leq_{\Psi-\gamma})$.

Observation 2. *Condition 6 is equivalent to the following condition:*

6'. If $\omega \in [\![\neg\gamma]\!]$ and $\omega' \in [\![\gamma]\!]$ then $\omega' \leq_{\Psi-\gamma} \omega \Rightarrow \omega' \leq_\Psi \omega$

Now we proceed to prove (iii). Note that, since α, β and γ are any formulas, using (C5), the postulate (C10) can be rewritten as follows:

If $\beta \vdash \gamma$ then $B((\Psi - \gamma) - \neg(\alpha \wedge \beta)) \vdash B((\Psi - \gamma) - \neg\alpha) \Rightarrow B(\Psi - \neg(\alpha \wedge \beta)) \vdash B(\Psi - \neg\alpha)$

First we prove that postulate (C10) entails condition 6 of an iterated assignment. By Observation 2, it is enough to prove that (C10) entails 6'. Thus, assume (C10) holds. Suppose $\omega \in [\![\neg\gamma]\!]$, $\omega' \in [\![\gamma]\!]$ and $\omega' \leq_{\Psi-\gamma} \omega$. We want to show that $\omega' \leq_\Psi \omega$.

Let α and β be formulas such that $[\![\alpha]\!] = \{\omega, \omega'\}$ and $[\![\beta]\!] = \{\omega'\}$. Note that $\{\omega'\} \subseteq \min([\![\alpha]\!], \leq_{\Psi-\gamma})$ because $\omega' \leq_{\Psi-\gamma} \omega$. We have also $\min([\![\alpha \wedge \beta]\!], \leq_{\Psi-\gamma}) = \{\omega'\}$.

Then, we have $[\![(\Psi - \gamma) - \neg(\alpha \wedge \beta)]\!] = [\![\Psi - \gamma]\!] \cup \min([\![\alpha \wedge \beta]\!], \leq_{\Psi-\gamma})$; the last expression is equal to $[\![\Psi - \gamma]\!] \cup \{\omega'\}$ which is a subset of $[\![\Psi - \gamma]\!] \cup \min([\![\alpha]\!], \leq_{\Psi-\gamma})$. But this last expression is $[\![(\Psi - \gamma) - \neg\alpha]\!]$. So, $[\![(\Psi - \gamma) - \neg(\alpha \wedge \beta)]\!] \subseteq [\![(\Psi - \gamma) - \neg\alpha]\!]$, that is $B((\Psi - \gamma) - \neg(\alpha \wedge \beta)) \vdash B((\Psi - \gamma) - \neg\alpha)$. Then, by (C10), we have $B(\Psi - \neg(\alpha \wedge \beta)) \vdash B(\Psi - \neg\alpha)$, that is $[\![\Psi]\!] \cup \{\omega'\} \subseteq [\![\Psi]\!] \cup \min(\{\omega, \omega'\}, \leq_\Psi)$. Therefore, $\omega' \in [\![\Psi]\!]$ or $\omega' \in \min(\{\omega, \omega'\}, \leq_\Psi)$. In both cases we get $\omega' \leq_\Psi \omega$ (In the first case we use the fact that $\leq_\Psi$ is a faithful assignment, in particular $[\![\Psi]\!] = \min(\mathcal{W}, \leq_\Psi)$).

Now we prove that Condition 6 entails Postulate (C10). Assume that $\beta \vdash \gamma$. We suppose that $\alpha \wedge \beta \nvdash \perp$ (the other case is trivial because the contraction by a tautology doesn't change the beliefs). Suppose $B((\Psi - \gamma) - \neg(\alpha \wedge \beta)) \vdash B((\Psi - \gamma) - \neg\alpha)$, that is

$$[\![(\Psi - \gamma) - \neg(\alpha \wedge \beta)]\!] \subseteq [\![(\Psi - \gamma) - \neg\alpha]\!] \tag{1}$$

