On the Frontier between Arbitration and Majority

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Abstract

We give in this paper new results on merging operators. Those operators aim to define the beliefs (or goals) of an agents' group from the individuals beliefs (goals). Using the logical framework of [KP99] we study the relationships between two important sub-families of merging operators: majority operators and arbitration operators. An open question was to know if those two families are disjoint or not. We show that there are operators that belong simultaneously to the two families. Furthermore the new family introduced allows the user to choose the "consensual level" he wants for his majority operator.

1 Introduction

When several agents interact in order to achieve a common task, they have to agree from time to time on what are the beliefs (or the goals) of the group. When some agents disagree on these common beliefs (goals), then one has to enter in a *negotiation* process. The problem is that sometimes the negotiation step do not rule out all the conflicts. But, even in this case, the group has to take a decision on what are its beliefs (goals) to carry on. So, in such cases, an *aggregation* step is needed between agents wishes.

So, formally, when a decision has to be taken about beliefs (goals) of the group, we can consider this as a two step process. First, a *negotiation* step allows agents to try to convince undecided or opponents. Then, when all agents have fixed opinions, an *aggregation* step states what are the common beliefs (goals) of the group.

The first step of this process has been extensively studied in multi-agents works (see *e.g.* [APM00] for an example of formalisation of negotiation by means of argumentation). But the second one is usually only quickly quoted. Indeed, in most of those works, when some conflict is not solved after the negotiation step, one uses expeditious means to solve the conflicts. For example, by supposing the existence of some oracle that decides what is the good solution, or by using a preference relation between agents denoting the relative reliability of each source. But, even if those solutions often allow to rule out the conflicts, the basic problem is not solved and there still are problems in some cases. For example, it is not realistic to suppose that an oracle exists and always knows the good answer. And in the case of using a reliability ordering, there are cases where some equally reliable agents disagree and we are back with our basic problem.

The formal framework for solve this belief (goals) *aggregation* step, is the use of knowledge merging operators [CH97, Cho98, BKMS92, LM99, Rev97, Kon00].

In some related works different sets of logical properties that knowledge merging operators have to satisfy have been proposed [Rev97, LS98, LM99, KP98, KP99]. Those logical characterisations are used to define a taxonomy of merging operators, that allows to compare different merging methods and to choose the method corresponding to the behaviour wanted in a particular application.

We will focus on the *merging with integrity constraints* characterisation given in [KP99, KP02]. This characterisation allows to make a distinction between two major sub-classes of merging operators: majority operators and arbitration operators. Majority operators solve conflicts using majority wishes, that is they try to satisfy the group as a whole. Whereas arbitration operators have a more consensual behaviour, trying to satisfy each agent as far as possible.

Consider the following example to illustrate these two behaviours (the following example is stated in terms of goals, but one can find similar ones concerning beliefs):

Example 1 *Ally, Brian and Charles have to decide what they will do this night. Brian and Ally want to go to the restaurant and to the cinema. Charles does not want to go out this night and so he does not want to go nor to the restaurant nor to the cinema. Taking some majority merging operator the result of the merging is that the group will decide to go both to the restaurant and to the cinema, whereas Charles would certainly have a bad night. If one takes an arbitration operator the result will be that the group has to decide to go either to the cinema or to the cinema but not both. So each member of the group will be satisfied as much as possible.*

So these two sub-classes have very different conflict resolution policies. An open question was to know if these two sub-classes are disjoint or not. And, though it seems natural to bet on a strict partition, we show in this paper that it is not the case. That is, there exists operators that belong simultaneously to the two sub-classes. We first give a trivial operator that straightforwardly satisfy this condition. But the real question was to know if more complex operators can satisfy it too. We show that, in the finite case, a whole family of (non-trivial) operators are both arbitration and majority operators. The new family of operators introduced, generalisation of a well known majority merging method [Rev97, LM99, KP99], allow to choose the "consensual level" that best fit the application needs.

The paper is organised as follow. In section 2, we give the definition of merging with integrity constraints operators, arbitration and majority operators are also defined. Then, we give in section 3 some concrete operators in order to illustrate the differences of behaviour between arbitration and majority operators. In section 4, we show that it is possible for an operator to be both a majority and an arbitration operator. We discuss briefly in section 5 of alternative expressions of arbitration behaviour. We conclude in section 6 with some open questions.

2 Merging with Integrity Constraints

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional atoms. An interpretation is a function from $\mathcal{P}$ to $\{0,1\}$. The set of all the interpretations is denoted $\mathcal{W}$. An interpretation I is a model of a formula if and only if it makes it true in the usual classical truth functional way. Let φ be a formula, $mod(\varphi)$ denotes the set of models of φ, *i.e.* $mod(\varphi) = \{I \in \mathcal{W}\ I \models \varphi\}$.

A *knowledge base* φ is a finite set of propositional formulae.

Let $\varphi_1, \ldots, \varphi_n$ be n knowledge bases (not necessarily different). We call *knowledge set* the multi-set Ψ consisting of those n knowledge bases: $\Psi = \{\varphi_1, \ldots, \varphi_n\}$. We note $\bigwedge \Psi$ the conjunction of the knowledge bases of Ψ, i.e. $\bigwedge \Psi = \varphi_1 \wedge \cdots \wedge \varphi_n$. The union of multi-sets will be noted $\sqcup$.

By abuse if φ is a knowledge base, φ will also denote the knowledge set $\Psi = \{\varphi\}$. For a positive integer n we will denote Ψ^n the multi-set when Ψ appears n times.

