# Belief base rationalization for propositional merging

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## Abstract

Existing belief merging operators take advantage of all the models from the bases, including those contradicting the integrity constraint. In this paper, we argue that this is not suited to every merging scenario, especially when the integrity constraint encodes physical laws. In that case the bases have to be 'rationalized' with respect to the integrity constraint during the merging process. We define several conditions characterizing the operators that are independent to such a rationalization process, and we show how these conditions interact with the standard IC postulates for belief merging. Especially, we give an independence-based axiomatic characterization of a distance-based operator.

*Keywords*: belief merging, rationalization, belief revision, belief update, independence to rationalization.

## 1 Introduction

Belief merging operators [15, 17] aim at computing the beliefs of a group of agents from a profile of belief bases representing the individual beliefs of the agents and some integrity constraint which typically represent physical laws or norms.

There are usually several ways to merge a profile of belief bases given some integrity constraint. The rational ones are characterized by a set of rationality postulates, the IC postulates [15], that merging operators should satisfy. Such operators are called IC merging operators.

Existing IC merging operators take advantage of every model from every base of the input profile, including those contradicting the given integrity constraint. However, this is not suited to every merging scenario. Especially, when the integrity constraint encodes knowledge about the underlying system, or structural laws due to the choice of encoding, the exploitation in the merging process of 'infeasible' worlds (i.e. conflicting with the constraints) can be questioned.

For instance, Condotta *et al.* [5] developed a framework for merging qualitative spatial or temporal information expressed in propositional logic. In this setting, an integrity constraint is used for encoding the spatial and/or temporal laws. The point is that if some variables encode the order between some time points, some worlds correspond to physically infeasible scenarios. For instance, consider three instants $t_1, t_2$ and $t_3$, forming a scenario such that $t_1 <_{\mathcal{T}} t_2$, $t_2 <_{\mathcal{T}} t_3$ and $t_3 <_{\mathcal{T}} t_1$, under the constraint that the time line $\mathcal{T}$ is totally ordered. It is reasonable to assume that such infeasible scenarios should be discarded in such a way that they have no impact on the resulting

Vol. 28, No. 7, 
Advance Access Publication on 9 October 2018 doi:10.1093/logcom/exy029

merged base. The problem here comes from the fact that the integrity constraint (i.e. the fact that $<_{\mathcal{T}}$ is transitive and irreflexive) is not explicitly represented, but this may also happen if this was the case. Let us illustrate it by focusing on a simple standard merging scenario.

EXAMPLE 1.1
Alice and Bob are asked to express their beliefs about the relative location of three French cities: Lyon, Marseille and Aix-en-Provence. Alice believes that Lyon is located north of Aix-en-Provence, and that Aix-en-Provence is located north of Marseille. Bob believes that Aix-en-Provence is the most southern city of the three. The way informal propositions of interest are associated with propositional symbols is not ruled by logic or by the merging process, and thus there are several ways to represent this kind of information. For instance, let us consider the three propositional symbols $p_{LA}$, $p_{AM}$ and $p_{LM}$, which respectively stand for 'Lyon is north of Aix-en-Provence', 'Aix-en-Provence is north of Marseille', and 'Lyon is north of Marseille'. Then Alice's beliefs can be encoded by the propositional formula $\varphi_1 = p_{LA} \wedge p_{AM}$, and Bob's beliefs by $\varphi_2 = p_{LA} \wedge \neg p_{AM}$. From the three propositional variables considered in this encoding, we obtain eight possible worlds. For instance, the world 101[1] represents the scenario where Lyon is the most northern city and Aix-en-Provence is the most southern one. This encoding is expressive enough to represent all six scenarios characterizing the relative positions of the three cities w.r.t. the meridian, which correspond to the six worlds 000, 100, 010, 101, 011 and 111. But then, the two remaining worlds, 110 and 001, represent two scenarios that are 'inconceivable' in the underlying domain. This is because the relation 'north of' is transitive: for instance, if Lyon is north of Aix-en-Provence and Aix-en-Provence is north of Marseille, then Lyon cannot conceivably be south of Marseille. Such transitivity rule can be represented by the integrity constraint $\mu = ((p_{LA} \wedge p_{AM}) \Rightarrow p_{LM}) \wedge ((\neg p_{AM} \wedge \neg p_{LA}) \Rightarrow \neg p_{LM})$, and doing so, the models of $\mu$ characterize the set of all six possible worlds.

Now, what can be deduced from Alice and Bob's beliefs from a global viewpoint? The answer depends on the merging operator under consideration. Consider the IC merging operator based on the Hamming distance and sum as aggregation function ($\Delta^{d_H,\Sigma}$) [15]; for this operator, the models of the merged base representing the beliefs of the group are the models of the integrity constraint which are as close as possible to the profile consisting of the two sources of information, where the distance between two worlds is evaluated as the number of atomic facts on which they differ. Among the six models of the integrity constraint, the worlds 100, 101 and 111 are both models of one of the two sources, and at distance 1 from the other source (see Figure 1). The worlds 000, 010, 001 and 011 are both at distance 1 of one of the sources, and at distance 2 from the other source. Hence, using sum as aggregation function, we get that the three worlds 100, 101 and 111 are kept as models of the merged base. This corresponds to the models of the formula $\mu \wedge p_{LA}$ which characterize all possible scenarios where Lyon is north of Aix-en-Provence.

But it can be seen that whereas both sources provide the same 'amount' of information given the choice of encoding ($\varphi_1$ and $\varphi_2$ have two models each), $\varphi_1$ actually provides more precise information than $\varphi_2$ in the underlying domain. This is due to the transivity of the relation 'north of' encoded through the integrity constraint $\mu$. When Alice claims that Lyon is north of Aix-en-Provence ($p_{LA}$) and that Aix-en-Provence is north of Marseille ($p_{AM}$), she also implicitly states that Lyon is north of Marseille ($p_{LM}$). In comparison, by stating that Aix-en-Provence is the most southern city of the three, Bob clearly does not provide any information about the relative position of Lyon and Marseille. Then 100 is at distance 1 from $\varphi_1$ only because it is at distance 1 of the model 110 of $\varphi_1$, whereas

[1] In short, worlds are denoted as binary sequences following the ordering $p_{LA} < p_{AM} < p_{LM}$.

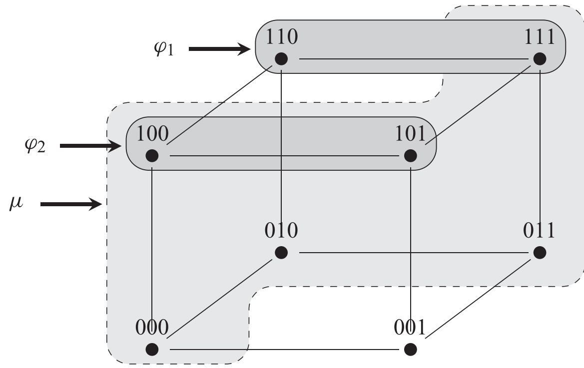


FIGURE 1. Graphical representation of the integrity constraint $\mu$ and the two sources ($\varphi_1$ and $\varphi_2$).

this model does not correspond to a conceivable world given the integrity constraint. Thus, it makes sense to disqualify this world. If we do it so that Alice now states that $p_{LA} \wedge p_{AM} \wedge p_{LM}$ (to be fully compatible with the integrity constraint), then the merged base obtained using the same belief merging operator corresponds to a formula equivalent to $p_{LA} \wedge p_{LM}$, i.e. it still states that Lyon is north of Aix-en-Provence (both sources agree on that), but it also states that Lyon is north of Marseille, which makes sense given that it can be derived from Alice's beliefs and it is not conflicting with Bob's beliefs.

As illustrated in this example, it may be desirable to 'rationalize' (i.e. refraining to take account for 'impossible' worlds) the input beliefs by the integrity constraint before merging them, in the context where these constraints encode the 'physical laws' of the considered domain. Indeed, such laws may not be explicitely provided by agents, because these laws are either common sense, or because the agents are not necessarily aware of the way the problem is encoded.

The 'infeasible' worlds considered in the above example may not be always inconceivable. Yet rationalization of the input beliefs may also be desirable, e.g. when the integrity constraint encodes some knowledge about the underlying domain.

EXAMPLE 1.2
One tries to seek in which city Charles lives. Bob believes that Charles lives in Rijsel, and Alice believes that Charles does not live in Lille. It turns out that the two names 'Lille' and 'Rijsel' refer to the same city ('Lille' is its French name and 'Rijsel' its Flemish name), and Bob knows it, and actually additionally mentions that Lille and Rijsel refer to the same city. If one chooses to represent both city names for the merging purpose, one can consider two propositions $p_L$ and $p_R$, respectively standing for 'Charles lives in Lille' and 'Charles lives in Rijsel'. Then the beliefs of Alice can be encoded as $\varphi'_1 = \neg p_L$, while Bob's ones can be expressed as $\varphi'_2 = p_L \wedge (p_L \Leftrightarrow p_R)$. Here, the integrity constraint expresses the fact that Lille and Rijsel refer to the same city, i.e. $\mu' = p_L \Leftrightarrow p_R$ (see Figure 2). Consider again the IC merging operator based on the Hamming distance and sum as aggregation function ($\Delta^{d_H,\Sigma}$) [15]. There are two possible worlds compatible with the integrity constraint, 00 (Charles does not live in Lille / Rijsel) and 11 (Charles lives in Lille / Rijsel). 00 is at distance 0 from $\varphi'_1$ (this world is a model of $\varphi'_1$) and at distance 2 from $\varphi'_2$. 11 is at distance 1 from $\varphi'_1$ (since only the fact that Charles lives in Lille conflicts with the information conveyed by Alice)

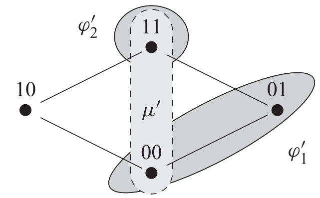


FIGURE 2. Graphical representation of the integrity constraint $\mu'$ and the two sources ($\varphi'_1$ and $\varphi'_2$).

and at distance 0 from $\varphi'_2$. Hence, using sum as aggregation function, we get that only the world 11 is kept so that the beliefs of the group are that Charles lives in Lille/Rijsel.

But 11 is at distance 1 from $\varphi'_1$ only because it is at distance 1 of the model 01 of $\varphi'_1$, whereas this model cannot correspond to a real-world situation given the integrity constraint. Thus, if we disqualify this world so that Alice's beliefs are rationalized w.r.t. the integrity constraint (i.e. one considers $\varphi'_1 \wedge \mu$ instead of $\varphi'_1$ in the merging process), then one cannot derive any conclusion from the merged base about the place where Charles lives. This is a more satisfactory result here: there is no reason to give more credit to Bob's beliefs only because he additionally mentioned some knowledge (the fact that 'Lille' and 'Rijsel' refer to the same city) that is irrelevant to the issue under consideration (the place where Charles lives).

In this paper, we do not make the distinction between the nature of those 'infeasible worlds' (i.e. those not satisfying the integrity constraint). We argue that in some situations, they should not play a role in the merging process, namely they should not have an impact on the resulting merged base. This calls for a new property for merging operators, which requires that when merging belief bases, the result should be equivalent to the one where every input belief base is rationalized with respect to the integrity constraint.

At this stage, it must be noted that many concepts of 'rationalization' have been considered in social sciences. Thus, in a decision making setting, a decision is rationalizable when it can be explained, for instance by pointing out a state-of-affairs under which it is the best decision (e.g. the one maximizing the utility of the decision maker) [4]. In voting theory, a voting rule is rationalizable whenever one can find a distance between the agents' preferences such that the collective preference generated by the voting rule is as close as possible w.r.t. this distance to the input profile of preferences [8]. Clearly enough, those notions of rationalization depart significantly from the one considered in this paper, which amounts to refrain from taking into account in a merging process the worlds which are irrelevant because they are 'infeasible'.

The contributions of the paper are as follows. We begin by introducing some preliminaries on belief merging, revision and update (Section 2). Then, in Section 3 we define a new class of so-called rationalization-driven merging operators. These operators preprocess as a first step the sources of information so as to fit the feasible worlds, i.e. the models of the integrity constraint; as a second step they take advantage of a standard merging operator applied to the rationalized sources of information to compute the merged result. Three types of rationalization are considered: rationalization by expansion, by revision and by update. The standard merging operators considered here are the IC merging operators [15]. As the IC operators assume consistent sources of information and a rationalization by expansion may lead the resulting sources of information to be inconsistent, we introduce the notion of EIC (for Extended IC) merging operator by considering in addition to the standard IC postulates a new, harmless condition, so that inconsistent sources can be properly handled. We show in Section 4 that investigating the extent to which rationalization-driven operators satisfy all IC postulates comes down to study the compatibility between the IC postulates and an 'independence to rationalization' postulate, one for each type of rationalization. We present an impossibility theorem when the rationalization step is update-based, and two characterization theorems for the two remaining rationalization techniques, i.e. expansion and revision. Lastly, Section 5 provides an independence-based axiomatic characterization of the distance-based operator based on the drastic distance and on the sum aggregation function ($\Delta^{d_D,\Sigma}$).

For the sake of readability, proofs of propositions are reported in a final appendix.

## 2 Formal preliminaries

We consider a propositional language $\mathcal{L}$ defined from a finite set of propositional variables $\mathcal{P}$ and the usual connectives. $\bot$ (resp. $\top$) is the Boolean constant always false (resp. true.)

An interpretation (or world) is a total function from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations is denoted $\mathcal{W}$. A world $I$ is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $mod(\phi)$ denotes the set of models of formula $\phi$, i.e. $mod(\phi) = \{I \in \mathcal{W} \mid I \models \phi\}$. Let $M$ be a set of worlds; $\varphi_M$ denotes a formula from $\mathcal{L}$ whose models are $M$.

### *2.1 Belief merging*

Belief merging aims at defining a belief base (the merged base) which represents the beliefs of a group of agents given their individual belief bases, and some integrity constraint.

A belief base (base for short) denotes the set of beliefs of an agent. In this work, a base is a finite set of propositional formulae, and its logical closure is the set of beliefs of the corresponding agent. Belief bases are interpreted conjunctively, which means that a base can always be considered as a single formula $\varphi$ up to logical equivalence. This formula is the conjunction of the elements of the base (it is well defined since a base contains finitely many elements).

A profile $\langle 1, \ldots, n\rangle$ is a (possibly empty) vector of agents involved in the merging process, where $n$ is any integer $n \geq 0$. A belief profile $\mathcal{K} = \langle \varphi_1, \ldots, \varphi_n\rangle$ is a (possibly empty) vector of bases, each base $\varphi_i$ representing the beliefs of agent $i$. When it is harmless, one usually does not distinguish the notions of profile and belief profile, i.e. each base is identified with the agent providing it, and the term 'profile' is used as a short for 'belief profile'. A profile is said to be p-consistent if it is a non-empty vector of consistent bases. We denote by $\mathcal{B}$ the set of all profiles, and we denote by $\mathcal{B}^\star$ the set of all p-consistent profiles. The integrity constraint is represented by a formula, often denoted by $\mu$ in this paper. The symbol $\sqcup$ denotes the concatenation of profiles, i.e. if $\mathcal{K}_1 = \langle \varphi_1, \ldots, \varphi_n\rangle$ and $\mathcal{K}_2 = \langle \varphi_{n+1}, \ldots, \varphi_{n+m}\rangle$, then $\mathcal{K}_1 \sqcup \mathcal{K}_2 = \langle \varphi_1, \ldots, \varphi_{n+m}\rangle$. The symbol $\equiv$ denotes the equivalence of profiles, i.e. two profiles are equivalent when there is a bijection between them so that each base from a profile is equivalent to its image in the other profile. Lastly, $\bigwedge \mathcal{K}$ denotes the conjunction of the belief bases of $\mathcal{K}$, i.e. $\bigwedge \mathcal{K} = \bigwedge\{\varphi_i \mid \varphi_i \in \mathcal{K}\}$, and given a base $\varphi$, the notation $\langle\varphi\rangle^n$ stands for the profile $\langle\underbrace{\varphi, \ldots, \varphi}_{n}\rangle$.

Let us formalize the two examples drafted in the introduction.