We want to show that $B(\Psi - \neg(\alpha \wedge \beta)) \vdash B(\Psi - \neg\alpha)$, that is to say

$$[\![\Psi - \neg(\alpha \wedge \beta)]\!] \subseteq [\![\Psi - \neg\alpha]\!] \tag{2}$$

By Theorem 3, Eqs. (1) and (2) can be rewritten, respectively, as

$$[\![\Psi - \gamma]\!] \cup \min([\![\alpha \wedge \beta]\!], \leq_{\Psi-\gamma}) \subseteq [\![\Psi - \gamma]\!] \cup \min([\![\alpha]\!], \leq_{\Psi-\gamma}) \tag{3}$$

and

$$[\![\Psi]\!] \cup \min([\![\alpha \wedge \beta]\!], \leq_\Psi) \subseteq [\![\Psi]\!] \cup \min([\![\alpha]\!], \leq_\Psi) \tag{4}$$

First we are going to prove that (3) entails

$$\min([\![\alpha \wedge \beta]\!], \leq_{\Psi-\gamma}) \subseteq \min([\![\alpha]\!], \leq_{\Psi-\gamma}) \tag{5}$$

In order to see that, take $\omega \in \min([\![\alpha \wedge \beta]\!], \leq_{\Psi-\gamma})$. Then, by Eq. (3), we have either $\omega \in [\![\Psi - \gamma]\!]$ or $\omega \in \min([\![\alpha]\!], \leq_{\Psi-\gamma})$. In the first case, by the fact of having a faithful assignment, $\omega \in \min(\mathcal{W}, \leq_{\Psi-\gamma})$ and therefore $\omega \in \min([\![\alpha]\!], \leq_{\Psi-\gamma})$. In the second case is trivial. Thus, in any case we have 5.

Now, towards a contradiction, suppose that (4) doesn't hold. Thus, there exists $\omega \in \min([\![\alpha \wedge \beta]\!], \leq_\Psi)$ such that $\omega \notin \min([\![\alpha]\!], \leq_\Psi)$. So there is $\omega' \in \min([\![\alpha]\!], \leq_\Psi)$ such that

$$\omega' <_\Psi \omega \tag{6}$$

Note that $\omega \models \beta$, and, by hypothesis, $\beta \vdash \gamma$, thus $\omega \models \gamma$. We are going to consider the following two cases: $\omega' \models \gamma$ and $\omega' \models \neg\gamma$

Suppose we are in the first case, *i.e.* $\omega' \models \gamma$. Then, by Condition 4 (equivalent to our assumption of Postulate (C8)), $\omega' <_{\Psi-\gamma} \omega$. Now, suppose we are in the second case, *i.e.* $\omega' \models \neg\gamma$. Then, because $\omega \models \gamma$ and (6), by Condition 6, $\omega' <_{\Psi-\gamma} \omega$. Thus in any case we have

$$\omega' <_{\Psi-\gamma} \omega \tag{7}$$

Since $\alpha \wedge \beta \vdash \gamma$, by Observation 1, $\omega \in \min([\![\alpha \wedge \beta]\!], \leq_{\Psi-\gamma})$. Then, by (5), $\omega \in \min([\![\alpha]\!], \leq_{\Psi-\gamma})$. But this is a contradiction with (7).

5 Iterated Contraction Vs Iterated Revision

We would like to investigate now the relationship between iterated contraction and iterated revision.

A natural tendency would be to try to generalize Levi and Harper Identity to the iterated case. In fact some related works followed this path [2,5,17].

In the following we will first argue and show that it is not so simple. We will also show that there are some problems when one follows this way for connecting iterated contraction and iterated revision. Actually, we will show that in the iterated case, they are not two sides of a same coin (i.e. two classes of operators linked by a bijection), but that they rather are two instances of a same kind of change operators, and that the link and difference is just a matter of degree of change.