Definition 1 *Let Ψ_1, Ψ_2 be two knowledge sets. Ψ_1 and Ψ_2 are equivalent, noted $\Psi_1 \leftrightarrow \Psi_2$, iff there exists a bijection f from $\Psi_1 = \{\varphi_1^1, \ldots, \varphi_n^1\}$ to $\Psi_2 = \{\varphi_1^2, \ldots, \varphi_n^2\}$ such that $\vdash f(\varphi) \leftrightarrow \varphi$.*

A pre-order $\leq$ is a reflexive and transitive relation. A pre-order is total if $\forall I, J\ I \leq J$ or $J \leq I$. Let $\leq$ be a pre-order, we define $<$ as follows: $I < J$ iff $I \leq J$ and $J \not\leq I$, and $\simeq$ as $I \simeq J$ iff $I \leq J$ and $J \leq I$. We wrote $I \in min(mod(\varphi), \leq)$ iff $I \models \varphi$ and $\forall J \in mod(\varphi)\ I \leq J$.

Once these definitions are stated, we can define merging operators. A knowledge base φ will denote the beliefs [1] of an agent. A knowledge set Ψ will denote a group of agents. The aim of merging operators is to define what are the beliefs of the group from the individuals beliefs and the constraints imposed by the system (physical constraints, laws, etc.). So, a merging operator $\triangle$ is a function that maps a knowledge set Ψ and a knowledge base μ that denotes the integrity constraints of the system, to a knowledge base $\triangle_\mu(\Psi)$ that contains the beliefs of the group. Recall that we suppose that all the knowledge bases have the same importance (*i.e.* reliability, hierarchical importance, etc...) and that they denote the beliefs (goals) of independent sources (agents). See *e.g.* [BDL+98, Cho98] for examples on prioritised knowledge bases.

The logical properties that one could expect from a belief merging operator are [KP99]:

Definition 2 $\triangle$ *is a merging with integrity constraints operator (IC merging operator in short) if and only if it satisfies the following properties:*

(IC0) $\triangle_\mu(\Psi) \models \mu$

(IC1) *If μ is consistent, then $\triangle_\mu(\Psi)$ is consistent*

(IC2) *If $\bigwedge \Psi$ is consistent with μ, then* $\triangle_\mu(\Psi) \equiv \bigwedge \Psi \wedge \mu$

[1] in the following, we will call "beliefs" the beliefs or the goals of an agent

(IC3) *If* $\Psi_1 \equiv \Psi_2$ *and* $\mu_1 \equiv \mu_2$*, then* $\triangle_{\mu_1}(\Psi_1) \equiv \triangle_{\mu_2}(\Psi_2)$

(IC4) *If* $\varphi_1 \models \mu$ *and* $\varphi_2 \models \mu$*, then* $\triangle_{\mu}(\varphi_1 \sqcup \varphi_2) \wedge \varphi_1$ *is consistent iff* $\triangle_{\mu}(\varphi_1 \sqcup \varphi_2) \wedge \varphi_2$ *is consistent*

(IC5) $\triangle_{\mu}(\Psi_1) \wedge \triangle_{\mu}(\Psi_2) \models \triangle_{\mu}(\Psi_1 \sqcup \Psi_2)$

(IC6) *If* $\triangle_{\mu}(\Psi_1) \wedge \triangle_{\mu}(\Psi_2)$ *is consistent , then* $\triangle_{\mu}(\Psi_1 \sqcup \Psi_2) \models \triangle_{\mu}(\Psi_1) \wedge \triangle_{\mu}(\Psi_2)$

(IC7) $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$

(IC8) *If* $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ *is consistent, then* $\triangle_{\mu_1 \wedge \mu_2}(\Psi) \models \triangle_{\mu_1}(\Psi)$

The intuitive meaning of the properties is the following: (IC0) assures that the result of the merging satisfies the integrity constraints. (IC1) states that if the integrity constraints are consistent, then the result of the merging will be consistent. (IC2) states that if possible, the result of the merging is simply the conjunction of the knowledge bases with the integrity constraints. (IC3) is the principle of irrelevance of syntax, expressing the fact that the result of the merging has to depend only of the expressed opinions and not of their syntactical presentation. (IC4) is the fairness postulate, the point is that when we merge two knowledge bases, merging operators must not give preference to one of them. (IC5) expresses the following idea: if a group Ψ_1 compromises on a set of alternatives which A belongs to, and another group Ψ_2 compromises on another set of alternatives which contains A too, so A has to be in the chosen alternatives if we join the two groups. (IC5) and (IC6) together state that if you could find two subgroups which agree on at least one alternative, then the result of the global merging will be exactly those alternatives the two groups agree on. (IC7) and (IC8) state that the notion of closeness is well-behaved, *i.e.* that an alternative that is preferred among the possible alternatives (μ_1), will remain preferred if we restrict the possible choices ($\mu_1 \wedge \mu_2$).

One can notice that when the knowledge set is a singleton (*i.e.* $\Psi = \{\varphi\}$) doing the merging $\triangle_{\mu}(\{K\})$ is exactly the revision of φ by a new evidence μ. That is $\triangle_{\mu}(\{K\}) = \varphi \circ \mu$, where $\circ$ is an AGM belief revision operator [Gär88, AGM85, KM91]. Thus IC merging operators can be considered as a generalisation of belief revision operators. See [KP02] for more results on the relationship between merging and belief revision.