EXAMPLE 1.1 (continued).
We have $\mathcal{P} = \{p_{LA}, p_{AM}, p_{LM}\}$, when $p_{LA}$ stands for 'Lyon is north of Aix-en-Provence', $p_{AM}$ stands for 'Aix-en-Provence is north of Marseille' and $p_{LM}$ stands for 'Lyon is north of Marseille'; $\mathcal{K} =$

$\langle\varphi_1,\varphi_2\rangle$, with $\varphi_1 = p_{LA} \wedge p_{AM}$, $\varphi_2 = p_{LA} \wedge \neg p_{AM}$; $\mu = ((p_{LA} \wedge p_{AM}) \Rightarrow p_{LM}) \wedge ((\neg p_{AM} \wedge \neg p_{LA}) \Rightarrow \neg p_{LM})$.

EXAMPLE 1.2 (continued).
We have $\mathcal{P}' = \{p_L, p_R\}$ (when $p_L$ (resp. $p_R$) stands for 'Charles lives in Lille (resp. Rijsel)); $\mathcal{K}' = \langle\varphi'_1,\varphi'_2\rangle$, with $\varphi'_1 = \neg p_L$, $\varphi'_2 = p_L \wedge (p_L \Leftrightarrow p_R)$; $\mu' = p_L \Leftrightarrow p_R$.

A preorder $\leq$ is a reflexive and transitive relation, and $<$ is its strict counterpart, i.e $I < J$ if and only if $I \leq J$ and $J \not\leq I$. As usual, $\simeq$ is defined by $I \simeq J$ iff $I \leq J$ and $J \leq I$. A preorder is total if and only if $\forall I, J$, $I \leq J$ or $J \leq I$. A preorder that is not total is said to be partial.

The assumption that inconsistent belief bases do not provide any information for the merging process is standard, and as a consequence merging operators deal with p-consistent profiles (see [15, Remark 2]).

DEFINITION 2.1 (Merging operator).
A *merging operator* $\Delta$ is a mapping from $\mathcal{L} \times \mathcal{B}^\star$ to $\mathcal{L}$, i.e. it associates a formula $\mu$ (the integrity constraint) and a p-consistent profile $\mathcal{K}$ with a new base $\Delta_\mu(\mathcal{K})$ (the merged base).

Let us recall the standard logical properties which are expected for merging operators [15].

DEFINITION 2.2 (IC merging operator).
A merging operator $\Delta$ is an *IC merging operator* iff for any formulae $\mu, \mu_1, \mu_2$, for any p-consistent profiles $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and for any consistent belief bases $\varphi_1$, $\varphi_2$, it satisfies the following postulates:

(IC0) $\Delta_\mu(\mathcal{K}) \models \mu$;
(IC1) If $\mu$ is consistent, then $\Delta_\mu(\mathcal{K})$ is consistent;
(IC2) If $\bigwedge \mathcal{K} \wedge \mu$ is consistent, then $\Delta_\mu(\mathcal{K}) \equiv \bigwedge \mathcal{K} \wedge \mu$;
(IC3) If $\mathcal{K}_1 \equiv \mathcal{K}_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(\mathcal{K}_1) \equiv \Delta_{\mu_2}(\mathcal{K}_2)$;
(IC4) If $\varphi_1 \models \mu$, $\varphi_2 \models \mu$ and $\Delta_\mu(\{\varphi_1,\varphi_2\}) \wedge \varphi_1$ are consistent, then $\Delta_\mu(\{\varphi_1,\varphi_2\}) \wedge \varphi_2$ is consistent;
(IC5) $\Delta_\mu(\mathcal{K}_1) \wedge \Delta_\mu(\mathcal{K}_2) \models \Delta_\mu(\mathcal{K}_1 \sqcup \mathcal{K}_2)$;
(IC6) If $\Delta_\mu(\mathcal{K}_1) \wedge \Delta_\mu(\mathcal{K}_2)$ is consistent, then $\Delta_\mu(\mathcal{K}_1 \sqcup \mathcal{K}_2) \models \Delta_\mu(\mathcal{K}_1) \wedge \Delta_\mu(\mathcal{K}_2)$;
(IC7) $\Delta_{\mu_1}(\mathcal{K}) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(\mathcal{K})$;
(IC8) If $\Delta_{\mu_1}(\mathcal{K}) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(\mathcal{K}) \models \Delta_{\mu_1}(\mathcal{K}) \wedge \mu_2$.

We refer the reader to [15] for an intuitive explanation of these postulates.
Each IC merging operator can be characterized by a syncretic assignment [15].

DEFINITION 2.3 (Syncretic assignment).
A *syncretic assignment* is a mapping which associates with every p-consistent profile $\mathcal{K}$, a preorder $\leq_\mathcal{K}$ over worlds[2] and such that for every p-consistent profile $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and for every consistent belief base $\varphi_1$, $\varphi_2$, $\leq_\mathcal{K}$ satisfies the following conditions:[3]

(1) If $I \models \mathcal{K}$ and $J \models \mathcal{K}$, then $I \simeq_\mathcal{K} J$;
(2) If $I \models \mathcal{K}$ and $J \not\models \mathcal{K}$, then $I <_\mathcal{K} J$;

[2] For each preorder $\leq_\mathcal{K}$, $\simeq_\mathcal{K}$ denotes the corresponding indifference relation and $<_\mathcal{K}$ the corresponding strict ordering. When $\mathcal{K} = \langle\varphi\rangle$ consists of a single base $\varphi$, we write $\leq_\varphi$ instead of $\leq_{\langle\varphi\rangle}$ in order to alleviate the notations.
[3] See [15] for intuitions about these conditions.

(3) If $\mathcal{K}_1 \equiv \mathcal{K}_2$, then $\leq_{\mathcal{K}_1} = \leq_{\mathcal{K}_2}$;
(4) $\forall I \models \varphi_1, \exists J \models \varphi_2 \; J \leq_{\langle \varphi_1, \varphi_2 \rangle} I$;
(5) If $I \leq_{\mathcal{K}_1} J$ and $I \leq_{\mathcal{K}_2} J$, then $I \leq_{\mathcal{K}_1 \sqcup \mathcal{K}_2} J$;
(6) If $I <_{\mathcal{K}_1} J$ and $I \leq_{\mathcal{K}_2} J$, then $I <_{\mathcal{K}_1 \sqcup \mathcal{K}_2} J$.

THEOREM 2.1 ([15]).
A merging operator $\Delta$ is an IC merging operator iff there exists a syncretic assignment associating with every p-consistent profile $\mathcal{K}$ a total preorder $\leq_{\mathcal{K}}$ such that for every formula $\mu$, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$.

Several families of IC merging operators can be defined, including distance-based merging operators, i.e. operators characterized by a distance between worlds and an aggregation function $f$ (a mapping which associates with a tuple of non-negative real numbers a non-negative real number) [13].

DEFINITION 2.4 (Distance-based merging operator).
Let $d$ be a distance between worlds[4] and $f$ be an aggregation function. The *distance-based merging operator* $\Delta^{d,f}$ is defined for every p-consistent profile $\mathcal{K}$ and every formula $\mu$ by $mod(\Delta^{d,f}_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$, where the preorder $\leq_{\mathcal{K}}$ over worlds induced by $\mathcal{K} = \langle \varphi_1, \ldots, \varphi_n \rangle$ is defined by

- $I \leq_{\mathcal{K}} J$ if and only if $d(I, \mathcal{K}) \leq d(J, \mathcal{K})$,
- $d(I, \mathcal{K}) = f(\langle d(I, \varphi_1), \ldots, d(I, \varphi_n) \rangle)$ and
- $d(I, \varphi) = \min_{J \models \varphi} d(I, J)$.

In short, in the following we denote $f_{\varphi \in \mathcal{K}}(d(I, \varphi_i)) = f(\langle d(I, \varphi_1), \ldots, d(I, \varphi_n) \rangle)$.

Usual distances are the drastic distance ($d_D(I,J) = 0$ if $I = J$ and 1 otherwise), and the Hamming distance ($d_H(I,J) = n$ if $I$ and $J$ differ on $n$ variables.) Note that some distance-based operators are not IC merging ones. Indeed, conditions of non-decreasingness, minimality, identity, symmetry, composition and decomposition must be satisfied by $f$ in the general case [13, 17]. We recall the ones that will be used in the rest of this paper below and refer the reader to [13, 17] for details on these additional conditions.

DEFINITION 2.5 (Properties on aggregation functions).
An aggregation function $f$ satisfies

- (symmetry) iff for any permutation $\sigma$, $f(x_1, \ldots, x_n) = f(\sigma(x_1, \ldots, x_n))$;
- (composition) iff $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$ implies $f(x_1, \ldots, x_n, z) \leq f(y_1, \ldots, y_n, z)$;
- (decomposition) iff $f(x_1, \ldots, x_n, z) \leq f(y_1, \ldots, y_n, z)$ implies $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$;

Usual aggregation functions such as $\Sigma$, *Gmax* (*leximax*), etc. satisfy all necessary conditions and lead to IC merging operators [15]. Noteworthy, given a profile $\mathcal{K}$ and a formula $\mu$, $\Delta^{d_D,\Sigma}_\mu(\mathcal{K})$ is equivalent to any formula whose models are models of $\mu$ which are models of a maximal number of bases from $\mathcal{K}$.

[4] Actually, a pseudo-distance is enough, i.e. the triangular inequality is not necessary.

EXAMPLE 1.1 (continued).
Consider again our example about the relative location of three french cities, i.e. $\mathcal{K} = \langle \varphi_1, \varphi_2 \rangle$, with $\varphi_1 = p_{LA} \wedge p_{AM}$, $\varphi_2 = p_{LA} \wedge \neg p_{AM}$; $\mu = ((p_{LA} \wedge p_{AM}) \Rightarrow p_{LM}) \wedge ((\neg p_{AM} \wedge \neg p_{LA}) \Rightarrow \neg p_{LM})$. We have

- $\Delta_{\mu}^{d_H,\Sigma}(\mathcal{K}) \equiv \Delta_{\mu}^{d_D,\Sigma}(\mathcal{K}) \equiv p_{LA} \wedge (p_{AM} \Rightarrow p_{LM}) \equiv p_{LA} \wedge \mu$.

EXAMPLE 1.2 (continued).
Consider now our example with the place where Charles live, i.e. $\mathcal{K}' = \langle \varphi'_1, \varphi'_2 \rangle$, with $\varphi'_1 = \neg p_L$, $\varphi'_2 = p_L \wedge (p_L \Leftrightarrow p_R)$; $\mu' = p_L \Leftrightarrow p_R$. We have

- $\Delta_{\mu'}^{d_H,\Sigma}(\mathcal{K}') \equiv p_L \wedge p_R$;
- $\Delta_{\mu'}^{d_D,\Sigma}(\mathcal{K}') \equiv p_L \Leftrightarrow p_R$.

### 2.2 Belief revision

Belief revision consists in incorporating into an existing belief base a new evidence represented by a propositional formula. Alchourrón *et al*. [1] provided a logical characterization of belief revision by means of the so-called AGM postulates. In the AGM framework, agent's beliefs are represented as theories, i.e. sets of formulae which are deductively closed. Revising a belief set by a propositional formula results also results in a belief set. Here, we focus on Katsuno and Mendelzon's characterization of AGM belief revision over a finite propositional language, i.e. where an agent's beliefs are represented as a propositional formula (also called belief base) [11].

DEFINITION 2.6 (KM revision operator).
A *KM revision operator* $\circ$ is a mapping associating with a formula $\mu$ and a formula $\varphi$, a new formula $\varphi \circ \mu$, such that for every formula $\mu, \mu'$, for any consistent bases $\varphi, \varphi'$, it satisfies the following postulates:

(R1) $\varphi \circ \mu \models \mu$;
(R2) If $\varphi \wedge \mu$ is consistent, then $\varphi \circ \mu \equiv \varphi \wedge \mu$;
(R3) If $\mu$ is consistent, then $\varphi \circ \mu$ is consistent;
(R4) If $\varphi \equiv \varphi'$ and $\mu_1 \equiv \mu_2$, then $\varphi \circ \mu \equiv \varphi' \circ \mu'$;
(R5) $(\varphi \circ \mu) \wedge \mu' \models \varphi \circ (\mu \wedge \mu')$;
(R6) If $(\varphi \circ \mu) \wedge \mu'$ is consistent, then $\varphi \circ (\mu \wedge \mu') \models (\varphi \circ \mu) \wedge \mu'$.

As in the case of IC merging operators, KM revision operators can be characterized in terms of total preorders over worlds. Indeed, each KM revision operator can be associated with a faithful assignment [11].

DEFINITION 2.7 (Faithful assignment).
A *faithful assignment* is a mapping which associates with every formula $\varphi$ a preorder $\leq_\varphi$ over worlds and such that for every formulae $\varphi$, $\varphi_1$, $\varphi_2$, it satisfies the following conditions:

(1) If $I \models \varphi$ and $J \models \varphi$, then $I \simeq_\varphi J$;
(2) If $I \models \varphi$ and $J \not\models \varphi$, then $I <_\varphi J$;
(3) If $\varphi_1 \equiv \varphi_2$, then $\leq_{\varphi_1} = \leq_{\varphi_2}$.

THEOREM 2.2 ([10]).
A revision operator $\circ$ is a KM revision operator if and only if there exists a faithful assignment associating to every formula $\varphi$, a total preorder $\leq_K$ over worlds such that for every formula $\mu$,

$$mod(\varphi \circ \mu) = \min(mod(\mu), \leq_\varphi).$$

As in the belief merging case, some rational revision operators can be characterized using a notion of distance (but no aggregation function is needed here). Thus, Dalal's revision operator $\circ^{Dal}$ [6] is defined by considering the faithful assignment associating with $\varphi$ the total preorder $\leq_K^{Dal}$ defined by $I \leq_\varphi^{Dal} J$ iff the Hamming distance from $I$ to $\varphi$ (i.e. the Hamming distance between $I$ and the closest model of $\varphi$) is lower than or equal to the Hamming distance from $J$ to $\varphi$. Similarly, the drastic revision operator $\circ^D$ (full meet revision operator [1]) is defined by considering the faithful assignment associating with $\varphi$ the total preorder $\leq_\varphi^D$ defined by $I \leq_\varphi^D J$ iff the drastic distance from $I$ to $\varphi$ is lower than or equal to the drastic distance from $J$ to $\varphi$. Alternatively, the drastic revision operator $\circ^D$ can be defined as for every base $\varphi$ and every formula $\mu$,

$$\varphi \circ^D \mu = \begin{cases} \varphi \wedge \mu & \text{if } \varphi \wedge \mu \text{ is consistent,} \\ \mu & \text{otherwise.} \end{cases}$$

Finally, if $\Delta$ is an IC merging operator, then one can associate with it a KM revision operator $\circ_\Delta$.

DEFINITION 2.8 ($\circ_\Delta$).
Let $\Delta$ be a merging operator. Its corresponding revision operator, denoted $\circ_\Delta$, is given by $\varphi \circ_\Delta \mu = \Delta_\mu(\langle\varphi\rangle)$.

THEOREM 2.3 ([15]).
If $\Delta$ satisfies (IC0–IC3) and (IC7–IC8), then $\circ_\Delta$ is a KM revision operator (i.e. it satisfies (R1–R6)).

For instance, Dalal's revision operator $\circ^{Dal}$ (resp. the drastic revision operator $\circ^D$) corresponds to any distance-based merging operator $\Delta^{d_H,f}$ (resp. $\Delta^{d_D,f}$), for any aggregation function $f$ satisfying $f(\alpha) = \alpha$ for every $\alpha \in \mathbb{R}$.

## 2.3 Belief update

Another family of change operators that is considered in this paper consists of belief update operators [10]. Update operators perform a model-wise change, whereas belief revision operators make a change at the base level (see [10] for a discussion). Such operators are defined as follows.