5.1 Identities in the General Case

Let us first recall the Levy and Harper Identities:

Levi identity	$\Psi \star \alpha = (\Psi \div \neg\alpha) \oplus \alpha$
Harper identity	$\Psi \div \alpha = \Psi \sqcap (\Psi \star \neg\alpha)$

Let us note the problems of using these identities for iterated contraction and revision in the epistemic state framework. First, in the AGM case, these two identities are definitional, that means that, for instance, using Levi identity one can obtain the revision operator $\star$ that defines the theory $\Psi \star \alpha$ from the right side of the identity using the contraction and expansion operators. But in our general framework, epistemic states are abstract objects, which can only be apprehended at the logical level from the projection function B. Thus, in this general framework, we do not fully know what $\Psi \div \neg\alpha$ is, and so we can not use it to define what should be $\Psi \star \alpha$. So using a definitional equality here is difficult. The only way to proceed seems to be choosing a particular representation of epistemic states to work with (such as total preorders over the interpretations), but then the results are given on this representation and not in the general case. Second, whereas $\oplus$ and $\sqcap$ have a clear meaning in the AGM framework, one has to figure out a definition in the epistemic state framework. That is by itself a non-trivial task (studying the possible definitions of $\sqcap$ is one of the main aims of [2]).

A possibility would be to restrict these identities to the beliefs of the epistemic states, as:

Belief Levi equivalence	$B(\Psi \star \alpha) \equiv B(\Psi \div \neg\alpha) \oplus \alpha$
Belief Harper equivalence	$B(\Psi \div \alpha) \equiv B(\Psi) \vee B(\Psi \star \neg\alpha)$

But we do not have identities anymore, but only equivalences that are not definitional. So one has to first identify two operators $\star$ and $\div$ and check that they are linked through these equivalences.

5.2 Identities Under the Canonical Representation

So to go further we have to commit to a particular representation of epistemic states. The canonical one, using total preorders (described in Sect. 2), can be used together with the faithful assignment, to define completely the new epistemic state after contraction (or revision). Thus, suppose that we have a faithful assignment $\Psi \mapsto \leq_\Psi$. We identify Ψ with $\leq_\Psi$, and we define $\leq_{\Psi-\gamma}$ satisfying the properties (4–7) of Theorem 4. Then, by Theorem 4, the operator defined by $\Psi - \gamma = \leq_{\Psi-\gamma}$ is an iterated contraction operator. In such a case we say that the operator $-$ is *given by* the assignment. We can proceed, in the same way when the assignment satisfies the requirements of an iterated assignment (for revision [6]) and then the operator defined by $\Psi \star \alpha = \leq_{\Psi\star\alpha}$ is an iterated revision operator.

Thus, one can restate the identities on the total pre-orders associated to the epistemic states by the operators[2]:

Tpo Levi identity	$\leq_{\Psi\star\alpha} = \leq_{\Psi\div\neg\alpha} \oplus \alpha$
Tpo Harper identity	$\leq_{\Psi\div\alpha} = \leq_\Psi \sqcap \leq_{\Psi\star\neg\alpha}$

So now we can define these pre-orders using the identities and check that we correctly obtain operators with the expected properties. The only remaining problem is to define the operators $\oplus$ and $\sqcap$ in this setting.