We will now define the two major sub-classes of merging operators: majority and arbitration operators : An IC merging operator is a majority operator if it satisfies the following property:

(Maj) $\exists n \ \triangle_{\mu}(\Psi_1 \sqcup \Psi_2^n) \vdash \triangle_{\mu}(\Psi_2)$

This postulate expresses the fact that if an opinion has a large audience, it will be the opinion of the group. So, majority operators try to satisfy the group as a whole[2]. On the other hand, arbitration operators try to satisfy each agent as far as possible : An IC merging operator is an arbitration operator if it satisfies the following property:

$$\textbf{(Arb)} \quad \left.\begin{array}{l} \triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2) \\ \triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2) \\ \mu_1 \nvdash \mu_2 \\ \mu_2 \nvdash \mu_1 \end{array}\right\} \Rightarrow \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle_{\mu_1}(\varphi_1)$$

This postulate ensures that this is the median possible choices that are preferred. It is much more intuitive when it is expressed in terms of syncretic assignment (*cf* condition 8 below). The point is that we can improve the result for one of the member of the group only if it does not make the result worth for an other member. This kind of behaviour is very close to egalitarism in social choice theory (see *e.g.* [Mou88]).

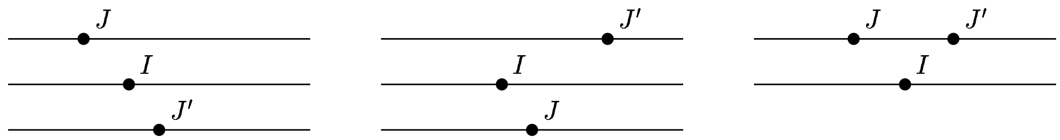
We will illustrate the (Arb) requirements on the following scenario:

Example 2 *Tom and David missed the soccer match yesterday between reds and yellows. So they don't know the result of the match. Tom listened in the morning that reds made a very good match. So he thinks that a win of reds is more plausible than a draw and that a draw is more reliable than a win of yellows. David was told that after that match yellows have now a lot of chances of winning the championship. From this information he infers that yellows win the match, or otherwise at least take a draw. Confronting their point of view, Tom and David agree on the fact that the two teams are of the same strength, and that they had the same chances of winning the match. What arbitration demand is that, with those informations, Tom and David have to agree that a draw between the two teams is the more plausible result.*

Now we will give a representation theorem for those operators in terms of pre-orders on interpretations. It provides a more constructive definition of those operators. We need first some definitions:

Definition 3 *A* syncretic assignment *is a function mapping each knowledge set* Ψ *to a total pre-order* $\leq_{\Psi}$

[2]Remark that, with the other postulates, (Maj) implies some kind of monotony property when the number of proponents grows: $\exists n_0 \ \forall n > n_0 \ \triangle_{\mu}(\Psi_1 \sqcup \Psi_2^n) \vdash \triangle_{\mu}(\Psi_2)$.



(a) φ_1 (b) φ_2 (c) $\varphi_1 \sqcup \varphi_2$

Figure 1: Arbitration

over interpretations such that for any knowledge sets Ψ, Ψ_1, Ψ_2 *and for any knowledge bases* φ_1, φ_2*:*

1. *If* $I \models \Psi$ *and* $J \models \Psi$*, then* $I \simeq_\Psi J$
2. *If* $I \models \Psi$ *and* $J \not\models \Psi$*, then* $I <_\Psi J$
3. *If* $\Psi_1 \equiv \Psi_2$*, then* $\leq_{\Psi_1} = \leq_{\Psi_2}$
4. $\forall I \models \varphi_1 \; \exists J \models \varphi_2 \; J \leq_{\varphi_1 \sqcup \varphi_2} I$
5. *If* $I \leq_{\Psi_1} J$ *and* $I \leq_{\Psi_2} J$*, then* $I \leq_{\Psi_1 \sqcup \Psi_2} J$
6. *If* $I <_{\Psi_1} J$ *and* $I \leq_{\Psi_2} J$*, then* $I <_{\Psi_1 \sqcup \Psi_2} J$

A majority syncretic assignment *is a syncretic assignment which satisfies the following:*

7. *If* $I <_{\Psi_2} J$*, then* $\exists n \; I <_{\Psi_1 \sqcup \Psi_2{}^n} J$

A fair syncretic assignment *is a syncretic assignment which satisfies the following:*

8. $\left.\begin{array}{l} I <_{\varphi_1} J \\ I <_{\varphi_2} J' \\ J \simeq_{\varphi_1 \sqcup \varphi_2} J' \end{array}\right\} \Rightarrow I <_{\varphi_1 \sqcup \varphi_2} J$

The two first conditions ensure that the models of the knowledge set (if any) are the more plausible interpretations for the pre-order associated to the knowledge set. The third condition states that two equivalent knowledge sets have the same associated pre-orders. Those three conditions are very closed to the ones existing in belief revision for faithful assignments [KM91]. The fourth condition states that, when merging two belief bases, for each model of the first one, there is a model of the second one that is at least as good than the first one. It ensures that the two knowledge bases are given the same consideration. The fifth condition says that if an interpretation I is at least as plausible as an interpretation J for a knowledge set Ψ_1 and if I is at least as plausible as J for a knowledge set Ψ_2, then if one joins the two knowledge sets, then I will still be at least as plausible as J. The sixth condition strengthen the previous condition by saying that an interpretation I is at least as plausible as an interpretation J for a knowledge set Ψ_1 and if I is strictly more plausible than J for a knowledge set Ψ_2, then if one joins the two knowledge sets, then I will be strictly more plausible than J. These two previous conditions corresponds to Pareto conditions in Social Choice Theory [Arr63, Kel78]. Condition 7 says that if an interpretation I is strictly more plausible than an interpretation J for a knowledge set Ψ_2, then there is a quorum n of repetitions of the knowledge set from which I will be more plausible than J for the larger knowledge set $\Psi_1 \sqcup \Psi_2{}^n$. This condition seems to be the weakest form of "majority" condition one could state. Condition 8 states that if an interpretation I is more plausible than an interpretation J for a belief base φ_1, if I is more plausible than J' for an other base φ_2, and if J and J' are equally plausible for the knowledge set $\varphi_1 \sqcup \varphi_2$, then I has to be more plausible than J and J' for $\varphi_1 \sqcup \varphi_2$. This requirement is illustrated figure 1 (the lower appears an interpretation, the more preferred it is). See also Example 2 for an intuitive explanation.