DEFINITION 2.9 (KM update operator).
A *KM update operator* $\diamond$ is a mapping associating with a formula $\mu$ and a formula $\varphi$, a new formula $\varphi \diamond \mu$, such that for any formulae $\mu, \mu_1, \mu_2$, for any consistent fomulas $\varphi, \varphi_1, \varphi_2$, it satisfies the following postulates:

(U1) $\varphi \diamond \mu \models \mu$;
(U2) If $\varphi \models \mu$, then $\varphi \diamond \mu \equiv \varphi$;
(U3) If $\varphi$ is consistent and $\mu$ is consistent, then $\varphi \diamond \mu$ is consistent;
(U4) If $\varphi_1 \equiv \varphi_2$ and $\mu_1 \equiv \mu_2$, then $\varphi_1 \diamond \mu_1 \equiv \varphi_2 \diamond \mu_2$;
(U5) $(\varphi \diamond \mu_1) \wedge \mu_2 \models \varphi \diamond (\mu_1 \wedge \mu_2)$;
(U6) If $(\varphi \diamond \mu_1) \models \mu_2$ and $(\varphi \diamond \mu_2) \models \mu_1$, then $\varphi \diamond \mu_1 \equiv \varphi \diamond \mu_2$;

(U7) If $\varphi$ is a complete[5] formula, then $(\varphi \diamond \mu_1) \wedge (\varphi \diamond \mu_2) \models \varphi \diamond (\mu_1 \vee \mu_2)$;
(U8) $(\varphi_1 \vee \varphi_2) \diamond \mu \equiv (\varphi_1 \diamond \mu) \vee (\varphi_2 \diamond \mu)$;
(U9) If $\varphi$ is a complete formula and $(\varphi \diamond \mu_1) \wedge \mu_2$ is consistent, then $\varphi \diamond (\mu_1 \wedge \mu_2) \models (\varphi \diamond \mu_1) \wedge \mu_2$.

There is also a characterization theorem for KM update operators in terms of total preorders over worlds.

THEOREM 2.4 ([10]).
An update operator $\diamond$ is a KM update operator if and only if there exists a faithful assignment associating every world $I$ with a total preorder $\leq_{\varphi\{I\}}$ such that for every fomula $\varphi$ and formula $\mu$,

$$mod(\varphi \diamond \mu) = \bigcup_{I \models \varphi} \min(mod(\mu), \leq_{\varphi\{I\}}).$$

As in the case of belief revision, some rational update operators can be characterized using a notion of distance. Thus, Forbus' operator $\diamond^F$ [9] is defined by considering the faithful assignment associating with any world $I$ the preorder $\leq^{Dal}_{\varphi\{I\}}$ (thus, $\diamond^F$ is based on the Hamming distance between worlds). Similarly, the drastic update operator $\diamond^D$ is defined by considering the faithful assignment associating with $I$ the total preorder $\leq^{D}_{\varphi\{I\}}$ (thus, $\diamond^D$ is based on the drastic distance between worlds). The symbol $\diamond^D$ can also be defined as follows: for every base $\varphi$ and every formula $\mu$,

$$\varphi \diamond^D \mu \equiv \begin{cases} \varphi & \text{if } \varphi \models \mu, \\ \mu & \text{otherwise.} \end{cases}$$

Using the formal links between revision and update (see for instance [16]), one can also define the update operator $\diamond_\Delta$ induced from an IC merging operator $\Delta$.

DEFINITION 2.10 ($\diamond_\Delta$).
$\varphi \diamond_\Delta \mu = \bigvee_{I \models \varphi} \Delta_\mu(\langle \varphi_{\{I\}} \rangle)$.

A direct consequence of Theorems 5 and 6 from [16] is as follows.

THEOREM 2.5 ([16]).
If $\Delta$ is an IC merging operator (i.e. it satisfies (IC0–IC8)), then $\diamond_\Delta$ is a KM update operator (i.e. it satisfies (U1–U9)).

For instance, Forbus' operator $\diamond^F$ (resp. the drastic update operator $\diamond^D$) corresponds to any distance-based merging operator $\Delta^{d_H,f}$ (resp. $\Delta^{d_D,f}$), for any aggregation function $f$ satisfying $f(x) = x$ for every $x$.

From now on, when $\Delta$ denotes a merging operator, $\circ_\Delta$ and $\diamond_\Delta$ denote its corresponding revision operator and update operator, respectively.

## 3 Rationalization-driven merging operators

As argued in the introduction, in a merging scenario where the integrity constraint represent physical laws or norms it is often preferable to discard 'infeasible' worlds (i.e. interpretations that are not models of the integrity constraint) from the merging process. A simple way to achieve this is to

[5] A formula is complete if it admits exactly one model.

make as a first step an explicit rationalization of all the belief bases from the input profile $\mathcal{K}$ with respect to the integrity constraint $\mu$, so as to get a new profile $\mathcal{K}'$:

$$\mathcal{K}, \mu \xrightarrow{\textit{rationalization of } \mathcal{K} \textit{ w.r.t. } \mu} \mathcal{K}', \mu \xrightarrow{\textit{merging } \mathcal{K}' \textit{ under } \mu} \Delta_\mu(\mathcal{K}').$$

So the problem is to know if the corresponding obtained operator $\Delta'$ satisfies minimal rationality postulates for merging:

$$\mathcal{K}, \mu \xrightarrow{\textit{rationalization-driven merging of } \mathcal{K} \textit{ w.r.t. } \mu} \Delta'_\mu(\mathcal{K}) \qquad (1)$$

Rationalizing a belief base with respect to some integrity constraint can take various forms but such a modification fundamentally consists in modifying a base to fit the conceivable worlds according to the integrity constraint. Merging a profile using $\Delta'$ then consists in computing the same merged base as the one obtained using $\Delta$ by first 'removing' from every base the worlds not satisfying the integrity constraint.

Such an 'expansion-based' rationalizing process raises an issue for merging operators. Indeed, whenever a belief base is inconsistent with the integrity constraint, its rationalization by expansion leads to an inconsistent base, so that rationalized profiles may not be p-consistent anymore. Yet merging operators are mappings from $\mathcal{L} \times \mathcal{B}^\star$ to $\mathcal{L}$ (cf. Definition 2.1), i.e. they deal with p-consistent profiles only. To solve this problem, the domain of merging operators must be extended.

### *3.1 Extended merging operators*

DEFINITION 3.1 (Extended merging operator).
An *extended merging operator* $\Delta$ is a mapping from $\mathcal{L} \times \mathcal{B}$ to $\mathcal{L}$, i.e. it associates a formula $\mu$ and a profile $\mathcal{K}$ with a new base $\Delta_\mu(\mathcal{K})$.

Remark that the existing IC postulates (cf. Definition 2.2) do not rule the case when one has to deal with a profile containing inconsistent bases. In order to fix this problem, we slightly extend the IC postulates.

DEFINITION 3.2 (EIC merging operator).
An extended merging operator $\Delta$ is an *EIC merging operator* iff for any formulae $\mu$, $\mu_1$, $\mu_2$, for any profiles[6] $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}_2$ and for any consistent belief bases $\varphi_1$, $\varphi_2$, it satisfies (IC0–IC8) and the following additional postulate, for any $n \geq 0$:

(Inc) $\Delta_\mu(\langle\bot\rangle^n) \equiv \mu$.

According to (Inc), merging a (possibly empty) trivial profile consisting of one or several instances of the canonical inconsistent base $\bot$ must lead to a merged base equivalent to the integrity constraint. This postulate, which is not very demanding, echoes what is achieved by IC merging operators when dealing with a trivial profile consisting only of logically valid bases (indeed, (IC2) ensures that the merged base for such profiles is also equivalent to the integrity constraint). It is indeed natural to assume that a profile $\langle\bot\rangle^n$ ($n \geq 0$) does not provide more information than a profile $\langle\top\rangle^n$.

[6] Note that in contrast to the definition of usual IC merging operators (Definition 2.2), p-consistency of the profiles is not required here.

(Inc) as given in Definition 3.1 is given in a canonical form, in the sense that it tells how an extended merging operator should behave when merging trivial profiles, but it does not explicitly say anything when only some bases of the input profile are inconsistent. Nevertheless, extended merging operators satisfying (Inc) together with some (IC) postulates (in particular, EIC merging operators) lead to a merged base equivalent to the one obtained by first removing inconsistent bases from the profile. This is formally stated by the following proposition.

PROPOSITION 3.1
Let $\Delta$ be an extended merging operator satisfying (Inc), (IC0), (IC1), (IC5) and (IC6). Then for every profile $\mathcal{K}$, for every formula $\mu$ and for every $m \geq 0$,

$$\Delta_\mu(\mathcal{K} \sqcup \langle\bot\rangle^m) \equiv \Delta_\mu(\mathcal{K}).$$

Now, in order to derive a representation theorem for EIC operators, one needs the following notion of assignment.

DEFINITION 3.3 (Extended syncretic assignment).
An *extended syncretic assignment* is a mapping which associates with every profile $\mathcal{K}$ a preorder $\leq_\mathcal{K}$ over worlds which satisfies conditions (1–6) (cf. Definition 2.3) and the following additional condition, for every $n \geq 0$:

(0) $I \simeq_{\langle\bot\rangle^n} J$.

Then the standard representation theorem for IC merging operators can be extended to EIC operators.

PROPOSITION 3.2
An extended merging operator $\Delta$ is an EIC merging operator iff there exists an extended syncretic assignment associating every profile $\mathcal{K}$ with a total preorder $\leq_\mathcal{K}$ such that for every formula $\mu$, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_\mathcal{K})$.

Let us remark that when distance-based merging operators (cf. Definition 2.4) are applied to p-consistent profiles, the distance between a world and a base $\varphi$ is always defined. We now extend the definition of a distance-based merging operator so that inconsistent bases can be handled in a convenient way. To do so, it is enough to set the distance between any world and an inconsistent base to be 0.

DEFINITION 3.4 (Extended distance-based merging operator).
Let $d$ be a distance between worlds[7] and $f$ be an aggregation function. The *extended distance-based merging operator* $\Delta^{d,f}$ is defined for every profile $\mathcal{K}$ and every formula $\mu$ by $mod(\Delta^{d,f}_\mu(\mathcal{K})) = \min(mod(\mu), \leq_\mathcal{K})$, where the preorder $\leq_\mathcal{K}$ over worlds induced by $\mathcal{K}$ is defined by

- $I \leq_\mathcal{K} J$ if and only if $d(I,\mathcal{K}) \leq d(J,\mathcal{K})$,
- $d(I,\mathcal{K}) = f_{\varphi_i \in \mathcal{K}}(d(I,\varphi_i))$
- $d(I,\varphi) = \begin{cases} min_{J \models \varphi} d(I,J) & \text{If } \varphi \text{ is consistent,} \\ 0 & \text{otherwise.} \end{cases}$

By convention, we set $f_{\varphi_i \in \mathcal{K}}(\mathcal{K}) = 0$ when $\mathcal{K}$ is empty.

[7] As in Definition 2.4, a pseudo-distance is enough, i.e. the triangular inequality is not necessary.

It is easy to check that every extended distance-based merging operator satisfies (Inc).

Any merging operator defined on p-consistent profiles can be associated with an extended merging operator that satisfies (Inc) and preserves each IC property satisfied by $\Delta$.

DEFINITION 3.5 (Canonical extension of a merging operator).
Let $\Delta$ be a merging operator. The *canonical extension* of $\Delta$ is the extended merging operator $\Delta'$ defined for any profile $\mathcal{K}$ and any formula $\mu$ as

$$\Delta'_\mu(\mathcal{K}) = \begin{cases} \Delta_\mu(\mathcal{K}_\star), & \text{if } \mathcal{K} \text{ contains a consistent base,} \\ \mu, & \text{otherwise,} \end{cases}$$

where $\mathcal{K}_\star$ is the profile formed of the consistent bases from $\mathcal{K}$ given in the same order.

PROPOSITION 3.3
Let $\Delta$ be a merging operator and $\Delta'$ its canonical extension. Then $\Delta'$ satisfies (Inc), and for $x \in \{0,\dots,8\}$, if $\Delta$ satisfies (ICx) then $\Delta'$ satisfies (ICx).

This is why in the rest of the paper, for simplicity and without any harm, we will focus on canonical extensions of merging operators only and the term 'extended' will be omitted when referring to extended merging operators.

### 3.2 Expansion-based rationalization

We are now ready to introduce the definition of our *expansion-based rationalizing merging operators* $\Delta^E$.

DEFINITION 3.6 (Expansion-based rationalizing merging operator).
A merging operator $\Delta^E$ is said to be an *expansion-based rationalizing merging operator* if it satisfies (Inc) and if there exists a merging operator $\Delta$ such that if for any formula $\mu$, and any profile $\langle\varphi_1,\dots,\varphi_n\rangle$,

$$\Delta^E_\mu(\langle\varphi_1,\dots,\varphi_n\rangle) \equiv \Delta_\mu(\langle\varphi_1\wedge\mu,\dots,\varphi_n\wedge\mu\rangle).$$

We will note $\Delta^{E[\Delta]}$ the expansion-based rationalizing merging operator induced by $\Delta$.

Our running examples show that in the general case, $\Delta^{E[\Delta]}$ is 'different' from its associated merging operator $\Delta$. Consider for instance the merging operator $\Delta^{d_H,\Sigma}$ and its induced expansion-based rationalizing merging operator $\Delta^{E[\Delta^{d_H,\Sigma}]}$.

EXAMPLE 1.1 (continued).

- $\Delta^{d_H,\Sigma}_\mu(\langle\varphi_1,\varphi_2\rangle) \equiv p_{LA}\wedge(p_{AM}\Rightarrow p_{LM})$;
- $\Delta^{E[\Delta^{d_H,\Sigma}]}_\mu(\langle\varphi_1,\varphi_2\rangle) \equiv \Delta^{d_H,\Sigma}_\mu(\langle\varphi_1\wedge\mu,\varphi_2\wedge\mu\rangle) \equiv \Delta^{d_H,\Sigma}_\mu(\langle p_{LA}\wedge p_{AM}\wedge p_{LM}, p_{LA}\wedge\neg p_{AM}\rangle) \equiv p_{LA}\wedge p_{LM}$.

We can observe that $\Delta^{E[\Delta^{d_H,\Sigma}]}_\mu(\langle\varphi_1,\varphi_2\rangle) \not\equiv \Delta^{d_H,\Sigma}_\mu(\langle\varphi_1,\varphi_2\rangle)$.

EXAMPLE 1.2 (continued).

$$
\begin{aligned}
\text{- } \Delta_{\mu'}^{d_H,\Sigma}(\langle\varphi_1',\varphi_2'\rangle) &\equiv p_L \wedge p_R;\\
\text{- } \Delta_{\mu'}^{E[\Delta^{d_H,\Sigma}]}(\langle\varphi_1',\varphi_2'\rangle) &\equiv \Delta_{\mu'}^{d_H,\Sigma}(\langle\varphi_1' \wedge \mu',\varphi_2' \wedge \mu'\rangle)\\
&\equiv \Delta_{\mu'}^{d_H,\Sigma}(\langle\neg p_L \wedge \neg p_R, p_L \wedge p_R\rangle)\\
&\equiv p_L \Leftrightarrow p_R.
\end{aligned}
$$

Again here, we can see that $\Delta_{\mu'}^{E[\Delta^{d_H,\Sigma}]}(\langle\varphi_1',\varphi_2'\rangle) \not\equiv \Delta_{\mu'}^{d_H,\Sigma}(\langle\varphi_1',\varphi_2'\rangle)$.

In the general case, the fact that $\Delta$ is an EIC merging operator is not sufficient to ensure that its induced expansion-based rationalizing merging operator $\Delta^{E[\Delta]}$ satisfies all IC postulates.

PROPOSITION 3.4
Let $\Delta^{E[\Delta]}$ be the expansion-based rationalizing operator induced by $\Delta$. For $x \in \{0,\ldots,6\}$, if $\Delta$ satisfies (ICx) then $\Delta^{E[\Delta]}$ satisfies (ICx). This is not true in the general case for $x \in \{7,8\}$.

This result shows that expanding all bases from an input profile with the integrity constraint before merging them does not lead to a merging process satisfying all IC postulates. However, the situation is different for distance-based merging operators based on the drastic distance, when some conditions on the aggregation function are met.

PROPOSITION 3.5
For any aggregation function $f$ which satisfies (symmetry), (composition) and (decomposition), $\Delta^{d_D,f}$ is an expansion-based rationalizing merging operator, induced by itself.

As a consequence, since $\Delta^{d_D,\Sigma}$ satisfies all IC postulates [15], we get the following.