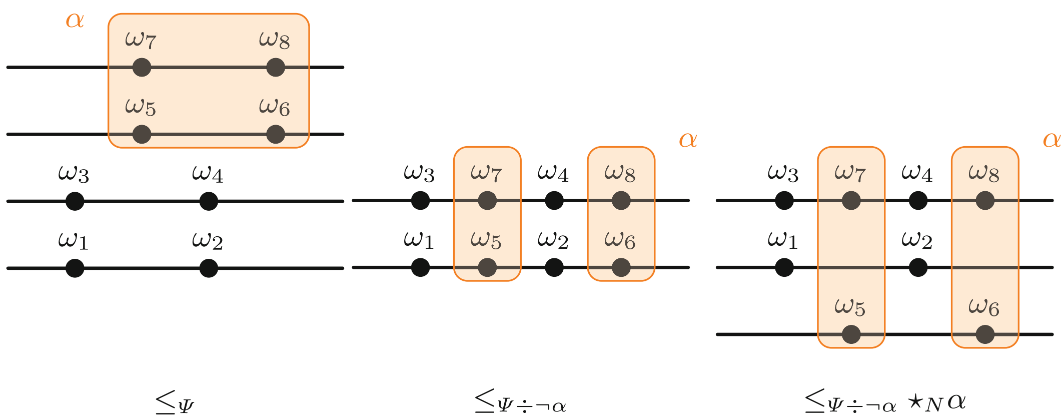

Fig. 1. From contraction to revision

As for $\oplus$ let us show that using Boutilier natural revision operator $\star_N$ [3] is a correct option, in the sense that using this operator as $\oplus$ we obtain a DP (Darwiche and Pearl [6]) revision operator (see Proposition 1).

Let us recall the definition of this operator on total pre-orders, that amounts to look at the most plausible models of the new piece of information and define them as the new most plausible interpretations while nothing else changes: Let $\leq_\Psi$ be the pre-order associated to the epistemic state Ψ by the faithful assignment, and let α be the new piece of information, then $\leq_{\Psi\star_N\alpha}$ (we will also use the equivalent notation $\leq_\Psi \star_N \alpha$) is defined as:

[2] Tpo means Total pre-order.

- If $\omega \models \min(\llbracket\alpha\rrbracket, \leq_\Psi)$ and $\omega' \not\models \min(\llbracket\alpha\rrbracket, \leq_\Psi)$, then $\omega <_{\Psi\star_N\alpha} \omega'$
- In all the other cases $\omega \leq_{\Psi\star_N\alpha} \omega'$ iff $\omega \leq_\Psi \omega'$

Then one can show that the Tpo Levi identity holds for the iterated case:

Proposition 1. *Let $\div$ be an iterated contraction operator given by its assignment $\Psi \mapsto \leq_\Psi$. Then the assignment defined by $\leq_{\Psi\star\alpha} = \leq_{\Psi\div\neg\alpha} \star_N \alpha$ satisfies properties (CR1)–(CR4), and can be used to define a Darwiche and Pearl iterated revision operator in the framework of the canonical representation of epistemic states.*

This proposition implies in particular that to each iterated contraction operator one can associate a corresponding iterated revision operator. So, this means that the cardinality of the class of iterated revision operators obtained via the Tpo Levi identity is at least equal to the cardinality of the class of iterated contraction operators. Note that this observation does not depend on the interpretation of the symbol $\oplus$ utilized.

The following example illustrates the use of our concrete Tpo Levi identity.

Example 1. Let us consider the total pre-order $\leq_\Psi$ represented in Fig. 1. So in that figure $\llbracket\leq_\Psi\rrbracket = \{\omega_1, \omega_2\}$ and $\llbracket\alpha\rrbracket = \{\omega_5, \omega_6, \omega_7, \omega_8\}$. In this Figure the lower an interpretation is, the more plausible it is. For instance in $\leq_\Psi$ we have that $\omega_1 <_\Psi \omega_3$. An iterated contraction by $\neg\alpha$ is a change that increases (improves) the plausibility of the models of α, with the condition that the most plausible models of α in $\leq_\Psi$ joins (become as plausible as) the most plausible models of $\leq_\Psi$ (We give on such possibility for $\leq_{\Psi\div\neg\alpha}$). The relation between the models of α doesn't change after contraction and nor does the relation between the models of $\neg\alpha$. From this, to define a revision, one can just select these most plausible models of α and take them as the most plausible models using Boutilier's natural revision ($\leq_{\Psi\div\neg\alpha} \star_N \alpha$).

The converse process, that is, defining iterated contraction operators starting from iterated revision operators using the Tpo Harper identity, requires in particular to find a correct definition for $\sqcap$. This problem is investigated by Booth and Chandler [2], where they show that there is not a single, canonical way to proceed.