And the representation theorem is:

Theorem 1 *An operator is an IC merging operator (respectively IC majority merging operator or IC arbitration operator) if and only if there exists a syncretic assignment (respectively majority syncretic assignment or fair syncretic assignment) that maps each knowledge set* Ψ *to a total pre-order* $\leq_\Psi$ *such that*

$$mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$$

This theorem shows that a merging operator corresponds to a family of pre-orders. In fact, a lot of operators are defined directly from those pre-orders, using a function that maps each knowledge set to a pre-order. It is the case with all operators defined from a distance. We give some of them in the following section.

3 Some IC merging operators

We give in this section the definition of three families of operators. All those operators are based on a distance between interpretations that induces the pre-order associated to each knowledge set. We define also a new family of operators, that generalises the $\triangle^{d,\Sigma}$ family.

Let d be a distance between interpretations[3], that is a function $d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{N}$ such that :

- $d(I,J) = d(J,I)$
- $d(I,J) = 0$ iff $I = J$

For example, one can use the Dalal distance [Dal88], noted, d_H, that is the Hamming distance between two interpretations (the number of propositional letters on which the two interpretations differ). We will use this distance in the examples because it is a well known, easy to define, distance but one has to keep in mind that it is not the sole possible choice and that the logical properties do not depend of the chosen distance.

This distance between interpretations induces naturally a distance between an interpretation and a knowledge base as follows:

$$d(I,\varphi) = \min_{J \models \varphi} d(I,J)$$

The difference between the four families of operators we define next lie in the way this distance between an interpretation and a knowledge base is used in order to define the distance between an interpretation and the knowledge set. So, it is this aggregation step of the individual preferences (distances) in a global one that makes behaviour differences between the families.

The three families stated next are well known, the $\triangle^{d,Max}$ family has been used in [Rev93, Rev97], the $\triangle^{d,\Sigma}$ family in [Rev97, LM99, KP99], and the $\triangle^{d,GMax}$ family in [KP98, KP99].

Definition 4 *Let Ψ be a knowledge set, I be an interpretation and d be a distance between interpretations. The Max, Σ, GMax distances are defined respectively by:*

- $d_{d,Max}(I,\Psi) = \max_{\varphi \in \Psi} d(I,\varphi)$
- $d_{d,\Sigma}(I,\Psi) = \sum_{\varphi \in \Psi} d(I,\varphi)$
- *Suppose $\Psi = \{\varphi_1 \ldots \varphi_n\}$. For each interpretation I we build the list $(d_1^I \ldots d_n^I)$ of distances between this interpretation and the n knowledge bases in Ψ, i.e. $d_j^I = d(I,\varphi_j)$. Let $d_{d,GMax}(I,\Psi)$ be the list obtained from $(d_1^I \ldots d_n^I)$ by sorting it in descending order* [4].

So, let $f \in \{Max, \Sigma, GMax\}$, such a distance induces a pre-order on interpretations:

$$I \leq_\Psi^{d,f} J \textit{ iff } d_{d,f}(I,\Psi) \leq d_{d,f}(J,\Psi)$$

And the corresponding merging operator is defined by:

$$mod(\triangle_\mu^{d,f}(\Psi)) = \min(mod(\mu), \leq_\Psi^{d,f})$$

Those operators satisfy the following properties:

Theorem 2 *$\triangle^{d,Max}$ operators satisfy (IC1-IC5), (IC7), (IC8) and (Arb). $\triangle^{d,GMax}$ operators are arbitration operators. $\triangle^{d,\Sigma}$ operators are majority operators.*

It is possible to generalise the $\triangle^{d,\Sigma}$ family in the following $\triangle^{d,\Sigma^n}$ operators:

Definition 5 $d_{d,\Sigma^n}(I,\Psi) = \sum_{\varphi \in \Psi} d(I,\varphi)^n$.

Then the corresponding pre-order is:

$$I \leq_\Psi^{d,\Sigma^n} J \textit{ iff } d_{d,\Sigma^n}(I,\Psi) \leq d_{d,\Sigma^n}(J,\Psi)$$

And the $\triangle^{d,\Sigma^n}$ operator is defined by:

$$mod(\triangle_\mu^{d,\Sigma^n}(\Psi)) = \min(mod(\mu), \leq_\Psi^{d,\Sigma^n})$$

It is easy to show then that:

Theorem 3 *$\triangle^{d,\Sigma^n}$ operators are majority operators.*

Now we illustrate the behaviour of these families on an example:

Example 3 *At a meeting of a block of flats co-owners, the chairman proposes for the coming year the construction of a swimming-pool, a tennis-court and a private-car-park. But if two of these three items are build, the rent will increase significantly. We will*

[3] Remark that the triangular inequality $d(I,J) \leq d(I,J') + d(J',J)$ is not required.

[4] the d_{GMax} distance do not strictly obey to the requirements of a distance, since it does not give numbers. In fact there is a natural mapping: choose a sufficiently big number N (where sufficiently means strictly bigger than all possible distances $d(I,\varphi_i)$, it is always possible since we work in the finite case), and then define $d_{d,GMax} = \Sigma_{j=1\ldots n}(d_{i_j}^I * N^{n-j})$, where i_j denotes the jth element in the sorted list.