COROLLARY 3.1
$\Delta^{d_D,\Sigma}$ is an expansion-based rationalizing EIC merging operator.

### *3.3 Revision-based and update-based rationalization*

Rationalization by expansion could be considered as a rather drastic process; when merging a profile containing a belief base such that no model of it satisfies the integrity constraint, one simply removes this base from the profile as an upstream step of the merging process. A more cautious behavior would be to consider as still relevant the models of every base of the input profile, even when the base is inconsistent with the integrity constraint $\mu$; instead of removing such bases from the profiles, one could repair them. For this purpose, one can take advantage of belief change operators in order to derive, for each base inconsistent with $\mu$, the closest base that is fully compatible with $\mu$. Two kinds of belief change operators appear as valuable candidates in this objective; KM revision operators [1, 11] (cf. Definition 2.6) if one wants to repair the bases globally and KM update operators [10] (cf. Definition 2.9) if one wants to repair the bases locally, in a model-wise fashion. The corresponding rationalization-driven merging operators are defined as follows.[8]

DEFINITION 3.7 (Revision-based rationalizing merging operator).
A merging operator $\Delta^R$ is said to be a *revision-based rationalizing merging operator* if it satisfies (Inc), if one can associate with each agent $i$ a KM revision operator (cf. Definition 2.6) $\circ_i$ and if there

[8]Let us recall that each base of a profile corresponds to the beliefs of an agent.

exists a merging operator $\Delta$ such that for any formula $\mu$ and any profile $\langle\varphi_1,\dots,\varphi_n\rangle$,

$$\Delta^R_\mu(\langle\varphi_1,\dots,\varphi_n\rangle)\equiv\Delta_\mu(\langle\varphi_1\circ_1\mu,\dots,\varphi_n\circ_n\mu\rangle).$$

We denote by $\Delta^{R[\Delta,\langle\circ_1,\dots,\circ_n\rangle]}$ the revision-based rationalizing merging operator *induced by* $\Delta$ and $\langle\circ_1,\dots,\circ_n\rangle$.

DEFINITION 3.8 (Update-based rationalizing merging operator).
A merging operator $\Delta^U$ is said to be an *update-based rationalizing merging operator* if it satisfies (Inc), if one can associate with each agent $i$ a KM update operator $\diamond_i$ (cf. Definition 2.9) and if there exists a merging operator $\Delta$ such that for any formula $\mu$ and any profile $\langle\varphi_1,\dots,\varphi_n\rangle$,

$$\Delta^U_\mu(\langle\varphi_1,\dots,\varphi_n\rangle)\equiv\Delta_\mu(\langle\varphi_1\diamond_1\mu,\dots,\varphi_n\diamond_n\mu\rangle).$$

We denote by $\Delta^{U[\Delta,\langle\circ_1,\dots,\circ_n\rangle]}$ the update-based rationalizing merging operator *induced by* $\Delta$ and $\langle\diamond_1,\dots,\diamond_n\rangle$.

Note that in the above definitions, revision (resp. update) operators are KM ones, so in particular, they satisfy the postulate (R1) (resp. (U1)). They are used to 'repair' each base according to the integrity constraint, which is an essential characteristic of rationalization. Note that we do not impose any connection between the belief change operators $\circ_i$ (resp. $\diamond_i$) associated with agent $i$ and the merging operator $\Delta$ under consideration. In addition, we do not impose any homogeneity condition, i.e. the agents may have different revision/update policies. Hence, operators $\Delta^R$ and $\Delta^U$ depend not only on a merging operator $\Delta$ but also on the KM revision/KM update operators associated with each agent.

Now, as in the case of $\Delta^E$ operators, we investigate the extent to which postulates satisfied by $\Delta$ are also satisfied by the induced operators $\Delta^R$ and $\Delta^U$. It turns out that a proposition similar to Proposition 3.5 also holds for revision-based rationalizing merging operators, which means that there exists a revision-based rationalizing merging operator satisfying all IC postulates.

PROPOSITION 3.6
For any aggregation function $f$ which satisfies (symmetry), (composition) and (decomposition), $\Delta^{d_D,f}$ is a revision-based rationalizing merging operator, induced by itself and $\langle\circ_1,\dots,\circ_n\rangle$ with $\circ_1=\cdots=\circ_n=\circ^D$, where $\circ^D$ is the drastic (KM) revision operator.

As a direct consequence,

COROLLARY 3.2
$\Delta^{d_D,\Sigma}$ is a revision-based rationalizing EIC merging operator.

However, we can show that the case of revision-based rationalizing merging operators is similar to $\Delta^E$ operators in general, i.e. not every property satisfied by a merging operator can be 'transferred' to its induced revision-based rationalizing merging operator.

PROPOSITION 3.7
Let $\Delta^R$ be a revision-based rationalizing merging operator induced by $\Delta$ and $\circ_1,\dots,\circ_n$. For $x\in\{0,\dots,6\}$, if $\Delta$ satisfies (ICx) then $\Delta^R$ satisfies (ICx). This is not true in the general case for $x\in\{7,8\}$.

The results are even weaker for update-based rationalizing merging operators.

PROPOSITION 3.8
Let $\Delta^U$ be an update-based rationalizing merging operator induced by $\Delta$ and $\diamond_1, \ldots, \diamond_n$. For $x \in \{0, 1, 3, 4, 5, 6\}$, if $\Delta$ satisfies (ICx) then $\Delta^U$ satisfies (ICx). This is not true in the general case for $x \in \{2, 7, 8\}$.

Moreover, a direct adaptation of Corollary 3.2 to update-based rationalizing merging operators does not hold, as shown by the following examples:

EXAMPLE 1.1 (continued).
Consider the merging operator $\Delta^{d_D,\Sigma}$, let $\diamond_1 = \cdots = \diamond_n = \diamond^D$ where $\diamond^D$ is the drastic update operator, and let $\Delta^U$ be the update-based rationalizing merging operator induced by $\Delta^{d_D,\Sigma}$ and $\langle \diamond_1, \ldots, \diamond_n \rangle$. We have

$$\begin{aligned} &- \Delta_\mu^{d_D,\Sigma}(\langle \varphi_1, \varphi_2 \rangle) && \equiv p_{LA} \wedge (p_{AM} \Rightarrow p_{LM}); \\ &- \Delta_\mu^{U[\Delta^{d_D,\Sigma}]}(\langle \varphi_1, \varphi_2 \rangle) && \equiv \Delta_\mu^{d_D,\Sigma}(\langle \varphi_1 \diamond^D \mu, \varphi_2 \diamond^D \mu \rangle) \\ &&& \equiv \Delta_\mu^{d_D,\Sigma}(\langle \mu, \varphi_2 \rangle) \\ &&& \equiv \varphi_2 \equiv p_{LA} \wedge \neg p_{AM}. \end{aligned}$$

That is, $\Delta_\mu^{U[\Delta^{d_D,\Sigma}]}(\langle \varphi_1, \varphi_2 \rangle) \not\equiv \Delta_\mu^{d_D,\Sigma}(\langle \varphi_1, \varphi_2 \rangle)$.

EXAMPLE 1.2 (continued).
Consider again the merging operator $\Delta^{d_D,\Sigma}$, with $\diamond_1 = \cdots = \diamond_n = \diamond^D$, $\diamond^D$ being the drastic update operator, and let $\Delta^U$ be the update-based rationalizing merging operator induced by $\Delta^{d_D,\Sigma}$ and $\langle \diamond_1, \ldots, \diamond_n \rangle$:

$$\begin{aligned} &- \Delta_{\mu'}^{d_D,\Sigma}(\langle \varphi'_1, \varphi'_2 \rangle) && \equiv p_L \Leftrightarrow p_R; \\ &- \Delta_{\mu'}^{U[\Delta^{d_D,\Sigma}]}(\langle \varphi'_1, \varphi'_2 \rangle) && \equiv \Delta_{\mu'}^{d_D,\Sigma}(\langle \varphi'_1 \diamond^D \mu', \varphi'_2 \diamond^D \mu' \rangle) \\ &&& \equiv \Delta_{\mu'}^{d_D,\Sigma}(\langle p_L \Leftrightarrow p_R, p_L \wedge p_R \rangle) \\ &&& \equiv \varphi_1 \equiv p_L \wedge p_R. \end{aligned}$$

That is, $\Delta_{\mu'}^{U[\Delta^{d_D,\Sigma}]}(\langle \varphi'_1, \varphi'_2 \rangle) \not\equiv \Delta_{\mu'}^{d_D,\Sigma}(\langle \varphi'_1, \varphi'_2 \rangle)$.

Up to now, we have shown that performing a rationalization by expansion, revision or update of each base from an input profile does not lead to a merging process satisfying all IC conditions in the general case. We have also exhibited specific expansion-based and revision-based rationalizing operators that satisfy (Inc) all IC properties (the rationalizing operators induced by the merging operator based on the drastic distance and the sum function). We are interested now in characterizing the class of *all* expansion/revision/update-based rationalizing operators that satisfy all IC postulates. This is our goal in the following sections.

## 4 Independence to rationalization

In this section we investigate the compatibility of the rationalization-driven merging operators introduced in the previous section, i.e. the expansion-based, revision-based and update-based rationalizing merging operators, with respect to the IC postulates. We start by establishing a connection between expansion-based and revision-based rationalizing operators. Then, we introduce an incompatibility result in the case of rationalization by update. This will be done by taking advantage of independence to rationalization postulates, corresponding to the three considered types of rationalization.

### *4.1 Rationalization by expansion and revision*

We start with rationalization by expansion. Let us stress that each expansion-based rationalizing operator considered in Proposition 3.5 coincides with the one characterizing it: merging operators $\Delta^{d_D,f}$ based on the drastic distance $d_D$ and an aggregation function $f$ satisfying some basic conditions are induced by themselves. Formally, this precisely means that for every profile $\langle\varphi_1,\ldots,\varphi_n\rangle$ and every formula $\mu$, we have that

$$\Delta_\mu^{E[\Delta^{d_D,f}]}(\langle\varphi_1,\ldots,\varphi_n\rangle)\equiv\Delta_\mu^{d_D,f}(\langle\varphi_1,\ldots,\varphi_n\rangle)\equiv\Delta_\mu^{d_D,f}(\langle\varphi_1\wedge\mu,\ldots,\varphi_n\wedge\mu\rangle).$$

In other words, the operators $\Delta^{d_D,f}$ satisfy a certain condition of independence to rationalization by expansion, which states that merging a profile should lead to the same merged base (modulo logical equivalence) as the one obtained by expanding every base with the integrity constraint. This independence condition is formalized by the following postulate.

DEFINITION 4.1
(Ind) A merging operator $\Delta$ satisfies (Ind) iff for every formula $\mu$ and for every profile $\langle\varphi_1,\ldots,\varphi_n\rangle$,

$$\Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle)\equiv\Delta_\mu(\langle\varphi_1\wedge\mu,\ldots,\varphi_n\wedge\mu\rangle).$$

(Ind) is close to the property of independence of irrelevant alternatives condition (IIA) considered in social choice theory [2, 3] for aggregating preference relations. Condition (IIA) states that the (aggregated) preference between two alternatives depends only on the preferences of the individuals on these two alternatives, and not on the preferences with respect to other alternatives. For voting rules (IIA) can be expressed as the fact that two preference profiles which coincide when projected onto a given agenda should always lead to the same winner [12]. In our belief merging setting, the set of models of the integrity constraint plays the role of the agenda for voting. Accordingly, an (IIA) condition in the belief merging setting can be stated as follows.

DEFINITION 4.2
(IIA) A merging operator $\Delta$ satisfies (IIA) iff for every formula $\mu$ and for every profiles $\langle\varphi_1,\ldots,\varphi_n\rangle$, $\langle\varphi'_1,\ldots,\varphi'_n\rangle$: if $\langle\varphi_1\wedge\mu,\ldots,\varphi_n\wedge\mu\rangle\equiv\langle\varphi'_1\wedge\mu,\ldots,\varphi'_n\wedge\mu\rangle$, then $\Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle)\equiv\Delta_\mu(\langle\varphi'_1,\ldots,\varphi'_n\rangle)$.

Clearly, our (Ind) condition is equivalent to (IIA) for syntax-insensitive merging operators. Formally,

PROPOSITION 4.1
If $\Delta$ is a merging operator satisfying (IC3), then $\Delta$ satisfies (Ind) if and only if $\Delta$ satisfies (IIA).

Then, a direct consequence of Proposition 3.5 and Corollary 3.1 is that (Ind) is compatible with all IC properties.

COROLLARY 4.1
$\Delta^{d_D,\Sigma}$ is an EIC merging operator satisfying (Ind).

And of course, for syntax-insensitive merging operators (those satisfying (IC3)), being an expansion-based rationalizing operator ensures to satisfy (Ind).

PROPOSITION 4.2
Let $\Delta$ be a merging operator satisfying (IC3). If $\Delta$ is an expansion-based rationalizing merging operator, then it satisfies (Ind).

Let us now consider the case of rationalization by revision. We first introduce the independence property corresponding to rationalization by revision.

DEFINITION 4.3
(Ind-$\circ$) A merging operator $\Delta$ satisfies (Ind-$\circ$) iff one can associate with each agent $i$ a KM revision operator $\circ_i$ such that for every formula $\mu$ and for every profile $\langle\varphi_1,\ldots,\varphi_n\rangle$,

$$\Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle)\equiv\Delta_\mu(\langle\varphi_1\circ_1\mu,\ldots,\varphi_n\circ_n\mu\rangle).$$

As for $\Delta^E$ operators (Proposition 4.2), being a syntax-insensitive revision-based rationalizing operator implies satisfying (Ind).

PROPOSITION 4.2
Let $\Delta$ be a merging operator satisfying (IC3). If $\Delta$ is a revision-based rationalizing merging operator, then it satisfies (Ind-$\circ$).

An interesting issue now concerns the set of admissible 'rationalizing' revision operators which characterize a revision-based rationalizing merging operator. From Proposition 4.3 this boils down to looking at what revision operators should be associated with each agent so that (Ind-$\circ$) holds. Actually, this choice is very constrained.

PROPOSITION 4.3
Let $\Delta$ be a merging operator satisfying (IC2) and (Ind-$\circ$). Then every revision operator $\circ_i$ considered in (Ind-$\circ$) is the revision operator $\circ_\Delta$ corresponding to $\Delta$ in the sense of Definition 2.8.

The choice of such a revision operator $\circ_\Delta$ is even more restrained for EIC merging operators.

PROPOSITION 4.4
Let $\Delta$ be an EIC merging operator satisfying (Ind-$\circ$). Then every revision operator $\circ_i$ is $\circ^D$, the drastic revision operator.

As a noticeable consequence of this proposition, we have the following.

PROPOSITION 4.5
Let $\Delta$ be an EIC merging operator. $\Delta$ satisfies (Ind) if and only if $\Delta$ satisfies (Ind-$\circ$).

This last result shows that for EIC merging operators the two notions of independence (Ind) and (Ind-$\circ$) coincide, or equivalently speaking, that the sets of EIC expansion-based rationalizing merging operators and EIC revision-based rationalizing merging operators are the same ones.

### *4.2 Rationalization by update: an incompatibility result*

Let us now focus on rationalization by update. We are going to show that one of the most basic IC postulates is incompatible with it. First, let us introduce the independence property corresponding to rationalization by update.

DEFINITION 4.4
(Ind-$\diamond$) A merging operator $\Delta$ satisfies (Ind-$\diamond$) iff one can associate with each agent $i$ a KM update operator $\diamond_i$ such that for every formula $\mu$ and for every profile $\langle\varphi_1,\ldots,\varphi_n\rangle$,

$$\Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle) \equiv \Delta_\mu(\langle\varphi_1 \diamond_1 \mu,\ldots,\varphi_n \diamond_n \mu\rangle).$$

As in the case of $\Delta^E$ and $\Delta^R$ operators, the postulate (Ind-$\diamond$) is satisfied by every syntax-insensitive update-based rationalizing merging operator:

PROPOSITION 4.6
Let $\Delta$ be a merging operator satisfying (IC3). If $\Delta$ is an update-based rationalizing merging operator, then it satisfies (Ind-$\diamond$).