One can see this as an additional richness of the iterated framework. However, this richness of the epistemic state representation has its counterparts. In fact in the iterated case there are more revision operators than contraction ones. More precisely we have the following result:

Theorem 5. *There are more iterated revision operators than iterated contraction operators. In particular, this entails that it is impossible to find an interpretation of the expansion $\oplus$ in the Tpo Levi identity in order to obtain all the iterated revision operators via this identity.*

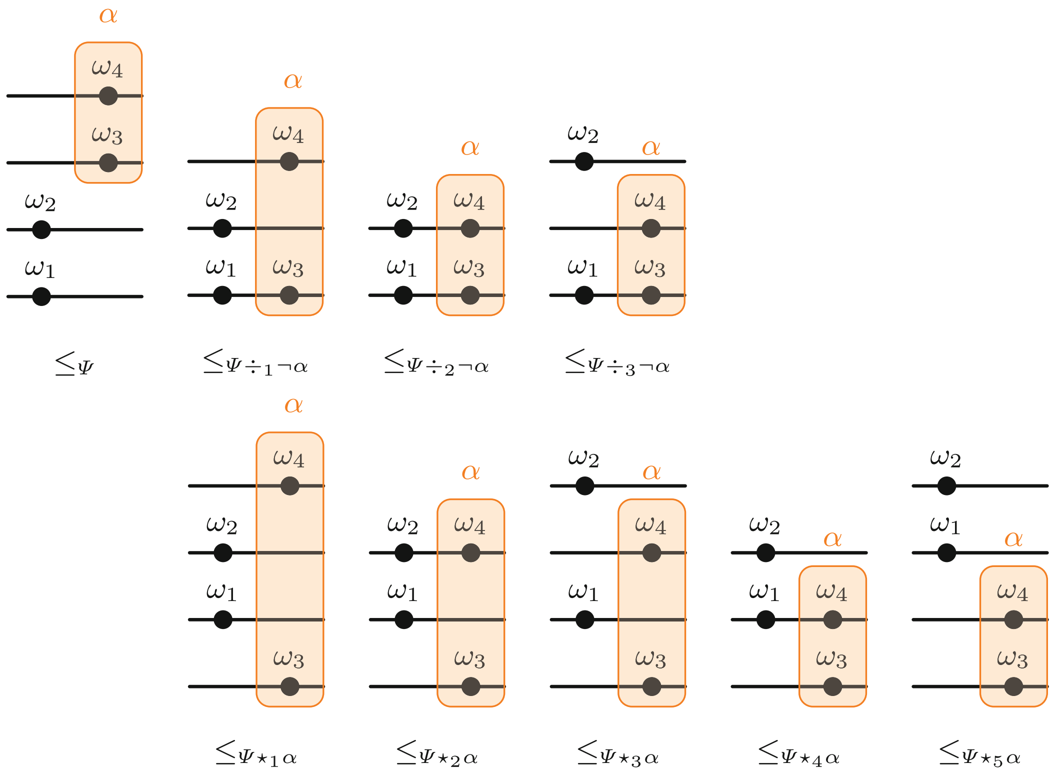

Fig. 2. All possible contractions by $\neg\alpha$ and revisions by α from $\leq_\Psi$

Proof. Let $\leq_\Psi$ and α be as illustrated in Fig. 2. Due to rigidity conditions for the iterated contraction, there are only three possible different outputs as results of contraction of $\leq_\Psi$ by $\neg\alpha$ ($\leq_{\Psi\div_1\neg\alpha}$, $\leq_{\Psi\div_2\neg\alpha}$ and $\leq_{\Psi\div_3\neg\alpha}$). Contrastingly there are five different possible output for revision. That is due to the rigidity postulates for iterated revision. Three of these possible revision outputs can be obtained from contraction outputs using natural revision as in the previous example ($\leq_{\Psi\star_1\alpha}$, $\leq_{\Psi\star_2\alpha}$, $\leq_{\Psi\star_3\alpha}$). The other two ones are $\leq_{\Psi\star_4\alpha}$ and $\leq_{\Psi\star_5\alpha}$, that are not related to any contraction result using the identity.