	φ_1	φ_2	φ_3	φ_4	$\mathbf{dist}_{Max}$	$\mathbf{dist}_{\Sigma}$	$\mathbf{dist}_{GMax}$	$\mathbf{dist}_{\Sigma^2}$
$(0,0,0,0)$	3	3	0	2	3	8	(3,3,2,0)	22
$(0,0,0,1)$	3	3	1	3	3	10	(3,3,3,1)	28
$(0,0,1,0)$	2	2	1	1	**2**	6	**(2,2,1,1)**	**10**
$(0,0,1,1)$	2	2	2	2	**2**	8	(2,2,2,2)	16
$(0,1,0,0)$	2	2	1	1	**2**	6	**(2,2,1,1)**	**10**
$(0,1,0,1)$	2	2	2	2	**2**	8	(2,2,2,2)	16
$(0,1,1,0)$	1	1	2	0	2	4	(2,1,1,0)	6
$(0,1,1,1)$	1	1	3	1	3	6	(3,1,1,1)	12
$(1,0,0,0)$	2	2	1	2	**2**	7	(2,2,2,1)	13
$(1,0,0,1)$	2	2	2	3	3	9	(3,2,2,2)	21
$(1,0,1,0)$	1	1	2	1	2	5	(2,1,1,1)	7
$(1,0,1,1)$	1	1	3	2	3	7	(3,2,1,1)	15
$(1,1,0,0)$	1	1	2	1	2	5	(2,1,1,1)	7
$(1,1,0,1)$	1	1	3	2	3	7	(3,2,1,1)	15
$(1,1,1,0)$	0	0	3	0	3	3	(3,0,0,0)	9
$(1,1,1,1)$	0	0	4	1	4	**5**	(4,1,0,0)	17

Table 1: Distances

denote by S, T, P respectively the construction of the swimming-pool, the tennis-court and the private-car-park. We will denote I the rent increase. The chairman outlines that build two items or more will have an important impact on the rent:

$$\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I$$

There is four co-owners $\Psi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$. Two of the co-owners want to build the three items and don't care about the rent increase: $\varphi_1 = \varphi_2 = S \wedge T \wedge P$. The third one thinks that build any item will cause at some time an increase of the rent and want to pay the lowest rent so he is opposed to any construction: $\varphi_3 = \neg S \wedge \neg T \wedge \neg P \wedge \neg I$. The last one thinks that the flat really needs a tennis-court and a private-car-park but don't want a high rent increase: $\varphi_4 = T \wedge P \wedge \neg I$.

The propositional letters S, T, P, I will be considered in that order for the valuations:

$$mod(\mu) = \mathcal{W} \setminus \{ (0,1,1,0), (1,0,1,0), (1,1,0,0), (1,1,1,0)\}$$
$$mod(\varphi_1) = \{(1,1,1,1), (1,1,1,0)\}$$
$$mod(\varphi_2) = \{(1,1,1,1), (1,1,1,0)\}$$
$$mod(\varphi_3) = \{(0,0,0,0)\}$$
$$mod(\varphi_4) = \{(1,1,1,0), (0,1,1,0)\}$$

We sum up the calculations in table 1. The lines shadowed correspond to the interpretations rejected by the integrity constraints. Thus the result has to be found among the interpretations that are not shadowed.

With the $\triangle^{d_H,Max}$ operator, the minimum distance is 2 and the chosen interpretations are $mod(\triangle_\mu^{d_H,Max}(\Psi)) = \{(0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1), (1,0,0,0)\}$. So the decision that best fit the group wishes is then not to increase the rent and to build one of the three items, or to increase the rent and build either the tennis court or the private car-park.

We can see on that example why $\triangle^{d,Max}$ operators are not IC merging operators. For example, the two interpretations $(0,0,1,0)$ and $(0,0,1,1)$ are chosen by $\triangle^{d_H,Max}$, although $(0,0,1,0)$ is better for φ_3 and φ_4 than $(0,0,1,1)$, whereas these two interpretations are equally preferred by φ_1 and φ_2. It seems then natural to globally prefer $(0,0,1,0)$ to $(0,0,1,1)$. It is in fact what demands (IC6).

The $\triangle^{d,GMax}$ family has been build with that idea of being more selective than the $\triangle^{d,Max}$ family. With the $\triangle^{d_H,GMax}$ operator the result is $mod(\triangle_\mu^{d_H,GMax}(\Psi)) = \{(0,0,1,0), (0,1,0,0)\}$, so the decision in this case will be to build either the tennis court or the car-park but without increasing the rent.

But if one chooses $\triangle^{d_H,\Sigma}$ for solving the conflict according to majority wishes, the result is then $mod(\triangle_\mu^{d_H,\Sigma}(\Psi)) = \{(1,1,1,1)\}$, and the decision will be to build the three items and to increase the rent.

Majority voting, *à la* $\triangle^{d,\Sigma}$, often seems more democratic than the other methods but, for example in this case, this only works if φ_3 accept to obey to this decision that is strictly opposed to its opinion. If φ_3 decides not to pay the rent increase, the works will perhaps not carry on because of a lack of money. So if a decision requires the approval of all the members a more consensual, arbitration like, method seems more adequate. These kind of issues are highly related with social choice theory [Arr63, Kel78, Mou88].

On this example, one can illustrate the use of the $\triangle^{d,\Sigma^n}$ family, since with the operator $\triangle^{d_H,\Sigma^2}$ we can see that the result (on this example) is the same as with the $\triangle^{d_H,GMax}$ operator. The reason is that the power used in the definition of the operator allows to be more consensual while keeping the majority behaviour.