Whereas no restriction is imposed on the choice of KM update operator associated with each agent, it turns out that there is no EIC merging operator satisfying (Ind-$\diamond$). More precisely, the property of independence to rationalization by update is not compatible with (IC2), which means that every update-based rationalizing merging operator violates one of the most basic IC postulates.

PROPOSITION 4.7
There is no merging operator satisfying (IC2) and (Ind-$\diamond$).

In this section we have shown that (i) the postulate (Ind) is consistent with all IC postulates (Corollary 4.1); (ii) the postulate (Ind-$\circ$) expresses a condition equivalent to (Ind) in the presence of all IC postulates (Corollary 14); (iii) the postulate (Ind-$\diamond$) is incompatible with one of the most basic IC postulates (Proposition 4.7). For these reasons, we only focus on (Ind) in the rest of the paper.

## 5 Rationalization by expansion: a characterization result

Corollary 3.1 states that the set of EIC merging operators satisfying (Ind) is not empty, by showing that $\Delta^{d_D,\Sigma}$ belongs to it. A key question is then to determine what are exactly the IC merging operators (not necessarily distance-based ones) satisfying (Ind). In the following, we give a representation theorem which answers this question. This requires a notion of *filtering* assignments, considering the conditions 0–6 of an extended syncretic assignment together with an additional condition.

DEFINITION 5.1 (Filtering assignment).
A *filtering assignment* is an extended syncretic assignment satisfying the following condition, for every belief base $\varphi_1,\varphi_2$:

(F) If $I <_{\varphi_1} J$ and $J <_{\varphi_2} I$, then $I \simeq_{\langle\varphi_1,\varphi_2\rangle} J$.

Condition (F) states that if a world is viewed as strictly more plausible than another world for a singleton profile, and the plausibility ordering is reversed for another singleton profile, then these worlds must be considered equally plausible with respect to the joint profile. Stated otherwise, when condition (F) holds together with conditions (1) and (2) (cf. Definition 2.3), it is sufficient to compare the plausibility of two distincts worlds $I,J$ with respect to two singleton profiles $\langle\varphi_1\rangle,\langle\varphi_2\rangle$ in order to determine the relative plausibility of $I$ and $J$ with respect to the doubleton profile $\langle\varphi_1,\varphi_2\rangle$.

Observe that condition (F) can be viewed as a stronger version of condition (4) (cf. Definition 2.3) in the presence of conditions (1) and (2).

PROPOSITION 5.1
Every mapping associating with every profile $\mathcal{K}$ a preorder $\leq_{\mathcal{K}}$ over worlds and satisfying conditions (1), (2) and (F) also satisfies condition (4).

Indeed, the additional constraint expressed by condition (F) with respect to condition (4) can be illustrated as follows. Consider three pairwise distinct models $I, J$and $L$ and two belief bases $\varphi_1 \equiv \varphi_{\{I,J\}}$ and $\varphi_2 \equiv \varphi_{\{I,L\}}$. When (1) and (2) hold, we have $I \simeq_{\varphi_1} J <_{\varphi_1} L$ and $I \simeq_{\varphi_2} L <_{\varphi_2} J$. Targeting an equity behavior, condition (4) alone does not require $J$ and $L$ to be equally plausible with respect to the profile $\langle \varphi_1, \varphi_2 \rangle$: we could have for instance $J <_{\langle \varphi_1, \varphi_2 \rangle} L$. Contrastingly, in such a case, condition (F) implies that $J \simeq_{\langle \varphi_1, \varphi_2 \rangle} L$.

Now, the following proposition shows that through a filtering assignment, all the worlds are ranked over a plausible ordering with at most two levels for any singleton profile $\langle \varphi \rangle$.

PROPOSITION 5.2
Every mapping associating with every profile $\mathcal{K}$ a preorder $\leq_{\mathcal{K}}$ over worlds and satisfying conditions (1), (2), (6) and (F) maps every singleton belief profile $\langle \varphi \rangle$ to a unique total preorder $\leq_{\varphi}$ over worlds defined for all worlds $I, J$ and every belief base $\varphi$ as $I <_{\varphi} J$ if and only if $I \models \varphi$ and $J \not\models \varphi$.

Proposition 5.2 is a key result to prove the following stronger result on filtering assignments. Let us denote $|I(\mathcal{K})| = |\{\varphi_i \in \mathcal{K} \mid I \models \varphi_i\}|$, i.e. the number of belief bases in $\mathcal{K}$ for which $I$ is a model. The following proposition holds.

PROPOSITION 5.3
Let $\leq_{\mathcal{K}}$ be the preorder over worlds associated with a profile $\mathcal{K}$ by a filtering syncretic assignment. We have $I <_{\mathcal{K}} J$ iff $|I(\mathcal{K})| > |J(\mathcal{K})|$.

An important consequence of Proposition 5.3 is the following representation theorem for EIC merging operators satisfying (Ind).

PROPOSITION 5.4
An EIC merging operator $\Delta$ satisfies (Ind) iff there exists a filtering syncretic assignment associating every profile $\mathcal{K}$ with a total preorder $\leq_{\mathcal{K}}$ such that for every formula $\mu$, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$.

Another consequence of Proposition 5.3 is given by the following corollary.

COROLLARY 5.1
There is only one filtering syncretic assignment.

Let us recall that given a profile $\mathcal{K}$ and a formula $\mu$, $\Delta_\mu^{d_D,\Sigma}(\mathcal{K})$ is equivalent to any formula whose models are models of $\mu$ which are models of a maximal number of bases from $\mathcal{K}$. Then, as a consequence of Corollaries 4.1 and 5.1 and Proposition 5.4, we get the following.

PROPOSITION 5.5
$\Delta^{d_D,\Sigma}$ is the only EIC merging operator satisfying (Ind).

This result gives a full axiomatic characterization of the EIC distance-based operator $\Delta^{d_D,\Sigma}$ in terms of independence to rationalization, or equivalently speaking, it shows that $\Delta^{d_D,\Sigma}$ is the only EIC merging operator which is also an expansion/revision-based rationalizing one.

Before concluding this section, let us point out an interesting relationship between our independence postulate (Ind) and the independence of irrelevant alternatives postulate (ESF-I) recently introduced in [7] when merging *epistemic states*. Postulate (ESF-I) requires the merging process to depend only on how the restrictions in the individual epistemic states are related. Though this postulate is stated in the case where the input profile consists of epistemic states (instead of propositional formulae), it puts conditions on the most 'entrenched beliefs' of the underlying epistemic states, and thus can be easily rephrased in our framework as follows.

DEFINITION 5.2 ((ESF-I) [7]).
A merging operator $\Delta$ satisfies (ESF-I) iff or all profiles $\mathcal{K} = \langle \varphi_1, \ldots, \varphi_n \rangle$, $\mathcal{K}' = \langle \varphi'_1, \ldots, \varphi'_n \rangle$ and for every formula $\mu$, if for every formula $\mu'$ such that $\mu' \models \mu$ we have that $\varphi_i \circ_\Delta \mu' \equiv \varphi'_i \circ_\Delta \mu'$ for each $i \in \{1, \ldots, n\}$, then we also have $\Delta_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K}')$.

PROPOSITION 5.6
Any extended distance-based merging operator $\Delta^{d,\Sigma}$ satisfies (ESF-I).

This shows that

- any EIC merging operator satisfying (Ind) also satisfies (ESF-I), since $\Delta^{d_D,\Sigma}$ is the only EIC merging operator satisfying (Ind) (cf. Corollary 21), and since it also satisfies (ESF-I) (cf. Proposition 5.6);
- the converse does not hold, since the EIC distance-based merging operator $\Delta^{d_H,\Sigma}$ does not satisfy (Ind) but satisfies (ESF-I) (cf. Proposition 5.6).

As a consequence, (ESF-I) is strictly less demanding than (Ind) for EIC merging operators.

## 6 Conclusion

In this paper, we have studied the case when belief bases are rationalized with respect to the integrity constraint during the merging process. Such a rationalization is expected in scenarios for which some IC merging operators can lead to unexpected merged bases because they give too much credit to infeasible worlds. This is especially important when the integrity constraint encodes strong constraints such as physical laws. In particular, when the propositional formulae are obtained via a translation from representations coming from a more expressive framework (such as qualitative temporal or spatial settings), the integrity constraint must be used to rule out infeasible worlds (the worlds created during the translation process) [5].

We have defined in formal terms several independence conditions for merging operators and studied how they interact with the standard IC postulates for belief merging. Especially, since rationalization by expansion may lead to inconsistent bases, we have extended the IC postulates with a new axiom (Inc) which constrains the behavior of merging operators applied to profiles consisting of inconsistent bases; we gave a representation theorem for the augmented set of postulates, called EIC, where the p-consistency condition on the profiles is relaxed. Then, we have shown that rationalization by update is impossible for EIC operators, since this independence property conflicts with some basic IC postulates. We have also shown that independence to rationalization by revision is equivalent to independence to rationalization by expansion for EIC operators. Finally, we have proven that there is a unique EIC operator satisfying the independence property to rationalization by expansion (or equivalently, by revision), namely the distance-based operator $\Delta^{d_D,\Sigma}$, where the drastic distance $d_D$ and sum as aggregation function are used.

In [18] Marquis and Schwind provided an alternative axiomatic characterization of the distance-based merging operator $\Delta^{d_D,\Sigma}$ in terms of *language* independence. They showed that $\Delta^{d_D,\Sigma}$ is the only IC merging operator $\Delta$ for which, whenever the representation language is modified in such a way that symbols in the original language correspond to formulae in the target language, if the input of $\Delta$ is modified so as to reflect the language change, then the output of $\Delta$ should be modified accordingly. In their framework, language change is modeled by symbol translations (i.e. substitutions) and language independence for a merging operator is its ability to be robust with respect to symbol translations. It is interesting to note that $\Delta^{d_D,\Sigma}$ is the only operator that fits these problems of rationalization and of independence of the choice of propositional language. So an interesting question is to determine whether these two notions are more deeply linked.

## Acknowledgements

This is a revised and extended version (including proofs of propositions given in a final appendix) of [14].

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## Appendix: Proofs

We introduce a few notations that are used in some proofs for convenience. Given a formula $\mu$ and a profile $\mathcal{K} = \langle \varphi_1, \ldots, \varphi_n \rangle$,

- $\mathcal{K}_\mu$ denotes the (possibly empty) vector formed of the bases from the profile $\mathcal{K}$, which are consistent with $\mu$, listed in the same order;
- $\mathcal{K}^{\wedge\mu}$ denotes the profile $\mathcal{K}^{\wedge\mu} = \langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle$;
- $\mathcal{K}^{\wedge\mu}_\mu$ is a shortcut for $(\mathcal{K}^{\wedge\mu})_\mu$;
- $\mathcal{K}^{\circ^D\mu}$ denotes the profile $\mathcal{K}^{\circ^D\mu} = \langle \varphi_1 \circ^D \mu, \ldots, \varphi_n \circ^D \mu \rangle$, where $\circ^D$ is the drastic revision operator.

For instance, if $\mu = a \wedge b$ and $\mathcal{K} = \langle a, \neg a \vee \neg b, b \rangle$, then $\mathcal{K}_\mu = \langle a, b \rangle$, $\mathcal{K}^{\wedge\mu} = \langle a \wedge b, \bot, a \wedge b \rangle$, $\mathcal{K}^{\wedge\mu}_\mu = \langle a \wedge b, a \wedge b \rangle$ and $\mathcal{K}^{\circ^D\mu} = \langle a \wedge b, a \wedge b, a \wedge b \rangle$.

PROPOSITION A.1
Let $\Delta$ be an extended merging operator satisfying (Inc), (IC0), (IC1), (IC5) and (IC6). Then for every profile $\mathcal{K}$, for every formula $\mu$ and for every $m \geq 0$,

$$\Delta_\mu(\mathcal{K} \sqcup \langle \bot \rangle^m) \equiv \Delta_\mu(\mathcal{K}).$$

PROOF. Let $\Delta$ be a merging operator satisfying (Inc), (IC0), (IC1), (IC5) and (IC6). Let $\mathcal{K}$ be a profile, $\mu$ be a formula and assume that $\mu$ is consistent (if $\mu$ is inconsistent, then the proof trivially follows from (IC0)). By (Inc) we have $\Delta_\mu(\langle \bot \rangle^m) \equiv \mu$. Yet, by (IC0) we have $\Delta_\mu(\mathcal{K}) \models \mu$. Hence, $\Delta_\mu(\mathcal{K}) \wedge \Delta_\mu(\langle \bot \rangle^m) \equiv \Delta_\mu(\mathcal{K})$, which is consistent by (IC1). Then by (IC5) and (IC6), we get $\Delta_\mu(\mathcal{K}) \wedge \Delta_\mu(\langle \bot \rangle^m) \equiv \Delta_\mu(\mathcal{K} \sqcup \langle \bot \rangle^m)$. Hence, $\Delta_\mu(\mathcal{K} \sqcup \langle \bot \rangle^m) \equiv \Delta_\mu(\mathcal{K})$. □

PROPOSITION A.2
An extended merging operator $\Delta$ is an EIC merging operator iff there exists an extended syncretic assignment associating every profile $\mathcal{K}$ with a total preorder $\leq_\mathcal{K}$ such that for every formula $\mu$, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_\mathcal{K})$.

PROOF. *(Only If)* Let $\Delta$ be an EIC merging operator. Let us consider the assignment mapping every profile $\mathcal{K}$ to a binary relation $\leq_\mathcal{K}$ over worlds, defined for all worlds $I, J$ as $I \leq_\mathcal{K} J$ iff $I \models \Delta_{\varphi_{\{I,J\}}}(\mathcal{K})$. The proof of Theorem 11 from [15] can be directly adapted to profiles here to show that $\leq_\mathcal{K}$ is a total preorder, that $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_\mathcal{K})$, and that it satisfies conditions 1–6 of a syncretic assignment. And by (Inc), for all worlds $I, J$, we have $\Delta_{\varphi_{\{I,J\}}}(\langle \bot \rangle^n) \equiv \varphi_{\{I,J\}}$, so $I \simeq_{\langle \bot \rangle^n} J$. Hence, condition 0 of an extended syncretic assignment (cf. Definition 3.3) is satisfied. *(If)* Consider an extended syncretic assignment mapping every profile $\mathcal{K}$ to a preorder $\leq_{\mathcal{K}}$ over worlds, and define the extended merging operator $\Delta$ by $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$, for every profile $\mathcal{K}$ and every formula $\mu$. One can directly adapt the proof of Theorem 11 from [15] to show that $\Delta$ satisfies (IC0–IC8). Yet by (IC0) we have $\Delta_\mu(\langle\bot\rangle^n) \models \mu$, and by condition 0 of an extended syncretic assignment and by definition of $\Delta$, we have that for every world $I \models \mu$, $I \models \Delta_\mu(\langle\bot\rangle^n)$. Hence, $\Delta$ satisfies (Inc). Therefore, $\Delta$ is an EIC merging operator. □

PROPOSITION A.3

Let $\Delta$ be a merging operator and $\Delta'$ its canonical extension. Then $\Delta'$ satisfies (Inc), and for $x \in \{0, \dots, 8\}$, if $\Delta$ satisfies (ICx) then $\Delta'$ satisfies (ICx).

PROOF. The proof is obvious; the fact that the canonical extension $\Delta'$ of a merging operator satisfies (Inc) each one of the IC postulates can be verified by definition of $\Delta'$. □

PROPOSITION A.4

Let $\Delta^{E[\Delta]}$ be the expansion-based rationalizing operator induced by $\Delta$. For $x \in \{0, \dots, 6\}$, if $\Delta$ satisfies (ICx) then $\Delta^{E[\Delta]}$ satisfies (ICx). This is not true in the general case for $x \in \{7, 8\}$.