The previous theorem is important because it tells us that in the iterated case the Levy identity has limitations. This theorem is also important since it gives us a true distinction between classical AGM framework and the iterated framework. In the classical AGM framework there is a bijection between revision and contraction operators. Contraction is often considered as a more fundamental operator since a revision can be defined, through Levi identity, as a contraction followed by a conjunction (expansion). In the iterated case there are more iterated revision than iterated contraction operators, so the more general change operator seems to be revision.

One can object that $\leq_{\Psi\star_4\alpha}$ and $\leq_{\Psi\star_5\alpha}$ could maybe be obtained through Levi identity by using another definition of $\oplus$ than Boutilier's natural revision operator, as we used here. But this does not change the fact that there is only three possible contraction results versus five possible revision results, and then with any alternative, there is still no way to define a bijection. We can just define a relation between $\div$ and a couple $(\star, \oplus)$, that is far from AGM original idea of this identity, and that does not contradicts our cardinality argument.

As a matter of fact there is a generalization of iterated revision operators, called improvement operators from which one can obtain iterated revision operators and at the same time iterated contraction operators. We make some brief comments about this in the next section.

6 Related Works

In a previous work [13,14] improvement operators are defined as a general class of iterative change operators, that contains Darwiche and Pearl iterated revision operators as special case. Actually, there is a more general class of improvement operators called *basic improvement* operators [16]. The postulates characterizing these operators say that at least a part of the new piece of information improves and the whole new piece of information does not worsen (this corresponds to postulates C3 and C4 of DP [6]).

Improvement operators are defined semantically on faithful assignments as an increase of plausibility of models of the new piece of information. This increase of plausibility can be more or less restricted, which leads to different families of operators [13]. But clearly the increase of plausibility of iterated contraction operators is limited due to the fact that the most plausible models of the new piece of information can not become more plausible than the models of the previous beliefs of the agent. Whereas for revision there is no such constraint, and so much more freedom is granted for improvement.

The following proposition says that our iterated contraction operators are also weak improvement operators (by the negation of the input).

Proposition 2. *Let $\div$ be an iterated contraction operator, then the operator $\hat{\div}$ defined as $\Psi \hat{\div} \alpha = \Psi \div \neg\alpha$ is a weak improvement operator [14], moreover it is a basic improvement operator.*

Thus, the previous proposition and the fact that iterated revision operators are also basic improvement operators seems to mean that this class of operator as the most primitive operators in iterated belief change.

Chopra et al. [5] also give postulates for iterated contraction, but they did it using iterated revision operators in their postulates, so the iterated contraction operators are not defined independently, but as a byproduct of iterated revision ones. Actually their starting point is a couple of given operators $*$ and $-$ that satisfy the Levi and Harper equivalences. Then, they characterize the four iterated semantic properties of Definition 4 in terms of syntactical postulates mixing the operators $*$ and $-$. In this work we propose a direct characterization of iterated contraction operators (not depending of any iterated revision operator).

Booth and Chandler [2] investigate the problem of the definition of iterated contraction through the Harper Identity for the concrete case of pre-orders on interpretations. The paper shows the richness of the question. In this paper we explain this richness by the fact that there are much more iterated revision operators than iterated contraction operators, so this means that several different iterated revision operators correspond, via the Harper identity, to the same iterated contraction operator.

The work of Ramachadran et al. [18] is very interesting. It concerns the characterization of three iterated contraction operator in the framework of the canonical representation of epistemic states. They give a pure syntactical characterization of these three operators. However they don't characterize the full class of iterated contraction operators.