4 Arbitration versus Majority

We show in this section that some operators are both majority and arbitration operators. We first show that with an (over)simple operator. Then, we show that a whole family of full sense operators (the $\triangle^{d,\Sigma^n}$ operators) satisfy also this condition.

4.1 Drastic Distance

The simplest distance between interpretations one can define is the following one:

$$d_{Dra}(I,J) = \begin{cases} 0 & \text{if } I = J \\ 1 & \text{otherwise} \end{cases}$$

The induced distance between an interpretation and a knowledge base is then also 0 or 1 if the interpretation respectively satisfy or not the knowledge base.

It is then easy to show that the operators given with this distance by the two families $\triangle^{d,GMax}$ and $\triangle^{d,\Sigma}$ are the same. And we have the following result:

Theorem 4 *The operator $\triangle^{d_{Dra},\Sigma} = \triangle^{d_{Dra},GMax}$ satisfies (IC0)-(IC8), (Maj) and (Arb).*

This easy to state result (consequence of theorem 2) is not very surprising. But the real question is to know if more elaborate distances can lead to such *"collision"* between majority and arbitration classes. We answer this question in the next section.

4.2 Graphical study

We show in this section that some $\triangle^{d,\Sigma^n}$ operators are simultaneously majority and arbitration operators. For an easy explanation, we will use a graphical construction showing the behaviour of the operators "at work". In order to have a 2D representation we will restrict ourselves to two knowledge bases.

The graphical construction is simple. We put the interpretations in the plane with their distance to the φ_2 base as abscissa and with their distance to φ_1 as ordinate. Then, the aim of the merging is to find the set of interpretations that are the closest to the (0,0) point. The differences between the operators lie in the chosen distance and in this definition of "closeness".

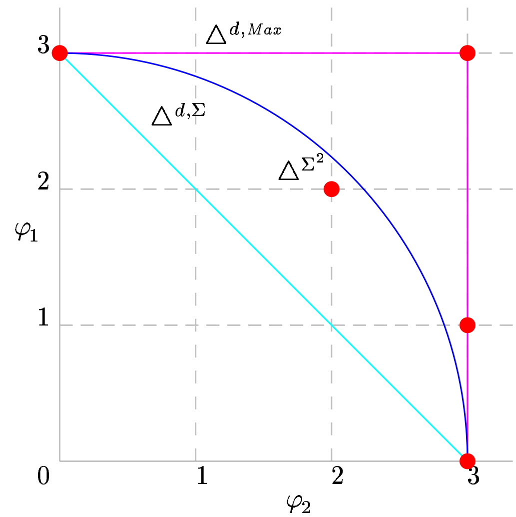


Figure 2: Merging of two knowledge bases

On figure 2, the curves represent the interpretations that are at a distance 3 from the knowledge set $\{\varphi_1, \varphi_2\}$ according to the operators $\triangle^{d,Max}$, $\triangle^{d,\Sigma}$ and $\triangle^{d,\Sigma^2}$. $\triangle^{d,Max}$ is represented by a square of size a, $\triangle^{d,\Sigma}$ by the line $x = a - y$, and $\triangle^{d,\Sigma^2}$ by a circle arc of radius $\sqrt{a}$, where a denotes the distance from the knowledge set. The $\triangle^{d,GMax}$ operator is hardly representable in this way, but one can figure out a curve that follows the one of $\triangle^{d,Max}$ but that prefers the interpretations that are closest to the axes. We will see soon how to approximate graphically the $\triangle^{d,GMax}$ operator. Then the result of the merging, using these three operators, is the set of interpretations that the respective curves meet first when a varies from 0 to ∞.

In particular, on this example, the result for $\triangle^{d,Max}$ and $\triangle^{d,\Sigma^2}$ is the interpretation placed in (2,2). And for $\triangle^{d,\Sigma}$ the result is the interpretations placed in (3,0) and (0,3). In the same way, one can rebuild the pre-orders $\leq_\Psi^{d,Max}$, $\leq_\Psi^{d,\Sigma}$ and $\leq_\Psi^{d,\Sigma^2}$ when one consider the order the interpretations are met by the curves (when a varies from 0 to ∞).

On the figure, we can see once again the problem of $\triangle^{d,Max}$, that do not make any distinction between the (3,0) and (3,3) points for example. It is why $\triangle^{d,Max}$ is not an IC merging operator.

On the other side, $\triangle^{d,\Sigma}$ do not make any distinction on the sources of disagreements. It is absolutely not consensual, since it allows to choose interpretations that satisfy completely one of the two bases and that dissatisfy completely the other one (the one placed in (0,3) for example), even if there are more consensual choices (an interpretation placed in (2,1) or in (2,2) for example). This behaviour can seem normal for a majority operator. But it is not systematic. Indeed, the operators $\triangle^{d,\Sigma^n}$ with $n > 1$ prefer more consensual choices, that is the ones closest to the line $x = y$. So, an interpretation placed in (2,2) would be prefer to one placed in (3,0).

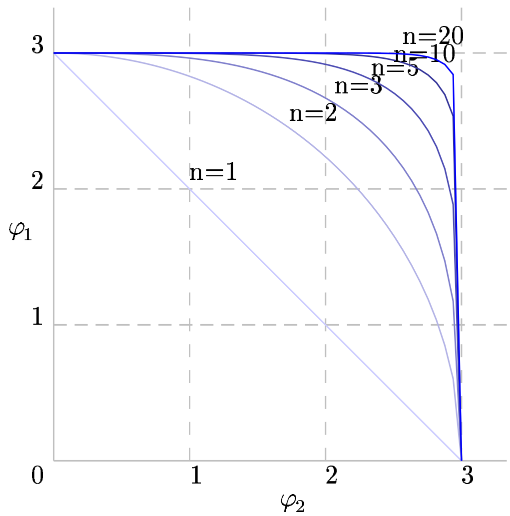


Figure 3: The $\triangle^{d,\Sigma^n}$ familly

Remark that the operator $\triangle^{d,\Sigma^2}$ is a particular operator of the $\triangle^{d,\Sigma^n}$ class, since it uses the Euclidean distance as distance between an interpretation and the knowledge set. This give a spherical distance, that is very natural and that obey to majority wishes but without the excesses of $\triangle^{d,\Sigma}$.