PROOF. Let $\Delta^E$ be an expansion-based rationalizing merging operator and $\Delta$ be its characterizing merging operator. In the following, $\mathcal{K}$ denotes any p-consistent profile $\langle\varphi_1, \dots, \varphi_n\rangle$, $\mu$ is any formula, $\varphi_1, \varphi_2$ are two consistent bases such that $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$ and $\mathcal{K}^{\wedge\mu}$ denotes the profile $\langle\varphi_1 \wedge \mu, \dots, \varphi_n \wedge \mu\rangle$. Assume $\Delta$ satisfies

- (IC0): then $\Delta^E_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K}^{\wedge\mu})$, and $\Delta_\mu(\mathcal{K}^{\wedge\mu}) \models \mu$ by (IC0), so $\Delta^E$ also satisfies (IC0).
- (IC1): assume $\mu$ is consistent. Then $\Delta^E_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K}^{\wedge\mu})$, which is consistent by (IC1), so $\Delta^E$ also satisfies (IC1).
- (IC2): assume $\mathcal{K} \wedge \mu$ is consistent. Then $\Delta^E_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K}^{\wedge\mu})$. Yet $\mathcal{K}^{\wedge\mu} \wedge \mu \equiv \mathcal{K} \wedge \mu$, thus $\mathcal{K}^{\wedge\mu} \wedge \mu$ is consistent, so by (IC2) $\Delta_\mu(\mathcal{K}^{\wedge\mu}) \equiv \mathcal{K}^{\wedge\mu} \wedge \mu \equiv \mathcal{K} \wedge \mu$. We get that $\Delta^E_\mu(\mathcal{K}) \equiv \mathcal{K} \wedge \mu$, so $\Delta^E$ also satisfies (IC2).
- (IC3): since $\mathcal{K}_1 \equiv \mathcal{K}_2$, we have $\mathcal{K}_1^{\wedge\mu} \equiv \mathcal{K}_2^{\wedge\mu}$, from which it follows that $\Delta^E$ also satisfies (IC3).
- (IC4): since $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$, we have $\Delta^E_\mu(\langle\varphi_1, \varphi_2\rangle) \equiv \Delta_\mu(\langle\varphi_1, \varphi_2\rangle)$, so $\Delta^E$ also satisfies (IC4).
- (ICx), $x \in \{5, 6\}$: we have $\Delta^E_\mu(\mathcal{K}_1) \wedge \Delta^E_\mu(\mathcal{K}_2) \equiv \Delta_\mu(\mathcal{K}_1^{\wedge\mu}) \wedge \Delta_\mu(\mathcal{K}_2^{\wedge\mu})$, and $\Delta^E_\mu(\mathcal{K}_1 \sqcup \mathcal{K}_2) \equiv \Delta_\mu(\mathcal{K}_1^{\wedge\mu} \sqcup \mathcal{K}_2^{\wedge\mu})$. Hence, since $\Delta$ satisfies (ICx), $\Delta^E$ also satisfies (ICx).

The following example shows that (IC7) (respectively, (IC8)) is not necessarily satisfied by $\Delta^E$ even if $\Delta$ satisfies it. Let $\Delta^{d_H,\Sigma}$ be the merging operator based on the Hamming distance and the sum function, and let $\Delta^E$ be the expansion-based rationalizing merging operator it characterizes. We know that $\Delta^{d_H,\Sigma}$ satisfies (IC7) and (IC8) (see [15, Theorem 20]). Let $\mu_1 = \top$, $\mu_2 = a \vee b$ and $\varphi_1 = \neg a$;

- Let $\varphi_2 = a$. On the one hand, $\Delta^E_{\mu_1}(\langle\varphi_1, \varphi_2\rangle) \equiv \Delta^E_\top(\langle\neg a, a\rangle) \equiv \Delta^{d_H,\Sigma}_\top(\langle\neg a,\ a\rangle) \equiv \top$, so $\Delta^E_{\mu_1}(\langle\varphi_1, \varphi_2\rangle) \wedge \mu_2 \equiv a \vee b$. On the other hand, $\Delta^E_{\mu_1 \wedge \mu_2}(\langle\varphi_1, \varphi_2\rangle) \equiv \Delta^E_{a \vee b}(\langle\neg a, a\rangle) \equiv \Delta^{d_H,\Sigma}_{a \vee b}(\langle\neg a \wedge (a \vee b), a \wedge (a \vee b)\rangle) \equiv \Delta^{d_H,\Sigma}_{a \vee b}(\langle\neg a \wedge b, a\rangle) \equiv b$. So we get that $\Delta^E_{\mu_1}(\langle\varphi_1, \varphi_2\rangle) \wedge \mu_2 \not\models \Delta^E_{\mu_1 \wedge \mu_2}(\langle\varphi_1, \varphi_2\rangle)$, which means that $\Delta^E$ does not satisfy (IC7).

- Let $\varphi_2 = a \wedge \neg b$. One the one hand, $\Delta^E_{\mu_1}(\langle \varphi_1, \varphi_2 \rangle) \equiv \Delta^E_{\top}(\langle \neg a, a \wedge \neg b \rangle) \equiv \Delta^{d_H,\Sigma}_{\top}(\langle \neg a, a \wedge \neg b \rangle) \equiv \neg b$, so $\Delta^E_{\mu_1}(\langle \varphi_1, \varphi_2 \rangle) \wedge \mu_2 \equiv a \wedge \neg b$. On the other hand, $\Delta^E_{\mu_1 \wedge \mu_2}(\langle \varphi_1, \varphi_2 \rangle) \equiv \Delta^E_{a \vee b}(\langle \neg a \wedge (a \vee b), a \wedge \neg b \wedge (a \vee b) \rangle) \equiv \Delta^{d_H,\Sigma}_{a \vee b}(\langle \neg a \wedge b, a \wedge \neg b \rangle) \equiv a \vee b$. So we get that $\Delta^E_{\mu_1}(\langle \varphi_1, \varphi_2 \rangle) \wedge \mu_2$ is consistent, and $\Delta^E_{\mu_1 \wedge \mu_2}(\langle \varphi_1, \varphi_2 \rangle) \not\models \Delta^E_{\mu_1}(\langle \varphi_1, \varphi_2 \rangle) \wedge \mu_2$, which means that $\Delta^E$ does not satisfy (IC8).

□

PROPOSITION A.5

For any aggregation function $f$ which satisfies (symmetry), (composition) and (decomposition), $\Delta^{d_D,f}$ is an expansion-based rationalizing merging operator, induced by itself.

PROOF. We use the notations introduced at the beginning of this appendix.

Let $f$ be an aggregation function which satisfies (symmetry), (composition) and (decomposition). We need to prove that $\Delta^{d_D,f}$ is an expansion-based rationalizing merging operator induced by itself, i.e. that for every formula $\mu$ and every p-consistent profile $\mathcal{K}$, $\Delta^{d_D,f}_{\mu}(\mathcal{K}) \equiv \Delta^{d_D,f}_{\mu}(\mathcal{K}^{\wedge\mu})$. Let $\mu$ be a formula and $\mathcal{K}$ be a profile. According to Definition 2.4 we need to prove that for all worlds $I, J \models \mu$, $f_{\varphi \in \mathcal{K}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}}(d_D(J, \varphi))$ if and only if $f_{\varphi \in \mathcal{K}^{\wedge\mu}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}^{\wedge\mu}}(d_D(J, \varphi))$.

Let $I, J$ be two worlds such that $I, J \models \mu$. For every base $\varphi$ such that $\varphi \wedge \mu$ is inconsistent, we have that $I, J \not\models \varphi$, thus $d_D(I, \varphi) = d_D(J, \varphi) = 1$. Hence, by (symmetry), (composition) and (decomposition) of $f$, we get that

$$f_{\varphi \in \mathcal{K}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}}(d_D(J, \varphi)) \text{ if and only if } f_{\varphi \in \mathcal{K}_\mu}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}_\mu}(d_D(J, \varphi)). \tag{A.2}$$

Note that every base $\varphi$ from $\mathcal{K}_\mu$ is consistent with $\mu$, i.e. $\varphi \wedge \mu$ is consistent. Moreover, for any world $L \models \mu$ and every base $\varphi$ such that $\varphi \wedge \mu$ is consistent we have that

- if $L \models \varphi$, then $L \models \varphi \wedge \mu$, so $d_D(L, \varphi) = d_D(L, \varphi \wedge \mu) = 0$;
- if $L \not\models \varphi$, then $L \not\models \varphi \wedge \mu$, so $d_D(L, \varphi) = d_D(L, \varphi \wedge \mu) = 1$.

Yet $I, J \models \mu$, so for every base $\varphi$ such that $\varphi \wedge \mu$ is consistent we have that $d_D(I, \varphi) = d_D(I, \varphi \wedge \mu)$ and $d_D(J, \varphi) = d_D(J, \varphi \wedge \mu)$. This means that $f_{\varphi \in \mathcal{K}_\mu}(d_D(I, \varphi)) = f_{\varphi \in \mathcal{K}_\mu^{\wedge\mu}}(d_D(I, \varphi))$ and $f_{\varphi \in \mathcal{K}_\mu}(d_D(J, \varphi)) = f_{\varphi \in \mathcal{K}_\mu^{\wedge\mu}}(d_D(J, \varphi))$. Hence, we get that

$$f_{\varphi \in \mathcal{K}_\mu}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}_\mu}(d_D(J, \varphi)) \text{ if and only if } f_{\varphi \in \mathcal{K}_\mu^{\wedge\mu}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}_\mu^{\wedge\mu}}(d_D(J, \varphi)). \tag{A.3}$$

Now, for every inconsistent base $\varphi$ we have $d_D(I, \varphi) = d_D(J, \varphi) = \mathsf{e}$. So by (symmetry), (composition) and (decomposition) of $f$, we get that

$$f_{\varphi \in \mathcal{K}_\mu^{\wedge\mu}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}_\mu^{\wedge\mu}}(d_D(J, \varphi)) \text{ if and only if } f_{\varphi \in \mathcal{K}^{\wedge\mu}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}^{\wedge\mu}}(d_D(J, \varphi)). \tag{A.4}$$

Equations A.2–A.4 together show that $f_{\varphi \in \mathcal{K}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}}(d_D(J, \varphi))$ if and only if $f_{\varphi \in \mathcal{K}^{\wedge\mu}}(d_D(I, \varphi)) \leq f_{\varphi \in \mathcal{K}^{\wedge\mu}}(d_D(J, \varphi))$, from which we can conclude that $\Delta^{d_D,f}_{\mu}(\mathcal{K}) \equiv \Delta^{d_D,f}_{\mu}(\mathcal{K}^{\wedge\mu})$. Therefore, $\Delta^{d_D,f}$ is an expansion-based rationalizing merging operator induced by itself. □

PROPOSITION A.6
For any aggregation function $f$ which satisfies (symmetry), (composition) and (decomposition), $\Delta^{d_D,f}$ is a revision-based rationalizing merging operator, induced by itself and $\langle \circ_1, \ldots, \circ_n \rangle$ with $\circ_1 = \cdots = \circ_n = \circ^D$, where $\circ^D$ is the drastic (KM) revision operator.

PROOF. We use the notations introduced at the beginning of this appendix.

Let $f$ be an aggregation function which satisfies (symmetry), (composition) and (decomposition). Let us prove that $\Delta^{d_D,f}$ satisfies (Ind-$\circ$). We need to show that one can associate with each agent $i$ a KM revision operator $\circ_i$ such that for every formula $\mu$ and every profile $\langle \varphi_1, \ldots, \varphi_n \rangle$, $\Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle) \equiv \Delta_\mu(\langle \varphi_1 \circ_1 \mu, \ldots, \varphi_n \circ_n \mu \rangle)$. Let us set $\circ_1 = \cdots = \circ_n = \circ^D$, the drastic revision operator, let $\mu$ be a formula and $\mathcal{K}$ be a profile. According to Definition 2.4 we need to prove that for all worlds $I, J \models \mu$, $f_{\varphi \in \mathcal{K}}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}}(d_D(J,\varphi))$ if and only if $f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(J,\varphi))$. Let $I, J$ be two worlds such that $I, J \models \mu$. Note that Equations A.2 and A.3 from the proof of Proposition 3.5 can be identically proved, so we only need to show that $f_{\varphi \in \mathcal{K}^{\wedge\mu}_\mu}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}^{\wedge\mu}_\mu}(d_D(J,\varphi))$ if and only if $f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(J,\varphi))$. Yet one can easily verify that the profile $\mathcal{K}_{\circ^D,\mu}$ consists of bases from the profile $\mathcal{K}^{\wedge\mu}_\mu$ and an arbitrary number of bases equal to $\mu$. And since $I, J \models \mu$, we have that $d_D(I,\mu) = d_D(J,\mu) = 0$. So by (symmetry), (composition) and (decomposition) of $f$, we get that

$$\begin{aligned} &f_{\varphi \in \mathcal{K}^{\wedge\mu}_\mu}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}^{\wedge\mu}_\mu}(d_D(J,\varphi)) \text{ if and only if} \\ &f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(J,\varphi)). \end{aligned} \tag{A.5}$$

Equations A.2 and A.3 from the proof of Proposition 3.5, together with Equation A.5 above show that $f_{\varphi \in \mathcal{K}}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}}(d_D(J,\varphi))$ if and only if $f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(I,\varphi)) \le f_{\varphi \in \mathcal{K}^{\circ^D\mu}}(d_D(J,\varphi))$, from which we can conclude that $\Delta^{d_D,f}_\mu(\mathcal{K}) \equiv \Delta^{d_D,f}_\mu(\mathcal{K}^{\circ^D\mu})$. Therefore, $\Delta^{d_D,f}$ satisfies (Ind-$\circ$). □

PROPOSITION A.7
Let $\Delta^R$ be a revision-based rationalizing merging operator induced by $\Delta$ and $\circ_1, \ldots, \circ_n$. For $x \in \{0, \ldots, 6\}$, if $\Delta$ satisfies (ICx) then $\Delta^R$ satisfies (ICx). This is not true in the general case for $x \in \{7, 8\}$.

PROOF. We use the notations introduced at the beginning of this appendix.

Let $\Delta^R$ be a revision-based rationalizing merging operator, induced by and $\Delta$ and $\langle \circ_1, \ldots, \circ_n \rangle$. In the following, $\mathcal{K}$ denotes any p-consistent profile $\langle \varphi_1, \ldots, \varphi_n \rangle$, $\mu$ is any formula and $\varphi_1, \varphi_2$ are two consistent bases such that $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$, $\mathcal{K}^{\wedge\mu}$ denotes the profile $\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle$ and $\mathcal{K}^{\circ\mu}$ denotes the profile $\langle K_1 \circ_1 \mu, \ldots, K_n \circ_n \mu \rangle$. Assume $\Delta$ satisfies

- (IC0): $\Delta^R_\mu(\mathcal{K}) \equiv \Delta_\mu(\langle K_1 \circ_1 \mu, \ldots, K_n \circ_n \mu \rangle)$, and $\Delta_\mu(\langle K_1 \circ_1 \mu, \ldots, K_n \circ_n \mu \rangle) \models \mu$ by (IC0), so $\Delta^R$ also satisfies (IC0).
- (IC1): assume $\mu$ is consistent. We have $\Delta^R_\mu(\mathcal{K}) \equiv \Delta_\mu(\langle K_1 \circ_1 \mu, \ldots, K_n \circ_n \mu \rangle)$, which is consistent by (IC1), so $\Delta^R$ also satisfies (IC1).
- (IC2): assume $\bigwedge \mathcal{K} \wedge \mu$ is consistent. This means that for each agent $i$, $\varphi_i \wedge \mu$ is consistent. Since each revision operator $\circ_i$ satisfies (R2), we have $\varphi_i \circ_i \mu \equiv \varphi_i \wedge \mu$ for each agent $i$. So $\Delta^R_\mu(\mathcal{K}) \equiv \Delta_\mu(\langle K_1 \circ_1 \mu, \ldots, K_n \circ_n \mu \rangle) \equiv \Delta_\mu(\mathcal{K}^{\wedge\mu})$. On the other hand, $\bigwedge \mathcal{K}^{\wedge\mu} \equiv \bigwedge \mathcal{K} \wedge \mu$, thus $\bigwedge \mathcal{K}^{\wedge\mu}$ is consistent. Hence, by (IC2) $\Delta_\mu(\mathcal{K}^{\wedge\mu}) \equiv \bigwedge \mathcal{K}^{\wedge\mu} \equiv \bigwedge \mathcal{K} \wedge \mu$. We get that $\Delta^R_\mu(\mathcal{K}) \equiv \bigwedge \mathcal{K} \wedge \mu$, so $\Delta^R$ also satisfies (IC2).