7 Conclusion

To sum up, in this paper we proposed the first direct logical characterization of the class of iterated contraction operators having an iterated behavior similar to the one proposed by Darwiche and Pearl for iterated revision operators. We stated a representation theorem in terms of total pre-orders on interpretations. We discussed the fact that there is no easy way to generalize the Levi and Harper identity in the iterated case, but more importantly, that this is not a primordial issue, since, conversely to the classical (AGM) case, these two classes of operators are not linked by a bijection in the iterated case. There are more iterated revision operators than iterated contraction operators, and both are special cases of the more general class of improvement operators, where iterated contractions produce a smaller change than iterated revision operators. So, these two classes of change operators are not different in nature, but in degree of change.

Acknowledgments. The authors would like to thank the reviewers for their helpful comments.

References

1. Alchourrón, C.E., Gärdenfors, P., Makinson, D.: On the logic of theory change: partial meet contraction and revision functions. J. Symbolic Logic **50**, 510–530 (1985)
2. Booth, R., Chandler, J.: Extending the harper identity to iterated belief change. In: Kambhampati, S., (ed.) Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence, IJCAI 2016, New York, NY, USA, 9–15 July 2016, pp. 987–993. IJCAI/AAAI Press (2016)
3. Boutilier, C.: Iterated revision and minimal change of conditional beliefs. J. Philos. Logic **25**(3), 262–305 (1996)
4. Caridroit, T., Konieczny, S., Marquis, P.: Contraction in propositional logic. In: Destercke, S., Denoeux, T. (eds.) ECSQARU 2015. LNCS (LNAI), vol. 9161, pp. 186–196. Springer, Cham (2015). doi:10.1007/978-3-319-20807-7_17
5. Chopra, S., Ghose, A., Meyer, T.A., Wong, K.-S.: Iterated belief change and the recovery axiom. J. Philos. Logic **37**(5), 501–520 (2008)
6. Darwiche, A., Pearl, J.: On the logic of iterated belief revision. Artif. Intell. **89**, 1–29 (1997)

7. Fermé, E., Hansson, S.O.: AGM 25 years. J. Philos. Logic **40**(2), 295–331 (2011)
8. Gärdenfors, P.: Knowledge in Flux. MIT Press, Cambridge (1988)
9. Hansson, S.O.: A Textbook of Belief Dynamics. Theory Change and Database Updating. Kluwer, Dordrecht (1999)
10. Harper, W.L.: Rational conceptual change. In: 1976 Proceedings of the Biennial Meeting of the Philosophy of Science Association, PSA, vol. 2, pp. 462–494. Philosophy of Science Association, East Lansing, Mich (1977)
11. Katsuno, H., Mendelzon, A.O.: Propositional knowledge base revision and minimal change. Artif. Intell. **52**, 263–294 (1991)
12. Katsuno, H., Mendelzon, A.O.: On the difference between updating a knowledge base and revising it. In: Belief Revision, pp. 183–203. Cambridge University Press (1992)
13. Konieczny, S., Medina Grespan, M., Pino Pérez, R.: Taxonomy of improvement operators and the problem of minimal change. In: Proceedings of the Twelfth International Conference on Principles of Knowledge Representation And Reasoning (KR 2010), pp. 161–170 (2010)
14. Konieczny, S., Pino Pérez, R.: Improvement operators. In: Proceedings of the Eleventh International Conference on Principles of Knowledge Representation and Reasoning (KR 2008), pp. 177–187 (2008)
15. Levi, I.: Subjunctives, dispositions and chances. Synthese **34**, 423–455 (1977)
16. Grespan, M.M., Pino Pérez, R.: Representation of basic improvement operators. In: Trends in Belief Revision and Argumentation Dynamics, pp. 195–227. College Publications (2013)
17. Nayak, A., Goebel, R., Orgun, M., Pham, T.: Taking Levi Identity seriously: a plea for iterated belief contraction. In: Lang, J., Lin, F., Wang, J. (eds.) KSEM 2006. LNCS (LNAI), vol. 4092, pp. 305–317. Springer, Heidelberg (2006). doi:10.1007/11811220_26
18. Ramachandran, R., Nayak, A.C., Orgun, M.A.: Three approaches to iterated belief contraction. J. Philos. Logic **41**(1), 115–142 (2012)