Furthermore, one can remark on figure 3 that, when one increases the value of n, the curve of $\triangle^{d,\Sigma^n}$ comes near to the one of $\triangle^{d,Max}$. And, with a sufficiently big n one can take the curve of $\triangle^{d,\Sigma^n}$ as an approximation of the one of $\triangle^{d,Max}$. But, for all n, an interpretation placed in (x, y) will always be preferred to an interpretation placed in $(x, y + 1)$ or in $(x + 1, y)$. And in fact, this way of following the $\triangle^{d,Max}$ curve but with a preference for the interpretations closest to the axes, is the one of the $\triangle^{d,GMax}$ curve. So, from some given n, we get $\triangle^{d,\Sigma^n} = \triangle^{d,GMax}$. More formally, we have the following result:

Theorem 5 *let Ψ be a knowledge set, $\exists n_0$ such that $\forall n > n_0$*

$$\triangle^{d,\Sigma^n}(\Psi) = \triangle^{d,GMax}(\Psi)$$

This result is an other answer to the partition between majority and arbitration operators. Since $\triangle^{d,\Sigma^n}$ operators, for all n greater than a given n_0 fixed by the maximum distance between an interpretation and a knowledge base, are both arbitration and majority operators. So the intersection between these two classes is not empty and it is possible, in a sense, to go continuously from one to the other.

5 Discussion

A possible conclusion that one can draw from the previous results is that maybe the (Arb) postulate do not capture exactly what we intend by *arbitration*. We hope we have sufficiently advocated in favour of (Arb), but let's see if we can find some alternatives.

Sometimes people take the following postulate as an expression of the arbitration behaviour (see *e.g.* [Mey01]):

(MI) $\forall n \;\; \triangle_\mu (\Psi_1 \sqcup {\Psi_2}^n) \leftrightarrow \triangle_\mu(\Psi_1 \sqcup \Psi_2)$

First we have to say that this postulate is not consistent with those of IC merging operators (see [KP02]). And, independently, we do not think that this postulate expresses the idea of being as close as possible to the wishes of each member of the group. It simply says that repetitions do not matter for the result of the merging (that is knowledge sets are no longer considered as multi-sets, but as simple sets). For example if the agents disagree only on the value of one propositional variable (for example consider Example 1, where the choice is simply to go to the cinema or not), then it is not possible to find compensations when building the result. So it seems sensible to take into account repetitions (that is different from being a majority operator).

An other way of thinking about arbitration operator is to allow the members to put a *right of veto* on some choices. The idea is to express the fact that the worst possible choices of each member of the group will not be chosen in the result (if possible). This can be express in this way :

(ArbV) If $\forall \varphi_i \; \exists \mu_i \; \forall \mu \; \triangle_{\mu_i \vee \mu} (\varphi_i) \vdash \mu$
and $\forall i \; \mu' \wedge \mu_i \vdash \bot$,
then $\triangle_{\mu' \vee \bigvee \mu_i}(\sqcup \varphi_i) \vdash \mu'$

Each μ_i corresponds to the worst possible choices of the agent φ_i. So what says this property is that if one can find an interpretation that does not belong to any μ_i, then no model from an μ_i will be in the result of the merging.

Even if this idea seems to be interesting, the property (ArbV) is really too complex. So it can not be checked directly, and it seems to be a very difficult task to find the corresponding condition on the syncretic assignment.

6 Conclusion

We have explored in this paper the frontier between two important subclasses of merging operators: arbitration and majority operators. The formers aiming to prefer consensual choices, whereas the latter referring to majority wishes.

An open question until now was to now if there is an intersection between these two classes or not. We have shown that it is the case, and that it is possible for an operator to be both an arbitration and a majority operator. Those operators seem to be a good compromise between democratic ideas lying in majority operators and consensual behaviour of arbitration operators.

We have introduced, in particular, a new family of operators (the $\triangle^{d,\Sigma^n}$ family), that allows to choose the *"consensual level"* of the majority operator according to the particular application needs.

An open question is to know if it is possible to characterised exactly what are the operators that belong simultaneously to the two classes.

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Appendix : Proof of Theorem 5

Proof : We will show that, given a knowledge set Ψ compound of m knowledge bases, and a distance d there exists n_0 such that $\forall n > n_0$ the two pre-orders $\leq_\Psi^{d,\Sigma^n}$ and $\leq_\Psi^{d,GMax}$ are the same. And we conclude by theorem 1.

We want to show that $I \leq_\Psi^{d,GMax} J$ iff $I \leq_\Psi^{d,\Sigma^n} J$.

(only if part) Consider two cases :

$I \simeq_\Psi^{d,GMax} J$, then the two sorted lists $(d^I_{\sigma(1)} \dots d^I_{\sigma(m)})$ and $(d^J_{\sigma(1)} \dots d^J_{\sigma(m)})$ are the same. And, for all n, the two distances $d_{d,\Sigma^n}(I,\Psi) = \Sigma_{i=1\dots m} {d^I_{\sigma(i)}}^n$ and $d_{d,\Sigma^n}(J,\Psi)$ are the same. So if $I \simeq_\Psi^{d,GMax} J$, then for all n $I \simeq_\Psi^{d,\Sigma^n} J$.