- (IC3): we have $\mathcal{K}_1 \equiv \mathcal{K}_2$, yet each revision operator $\circ_i$ satisfies (R4), so $\mathcal{K}_1^{\circ\mu} \equiv \mathcal{K}_2^{\circ\mu}$, from which it follows that $\Delta^R$ also satisfies (IC3).
- (IC4): for every agent *i*, every revision operator $\circ_i$ satisfies (R2), so $\varphi_i \models \mu$ implies $\varphi_i \circ_i \mu \equiv \varphi_i \wedge \mu \equiv \varphi_i$. Hence, since $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$, we have that $\Delta^R_\mu(\langle\varphi_1, \varphi_2\rangle) \equiv \Delta_\mu(\langle\varphi_1 \circ_1 \mu, \varphi_2 \circ_2 \mu\rangle) \equiv \Delta_\mu(\langle\varphi_1, \varphi_2\rangle)$, so $\Delta^R$ also satisfies (IC4).
- (ICx), $x \in \{5, 6\}$: we have $\Delta^R_\mu(\mathcal{K}_1) \wedge \Delta^R_\mu(\mathcal{K}_2) \equiv \Delta_\mu(\mathcal{K}_1^{\circ\mu}) \wedge \Delta_\mu(\mathcal{K}_2^{\circ\mu})$, and $\Delta^R_\mu(\mathcal{K}_1 \sqcup \mathcal{K}_2) \equiv \Delta_\mu(\mathcal{K}_1^{\circ\mu} \sqcup \mathcal{K}_2^{\circ\mu})$. Hence, since $\Delta$ satisfies (ICx), $\Delta^R$ also satisfies (ICx).

The example introduced in the proof of Proposition 3.4—which shows that (IC7) (respectively, (IC8)) is not necessarily satisfied by $\Delta^E$ even if $\Delta$ satisfies it—can also be used here to show that (IC7) (respectively, (IC8)) is not necessarily satisfied by $\Delta^R$ even if $\Delta$ satisfies it, with the following settings: consider again $\Delta^{d_H,\Sigma}$ be the merging operator based on the Hamming distance and the sum function, and additionally define $\circ_1 = \cdots = \circ_n = \circ^D$, where $\circ^D$ is the drastic (KM) revision operator; then let $\Delta^R$ be the revision-based rationalizing merging operator induced by $\Delta^{d_H,\Sigma}$ and $\langle\circ_1, \ldots, \circ_n\rangle$. Let $\mu_1 = \top$, $\mu_2 = a \vee b$ and $\varphi_1 = \neg a$. Then, one can easily verify that $\varphi_2 = a$ (resp. $\varphi_2 = a \wedge \neg b$) provides a counter-example for (IC7) (resp. (IC8)). □

PROPOSITION A.8

Let $\Delta^U$ be an update-based rationalizing merging operator induced by $\Delta$ and $\diamond_1, \ldots, \diamond_n$. For $x \in \{0, 1, 3, 4, 5, 6\}$, if $\Delta$ satisfies (ICx) then $\Delta^U$ satisfies (ICx). This is not true in the general case for $x \in \{2, 7, 8\}$.

PROOF. Let $\Delta^U$ be an update-based rationalizing merging operator, induced by and $\Delta$ and $\langle\diamond_1, \ldots, \diamond_n\rangle$. We use the notations introduced at the beginning of this appendix, and in addition, $\mathcal{K}^{\diamond\mu}$ denotes the profile $\langle K_1 \diamond_1 \mu, \ldots, K_n \diamond_n \mu\rangle$. In the following, $\mathcal{K}$ denotes any p-consistent profile $\langle\varphi_1, \ldots, \varphi_n\rangle$, $\mu$ is any formula and $\varphi_1, \varphi_2$ are two consistent bases such that $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$.

Assume $\Delta$ satisfies

- (IC0): $\Delta^U_\mu(\mathcal{K}) \equiv \Delta_\mu(\langle K_1 \diamond_1 \mu, \ldots, K_n \diamond_n \mu\rangle)$, and $\Delta_\mu(\langle K_1 \diamond_1 \mu, \ldots, K_n \diamond_n \mu\rangle) \models \mu$ by (IC0), so $\Delta^U$ also satisfies (IC0).
- (IC1): assume $\mu$ is consistent. We have $\Delta^U_\mu(\mathcal{K}) \equiv \Delta_\mu(\langle K_1 \diamond_1 \mu, \ldots, K_n \diamond_n \mu\rangle)$, which is consistent by (IC1), so $\Delta^U$ also satisfies (IC1).
- (IC3): we have $\mathcal{K}_1 \equiv \mathcal{K}_2$, yet each update operator $\diamond_i$ satisfies (U4), so $\mathcal{K}_1^{\diamond\mu} \equiv \mathcal{K}_2^{\diamond\mu}$, from which it follows that $\Delta^U$ also satisfies (IC3).
- (IC4): for every agent *i*, every update operator $\diamond_i$ satisfies (U2), so $\varphi_i \models \mu$ implies $\varphi_i \diamond_i \mu \equiv \varphi_i \wedge \mu \equiv \varphi_i$. Hence, since $\varphi_1 \models \mu$ and $\varphi_2 \models \mu$, we have that $\Delta^U_\mu(\langle\varphi_1, \varphi_2\rangle) \equiv \Delta_\mu(\langle\varphi_1 \diamond_1 \mu, \varphi_2 \diamond_2 \mu\rangle) \equiv \Delta_\mu(\langle\varphi_1, \varphi_2\rangle)$, so $\Delta^U$ also satisfies (IC4).
- (ICx), $x \in \{5, 6\}$: we have $\Delta^U_\mu(\mathcal{K}_1) \wedge \Delta^U_\mu(\mathcal{K}_2) \equiv \Delta_\mu(\mathcal{K}_1^{\diamond\mu}) \wedge \Delta_\mu(\mathcal{K}_2^{\diamond\mu})$, and $\Delta^U_\mu(\mathcal{K}_1 \sqcup \mathcal{K}_2) \equiv \Delta_\mu(\mathcal{K}_1^{\diamond\mu} \sqcup \mathcal{K}_2^{\diamond\mu})$. Hence, since $\Delta$ satisfies (ICx), $\Delta^U$ also satisfies (ICx).

The following example shows that (IC2) is not necessarily satisfied by $\Delta^U$ even if $\Delta$ satisfies it. Let $\Delta$ be any merging operator satisfying (IC2), and let $\diamond_1 = \cdots = \diamond_n = \diamond^D$, where $\diamond^D$ is the drastic (KM) update operator defined as

$$\varphi \diamond^D \mu = \begin{cases} \varphi & \text{if } \varphi \models \mu, \\ \mu & \text{otherwise.} \end{cases}$$

Let $\Delta^U$ be the update-based rationalizing merging operator induced by $\Delta$ and $\langle\diamond_1, \ldots, \diamond_n\rangle$. We have $\Delta^U_a(\langle b\rangle) \equiv \Delta_a(\langle b \diamond^D a\rangle) \equiv \Delta_a(\langle a\rangle)$, which is equivalent to $a$ since $\Delta$ satisfies (IC2). So $\Delta^U_a(\langle b\rangle) \not\equiv a \wedge b$, which means that $\Delta^U$ does not satisfy (IC2).

The example introduced in the proof of Proposition 3.4—which shows that (IC7) (respectively, (IC8)) is not necessarily satisfied by $\Delta^E$ even if $\Delta$ satisfies it—can also be used here to show that (IC7) (respectively, (IC8)) is not necessarily satisfied by $\Delta^U$ even if $\Delta$ satisfies it, with the following settings: consider again $\Delta^{d_H,\Sigma}$ be the merging operator based on the Hamming distance and the sum function, and additionally define $\diamond_1 = \cdots = \diamond_n = \diamond^D$, where $\diamond^D$ is the drastic (KM) update operator defined as above; then let $\Delta^U$ be the update-based rationalizing merging operator induced by $\Delta^{d_H,\Sigma}$ and $\langle \circ_1, \ldots, \circ_n \rangle$. Let $\mu_1 = \top$, $\mu_2 = a \vee b$ and $\varphi_1 = \neg a$. Then, one can easily verify that setting $\varphi_2 = a$ (resp. $\varphi_2 = a \wedge \neg b$) provides a counter-example for (IC7) (resp. (IC8)). □

PROPOSITION A.9
If $\Delta$ is a merging operator satisfying (IC3), then $\Delta$ satisfies (Ind) if and only if $\Delta$ satisfies (IIA).

PROOF. Let $\Delta$ be a merging operator satisfying (IC3). *(If)* Assume that $\Delta$ satisfies (IIA). Let $\varphi_1, \ldots, \varphi_n$ be $n$ belief bases and $\mu$ be a formula. We have $\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle \equiv \langle \varphi_1 \wedge \mu \wedge \mu, \ldots, \varphi_n \wedge \mu \wedge \mu \rangle$. From (IIA) we get $\Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle) \equiv \Delta_\mu(\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle)$. Hence, $\Delta$ satisfies (Ind). *(Only If)* Assume that $\Delta$ satisfies (Ind). Let $\varphi_1, \ldots, \varphi_n, \varphi'_1, \ldots, \varphi'_n$ be $2n$ belief bases, $\mu$ be a formula, and assume that $\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle \equiv \langle \varphi'_1 \wedge \mu, \ldots, \varphi'_n \wedge \mu \rangle$. From (IC3) we get that $\Delta_\mu(\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle) \equiv \Delta_\mu(\langle \varphi'_1 \wedge \mu, \ldots, \varphi'_n \wedge \mu \rangle)$. Yet from (Ind) we have $\Delta_\mu(\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle) \equiv \Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle)$ and $\Delta_\mu(\langle \varphi'_1 \wedge \mu, \ldots, \varphi'_n \wedge \mu \rangle) \equiv \Delta_\mu(\langle \varphi'_1, \ldots, \varphi'_n \rangle)$. Hence, $\Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle) \equiv \Delta_\mu(\langle \varphi'_1, \ldots, \varphi'_n \rangle)$. Therefore, $\Delta$ satisfies (IIA). □

PROPOSITION A.10
Let $\Delta$ be a merging operator satisfying (IC3). If $\Delta$ is an expansion-based rationalizing merging operator, then it satisfies (Ind).

PROOF. Let $\Delta$ be a merging operator satisfying (IC3). Let us assume that $\Delta$ is an expansion-based rationalizing merging operator and let us show that it satisfies (Ind). What we need to prove is that for every formula $\mu$ and every profile $\langle \varphi_1, \ldots, \varphi_n \rangle$, $\Delta_\mu(\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle) \equiv \Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle)$. Yet $\Delta$ is induced by a merging operator $\Delta'$, so for every formula $\mu$ and every profile $\langle \varphi_1, \ldots, \varphi_n \rangle$, $\Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle) \equiv \Delta'_\mu(\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle)$. Let $\mu$ be a formula and $\langle \varphi_1, \ldots, \varphi_n \rangle$ be a profile. We have

$$\begin{aligned} \Delta_\mu(\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle) &\equiv \Delta'_\mu(\langle \varphi_1 \wedge \mu \wedge \mu, \ldots, \varphi_n \wedge \mu \wedge \mu \rangle) \\ &\equiv \Delta'_\mu(\langle \varphi_1 \wedge \mu, \ldots, \varphi_n \wedge \mu \rangle) \\ &\equiv \Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle). \end{aligned}$$

This concludes the proof. □

PROPOSITION A.11
Let $\Delta$ be a merging operator satisfying (IC3). If $\Delta$ is a revision-based rationalizing merging operator, then it satisfies (Ind-$\circ$).

PROOF. Let us first emphasize an intermediate result. Let $\varphi$ be a base, $\mu$ be a formula and $\circ$ be a revision operator satisfying (R1) and (R2). By (R1) $\varphi \circ \mu \models \mu$. Since $(\varphi \circ \mu) \wedge \mu$ is consistent, by (R2) $(\varphi \circ \mu) \circ \mu \equiv (\varphi \circ \mu) \wedge \mu \equiv \varphi \circ \mu$. That is,

$$(\varphi \circ \mu) \circ \mu \equiv \varphi \circ \mu. \tag{A.6}$$

Now, let $\Delta$ be a merging operator satisfying (IC3). Let us assume that $\Delta$ is a revision-based rationalizing merging operator and let us show that it satisfies (Ind-$\circ$). What we need to prove is that one can associate with each agent $i$ a KM revision operator $\circ_i$ such that for every formula $\mu$ and every profile $\langle \varphi_1, \ldots, \varphi_n \rangle$, $\Delta_\mu(\langle \varphi_1 \circ_1 \mu, \ldots, \varphi_n \circ_n \mu \rangle) \equiv \Delta_\mu(\langle \varphi_1, \ldots, \varphi_n \rangle)$. Yet $\Delta$ is induced by a merging operator $\Delta'$ and $\langle \circ_1, \ldots, \circ_n \rangle$, where each $\circ_i$ is a KM revision operator. So for every formula $\mu$ and every profile $\langle\varphi_1,\ldots,\varphi_n\rangle$, $\Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle) \equiv \Delta'_\mu(\langle\varphi_1 \circ_1 \mu,\ldots,\varphi_n \circ_n \mu\rangle)$. Let $\mu$ be a formula and $\langle\varphi_1,\ldots,\varphi_n\rangle$ be a profile. We have

$$\begin{aligned}\Delta_\mu(\langle\varphi_1 \circ_1 \mu,\ldots,\varphi_n \circ_n \mu\rangle) &\equiv \Delta'_\mu(\langle(\varphi_1 \circ_1 \mu)\circ_1 \mu,\ldots,(\varphi_n \circ_n \mu)\circ_n \mu\rangle)\\ &\equiv \Delta'_\mu(\langle\varphi_1 \circ_1 \mu,\ldots,\varphi_n \circ_n \mu\rangle) \text{ (from Equation A.6)}\\ &\equiv \Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle).\end{aligned}$$

This concludes the proof. □

PROPOSITION A.12
Let $\Delta$ be a merging operator satisfying (IC2) and (Ind-$\circ$). Then every revision operator $\circ_i$ considered in (Ind-$\circ$) is the revision operator $\circ_\Delta$ corresponding to $\Delta$ in the sense of Definition 2.8.

PROOF. Let $\Delta$ be a merging operator satisfying (IC2) and (Ind-$\circ$). This means that one can associate with each agent $i$ a KM revision operator $\circ_i$ such that for every formula $\mu$ and every profile $\langle\varphi_1,\ldots,\varphi_n\rangle$,

$$\Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle) \equiv \Delta_\mu(\langle\varphi_1 \circ_1 \mu,\ldots,\varphi_n \circ_n \mu\rangle). \quad \text{(A.7)}$$

Let us first prove that for every KM revision operator $\circ_i$ involved in Equation A.7 above, we have $\circ_i = \circ_\Delta$. Let $\circ_i$ be any revision operator involved in Equation 7, $\varphi$ be a belief base and $\mu$ be a formula. By (R1) we have $\varphi \circ_i \mu \models \mu$, so by (Ind-$\circ$) and (IC2), we get $\Delta_\mu(\langle\varphi\rangle) \equiv \Delta_\mu(\langle\varphi \circ_i \mu\rangle) \equiv \varphi \circ_i \mu$. Hence, for every revision operator $\circ_i$, we have $\circ_i = \circ_\Delta$. □

PROPOSITION A.13
Let $\Delta$ be an EIC merging operator satisfying (Ind-$\circ$). Then every revision operator $\circ_i$ is $\circ^D$, the drastic revision operator.