$I <_\Psi^{d,GMax} J$. This mean that for the two sorted lists $(d^I_{\sigma(1)} \dots d^I_{\sigma(m)})$ and $(d^J_{\sigma(1)} \dots d^J_{\sigma(m)})$ there exists $k \leq m$ such that $\forall i < k$ $d^I_{\sigma(i)} = d^J_{\sigma(i)}$ and $d^I_{\sigma(k)} < d^J_{\sigma(k)}$. Consider the worst case, where $k = 1$ and such that $(d^I_{\sigma(1)} \dots d^I_{\sigma(m)}) = (x \dots x)$ and $(d^J_{\sigma(1)} \dots d^J_{\sigma(m)}) = (y\ 0 \dots 0)$ with $x < y$. The other cases will be directly retrieved by sum properties. Then the question is to find n_0 such that $\Sigma_{i=1\dots m} {d^I_{\sigma(i)}}^{n_0} < \Sigma_{i=1\dots m} {d^J_{\sigma(i)}}^{n_0}$, that is $m.x^{n_0} < y^{n_0}$. Once again, consider the worst case : $y = x + 1$, that gives $m.x^{n_0} < (x+1)^{n_0}$. It is enough to find n_0 such that $m.x^{n_0} < x^{n_0} + n.x^{n_0-1}$. That gives $n_0 > (m-1).x$. So let's note N the maximum value given by the distance d (so $x < N$). We can take $n_0 = m.N$, where m is the cardinality of Ψ. So $\forall n > n_0 = m.N$, if $I <_\Psi^{d,GMax} J$, then $I <_\Psi^{d,\Sigma^n} J$.

(if part) We want to show that, $\forall n > n_0$, if $I \leq_\Psi^{d,\Sigma^n} J$, then $I \leq_\Psi^{d,GMax} J$. Simply remark that the contraposition is : if $I <_\Psi^{d,GMax} J$, then $I <_\Psi^{d,\Sigma^n} J$, that is what we prove in the only if part (we show this with $n = n_0 = m.N$, and then derive the result for all $n > n_0$).

□

References

[AGM85] C. E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50:510–530, 1985.

[APM00] L. Amgoud, S. Parsons, and N. Maudet. Arguments, dialogue and negociation. In *Proceedings of the Fourteenth European Conference on Artificial Intelligence (ECAI'00)*, pages 338–342, 2000.

[Arr63] K. J. Arrow. *Social choice and individual values.* Wiley, New York, second edition, 1963.

[BDL+98] S. Benferhat, D. Dubois, J. Lang, H. Prade, A. Saffioti, and P. Smets. A general approach for inconsistency handling and merging information in prioritized knowledge bases. In *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, pages 466–477, 1998.

[BKMS92] C. Baral, S. Kraus, J. Minker, and V. S. Subrahmanian. Combining knowledge bases consisting of first-order theories. *Computational Intelligence*, 8(1):45–71, 1992.

[CH97] L. Cholvy and T. Hunter. Fusion in logic: a brief overview. In *Proceedings of the Fourth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'97)*, Lecture Notes in Computer Science 1244, pages 86–95, 1997.

[Cho98] L. Cholvy. Reasoning about merged information. In D. M. Gabbay and Ph. Smets, editors, *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, volume 3, pages 233–263. Kluwer Academic Publishers, 1998.

[Dal88] M. Dalal. Investigations into a theory of knowledge base revision: preliminary report. In *Proceedings of the National Conference on Artificial Intelligence (AAAI'88)*, pages 475–479, 1988.

[Gär88] P. Gärdenfors. *Knowledge in flux.* MIT Press, 1988.

[Kel78] J. S. Kelly. *Arrow impossibility theorems.* Series in economic theory and mathematical economics. Academic Press, New York, 1978.

[KM91] H. Katsuno and A. O. Mendelzon. Propositional knowledge base revision and minimal change. *Artificial Intelligence*, 52:263–294, 1991.

[Kon00] S. Konieczny. On the difference between merging knowledge bases and combining them. In *Proceedings of the Seventh International Conference on Principles of Knowledge Representation and Reasoning (KR'00)*, pages 135–144, 2000.

[KP98] S. Konieczny and R. Pino Pérez. On the logic of merging. In *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, pages 488–498, 1998.

[KP99] S. Konieczny and R. Pino Pérez. Merging with integrity constraints. In *Proceedings of the Fifth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'99)*, Lecture Notes in Artificial Intelligence 1638, pages 233–244, 1999.

[KP02] S. Konieczny and R. Pino Pérez. Merging information under constraints: a logical framework. *Journal of Logic and Computation*, 15, 2002. to appear.

[LM99] J. Lin and A. O. Mendelzon. Knowledge base merging by majority. In *Dynamic Worlds: From the Frame Problem to Knowledge Management.* Kluwer, 1999.

[LS98] P. Liberatore and M. Schaerf. Arbitration (or how to merge knowledge bases). *IEEE Transactions on Knowledge and Data Engineering*, 10(1):76–90, 1998.

[Mey01] T. Meyer. On the semantics of combination operators. *Journal of Applied Non-Classical Logics*, 11(1-2):59–84, 2001.

[Mou88] H. Moulin. *Axioms of cooperative decision making.* Monograph of the Econometric Society. Cambridge University Press, 1988.

[Rev93] P. Z. Revesz. On the semantics of theory change: arbitration between old and new information. In *Proceedings of the 12^{th} ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Databases*, pages 71–92, 1993.

[Rev97] P. Z. Revesz. On the semantics of arbitration. *International Journal of Algebra and Computation*, 7(2):133–160, 1997.