PROOF. Let $\Delta$ be an EIC merging operator satisfying (Ind-$\circ$). From Proposition 4.5 we know that every revision operator $\circ_i$ is $\circ_\Delta$, the revision operator corresponding to $\Delta$. Let us prove that $\circ_\Delta = \circ^D$, i.e. $\circ_\Delta$ is the drastic revision operator. On the one hand, we know that the faithful assignment associated with $\circ_\Delta$ coincides with the syncretic assignment associated with $\Delta$, restricted to singleton profiles. So since $\Delta$ is an EIC merging operator, by Theorem 2.1 it corresponds to an assignment satisfying conditions 1–6 (see Definition 2.3). On the other hand, $\circ^D$ corresponds to a faithful assignment defined such that for all worlds $I,J,L$, if $L \notin \{I,J\}$ then $I \simeq_{\varphi_{\{L\}}} J$. Towards a contradiction, assume that $\circ_\Delta \neq \circ^D$. This means that there exist three distinct worlds $I,J,L$ such that (i) $I <_{\varphi_{\{I\}}} J <_{\varphi_{\{I\}}} L$, i.e. there exists a total preorder $<_{\varphi_{\{I\}}}$ with at least 3 distinct levels. Let $\varphi \equiv \varphi_{\{J,L\}}$. By conditions 1 and 2, we have (ii) $J \simeq_\varphi L <_\varphi I$. Now

(a) By (i), we have $J <_{\varphi_{\{I\}}} L$ and by (ii), we have $J \simeq_\varphi L$. Thus, by condition 6 of a syncretic assignment, we get $J <_{\langle\varphi_{\{I\}},\varphi\rangle} L$.
(b) By (R2), $\varphi \circ_\Delta \varphi_{\{I,J\}} \equiv \varphi_{\{J\}}$, and $\varphi_{\{I\}} \circ_\Delta \varphi_{\{I,J\}} \equiv \varphi_{\{I\}}$. Then by (Ind-$\circ$), $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi,\varphi_{\{I\}}\rangle) \equiv \Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{J\}},\varphi_{\{I\}}\rangle)$. Hence, by (IC0), (IC1) and (IC4), we get $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi,\varphi_{\{I\}}\rangle) \equiv \varphi_{\{I,J\}}$. This means that $I \simeq_{\langle\varphi_{\{I\}},\varphi\rangle} J$.
(c) Using a similar reasoning as for (b) with $L$ instead of $J$, we get $I \simeq_{\langle\varphi_{\{I\}},\varphi\rangle} L$.

We have that (a), (b) and (c) together lead to a contradiction, which means that $\circ_\Delta = \circ^D$. □

PROPOSITION A.14
Let $\Delta$ be an EIC merging operator. $\Delta$ satisfies (Ind) if and only if $\Delta$ satisfies (Ind-$\circ$).

PROOF. We use the notations introduced at the beginning of this appendix.

Let $\Delta$ be an EIC merging operator. Let us first prove that for every profile $\mathcal{K}$, every formula $\mu$ and all $m \geq 0$, $\Delta_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K} \sqcup \langle\mu\rangle^m)$. So let $\mathcal{K}$ be a profile, $\mu$ be a formula and let $m \geq 0$. If $\mu$ is inconsistent, then the proof trivially follows from Proposition 3.1. So assume that $\mu$ is consistent. By (IC2) we have that $\Delta_\mu(\langle\mu\rangle^m) \equiv \mu$. Yet, by (IC0) we have $\Delta_\mu(\mathcal{K}) \models \mu$. Hence, $\Delta_\mu(\mathcal{K}) \wedge \Delta_\mu(\langle\mu\rangle^m) \equiv \Delta_\mu(\mathcal{K})$, which is consistent by (IC1). Then by (IC5) and (IC6), we get $\Delta_\mu(\mathcal{K}) \wedge \Delta_\mu(\langle\mu\rangle^m) \equiv \Delta_\mu(\mathcal{K} \sqcup \langle\mu\rangle^m)$. Hence,

$$\Delta_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K} \sqcup \langle\mu\rangle^m). \tag{A.8}$$

Now,

$$\begin{aligned}
\Delta_\mu(\mathcal{K}^{\wedge\mu}) &\equiv \Delta_\mu(\mathcal{K}_\mu^{\wedge\mu} \sqcup \langle\bot\rangle^m) && \text{by (IC3)}\\
&\equiv \Delta_\mu(\mathcal{K}_\mu^{\wedge\mu}) && \text{from Proposition 1}\\
&\equiv \Delta_\mu(\mathcal{K}_\mu^{\wedge\mu} \sqcup \langle\mu\rangle^m) && \text{from Equation 8}\\
&\equiv \Delta_\mu(\mathcal{K}^{\circ^D\mu}) && \text{by } (IC3) \text{ and by definition of } \circ_D,
\end{aligned}$$

where $m$ is the number of bases from $\mathcal{K}$ that are inconsistent with $\mu$.

Hence, we get that $\Delta$ satisfies (Ind) if and only if for every profile $\mathcal{K}$ and every formula $\mu$, $\Delta_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K}^{\wedge\mu})$ if and only if for every profile $\mathcal{K}$ and every formula $\mu$, $\Delta_\mu(\mathcal{K}) \equiv \Delta_\mu(\mathcal{K}^{\circ^D\mu})$ if and only if $\Delta$ satisfies (Ind-$\circ$).

This concludes the proof. □

PROPOSITION A.15
Let $\Delta$ be a merging operator satisfying (IC3). If $\Delta$ is an update-based rationalizing merging operator, then it satisfies (Ind-$\diamond$).

PROOF. Let us first emphasize an intermediate result. Let $\varphi$ be a base, $\mu$ be a formula and $\diamond$ be an update operator satisfying (U1) and (U2). By (U1) $\varphi \diamond \mu \models \mu$. So by (U2),

$$(\varphi \diamond \mu) \diamond \mu \equiv \varphi \diamond \mu. \tag{A.9}$$

Now, let $\Delta$ be a merging operator satisfying (IC3). Let us assume that $\Delta$ is an update-based rationalizing merging operator and let us show that it satisfies (Ind-$\diamond$). What we need to prove is that one can associate with each agent $i$ a KM revision operator $\diamond_i$ such that for every formula $\mu$ and every profile $\langle\varphi_1, \ldots, \varphi_n\rangle$, $\Delta_\mu(\langle\varphi_1 \diamond_1 \mu, \ldots, \varphi_n \diamond_n \mu\rangle) \equiv \Delta_\mu(\langle\varphi_1, \ldots, \varphi_n\rangle)$. Yet $\Delta$ is induced by a merging operator $\Delta'$ and $\langle\diamond_1, \ldots, \diamond_n\rangle$, where each $\diamond_i$ is a KM revision operator. So for every formula $\mu$ and every profile $\langle\varphi_1, \ldots, \varphi_n\rangle$, $\Delta_\mu(\langle\varphi_1, \ldots, \varphi_n\rangle) \equiv \Delta'_\mu(\langle\varphi_1 \diamond_1 \mu, \ldots, \varphi_n \diamond_n \mu\rangle)$. Let $\mu$ be a formula and $\langle\varphi_1, \ldots, \varphi_n\rangle$ be a profile. We have

$$\begin{aligned}
\Delta_\mu(\langle\varphi_1 \diamond_1 \mu, \ldots, \varphi_n \diamond_n \mu\rangle) &\equiv \Delta'_\mu(\langle(\varphi_1 \diamond_1 \mu) \diamond_1 \mu, \ldots, (\varphi_n \diamond_n \mu) \diamond_n \mu\rangle)\\
&\equiv \Delta'_\mu(\langle\varphi_1 \diamond_1 \mu, \ldots, \varphi_n \diamond_n \mu\rangle) \text{ (from Equation 9)}\\
&\equiv \Delta_\mu(\langle\varphi_1, \ldots, \varphi_n\rangle).
\end{aligned}$$

This concludes the proof. □

PROPOSITION A.16
There is no merging operator satisfying (IC2) and (Ind-$\diamond$).

PROOF. Towards a contradiction, assume that there exists a merging operator $\Delta$ satisfying (IC2) and (Ind-$\diamond$). By (Ind-$\diamond$) (cf. Definition 4.4) we can associate with every agent $i$ a KM update operator $\diamond_i$

such that for every profile $\langle\varphi_1,\ldots,\varphi_n\rangle$,

$$\Delta_\mu(\langle\varphi_1,\ldots,\varphi_n\rangle)\equiv\Delta_\mu(\langle\varphi_1\diamond_1\mu,\ldots,\varphi_n\diamond_n\mu\rangle). \tag{A.10}$$

Let us first show that every KM update operator $\diamond_i$ involved in Equation 10 above is the drastic update operator, i.e. $\diamond_i=\diamond^D$, where the drastic update operator $\diamond^D$ is defined for every belief base $\varphi$ and every formula $\mu$ as

$$\varphi\diamond^D\mu=\begin{cases}\varphi & \text{if } \varphi\models\mu,\\ \mu & \text{otherwise.}\end{cases}$$

According to Theorem 2.4, any KM update operator $\diamond$ corresponds to a faithful assignment associating every world $I$ with a preorder $\leq_{\varphi_{\{I\}}}$ such that for every base $\varphi$ and every formula $\mu$, $mod(\varphi\diamond\mu)=\bigcup_{I\models\varphi}\min(mod(\mu),\leq_{\varphi_{\{I\}}})$. Then, one can first remark that the drastic update operator $\diamond^D$ corresponds to a faithful assignment defined such that for all distinct worlds $I,J,L$, $I\not\leq_{\varphi_{\{L\}}}J$ (and $J\not\leq_{\varphi_{\{L\}}}I$ by symmetry). Indeed, if one would have that $J<_{\varphi_{\{L\}}}I$ for some distinct worlds $I,J,L$, then we would get by Theorem 2.4 that $\varphi_{\{L\}}\diamond^D\varphi_{\{I,J\}}\equiv\varphi_{\{J\}}$, which contradicts the definition of $\diamond^D$.

So now we want to show that for every KM update operator $\diamond_i$ involved in Equation 10 above, we have $\diamond_i=\diamond^D$. Towards a contradiction, assume that for some KM update operator $\diamond_i$ involved in Equation 10, $\diamond_i$ is not the drastic update operator, i.e. $\diamond_i\neq\diamond^D$. That is, we assume that there exist three distinct worlds $I,J,L$ such that $J<^{\diamond_i}_{\varphi_{\{L\}}}I$. On the one hand, by (U1), we have $\varphi_{\{L\}}\diamond_j\varphi_{\{I,J\}}\models\varphi_{\{I,J\}}$, and by (U3), $\varphi_{\{L\}}\diamond_j\varphi_{\{I,J\}}$ is consistent. Yet $J<^{\diamond_i}_{\varphi_{\{L\}}}I$ by hypothesis, thus (i) $\varphi_{\{L\}}\diamond_i\varphi_{\{I,J\}}\equiv\varphi_{\{J\}}$. On the other hand, by (U2), we have (ii) $\varphi_{\{I\}}\diamond_i\varphi_{\{I,J\}}\equiv\varphi_{\{I\}}$. Yet by (U8) we have $\varphi_{\{I,L\}}\diamond_i\varphi_{\{I,J\}}\equiv(\varphi_{\{I\}}\diamond_i\varphi_{\{I,J\}})\vee(\varphi_{\{L\}}\diamond_i\varphi_{\{I,J\}})$. Hence, by (i) and (ii) we get that (iii) $\varphi_{\{I,L\}}\diamond_i\varphi_{\{I,J\}}\equiv\varphi_{\{I\}}\vee\varphi_{\{J\}}\equiv\varphi_{\{I,J\}}$. Now, from (Ind-$\diamond$) and (iii) we get that (iv) $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I,L\}}\rangle)\equiv\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I,L\}}\diamond_i\varphi_{\{I,J\}}\rangle)\equiv\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I,J\}}\rangle)$. Yet by (IC2), we get that $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I,L\}}\rangle)\equiv\varphi_{\{I\}}$ and $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I,J\}}\rangle)\equiv\varphi_{\{I,J\}}$, which contradicts (iv). This proves that every KM update operator $\diamond_i$ involved in Equation 10 above is the drastic update operator $\diamond^D$.

Now, let $I,J,L$ be three distinct worlds. On the one hand, $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{J,L\}}\rangle)=\varphi_{\{J\}}$ by (IC2). On the other hand, $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{J,L\}}\diamond^D\varphi_{\{I,J\}}\rangle)$, which is equivalent to $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I,J\}}\rangle)$ by definition of $\diamond^D$, which is equivalent to $\varphi_{\{I,J\}}$ by (IC2). Hence, we have that $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{J,L\}}\rangle)\not\equiv\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{J,L\}}\diamond^D\varphi_{\{I,J\}}\rangle)$. This contradicts the initial assumption that $\Delta$ satisfies (Ind-$\diamond$), from which we can conclude that there is no merging operator $\Delta$ satisfying both (IC2) and (Ind-$\diamond$). □

PROPOSITION A.17
Every mapping associating with every profile $\mathcal{K}$ a preorder $\leq_{\mathcal{K}}$ over worlds and satisfying conditions (1), (2) and (F) also satisfies condition (4).

PROOF. Consider a mapping associating with every profile $\mathcal{K}$ a preorder $\leq_{\mathcal{K}}$ over worlds and satisfying conditions (1), (2) and (F). We need to prove that condition (4) is satisfied, i.e. let $\varphi_1,\varphi_2$ be two consistent belief bases, let $I$ be a world such that $I\models\varphi_1$ and let us prove that there exists a world $J$ such that $J\models\varphi_2$ and $J\leq_{\langle\varphi_1,\varphi_2\rangle}I$. This is trivially true if $I\models\varphi_2$, so assume $I\not\models\varphi_2$. Let $J$ be a world such that $J\models\varphi_2$. If $J\models\varphi_1$, then conditions 1 and 2 together imply that $J<_{\langle\varphi_1,\varphi_2\rangle}I$, so condition 4 is satisfied. Now, assume $J\not\models\varphi_1$. Then conditions 1 and 2 together imply that $I<_{\varphi_1}J$ and $J<_{\varphi_2}I$. So by condition F, we get $I\simeq_{\langle\varphi_1,\varphi_2\rangle}J$, thus condition 4 is satisfied. □

PROPOSITION A.18
Every mapping associating with every profile $\mathcal{K}$ a preorder $\leq_{\mathcal{K}}$ over worlds and satisfying conditions (1), (2), (6) and (F) maps every singleton belief profile $\langle\varphi\rangle$ to a unique total preorder $\leq_{\varphi}$ over worlds defined for all worlds $I, J$ and every belief base $\varphi$ as $I <_{\varphi} J$ if and only if $I \models \varphi$ and $J \not\models \varphi$.

PROOF. *(If)* Direct from condition 2. *(Only If)* Let $\varphi$ be a belief base. The result is trivial if $\varphi$ in inconsistent, so assume that $\varphi$ is consistent. Let us first show that if $I \not\models \varphi$ and $J \not\models \varphi$, then $I \simeq_{\langle\varphi\rangle} J$. Let $I \not\models \varphi$, $J \not\models \varphi$ and assume by contradiction that $I <_{\langle\varphi\rangle} J$. Let $L \models \varphi$. By condition 2 of a syncretic assignement we have $L <_{\langle\varphi\rangle} J$, $J <_{\varphi\{J\}} I$ and $J <_{\varphi\{J\}} L$. Thus, by condition F we get $I \simeq_{\{\varphi,\varphi\{J\}\}} J$ and $J \simeq_{\{\varphi,\varphi\{J\}\}} L$. Thus, $I \simeq_{\{\varphi,\varphi\{J\}\}} L$. Since by condition 2, $L <_{\langle\varphi\rangle} I$, then by condition 6 we get $I <_{\varphi\{J\}} L$. Yet by conditions 1 and 2 we have $I \simeq_{\varphi\{I,L\}} L <_{\varphi\{I,L\}} J$. Since $J <_{\varphi\{J\}} I$, $I <_{\varphi\{I,L\}} J$, $J <_{\varphi\{J\}} L$ and $L <_{\varphi\{I,L\}} J$, by condition F we get $I \simeq_{\{\varphi\{J\},\varphi\{I,L\}\}} J$ and $J \simeq_{\{\varphi\{J\},\varphi\{I,L\}\}} L$. Thus, $I \simeq_{\{\varphi\{J\},\varphi\{I,L\}\}} L$. But since $I <_{\varphi\{J\}} L$ and $I \simeq_{\varphi\{I,L\}} L$, by condition 6 we get $I <_{\{\varphi\{J\},\varphi\{I,L\}\}} L$, leading to a contradiction. Thus, we do not have $I <_{\langle\varphi\rangle} J$. By a similar reasoning, we can prove that we do not have $J <_{\langle\varphi\rangle} I$ either. Hence, we get

$$\text{If } I \not\models \varphi \text{ and } J \not\models \varphi, \text{ then } I \simeq_{\langle\varphi\rangle} J. \tag{A.11}$$

In the case where $I \models \varphi$ and $J \models \varphi$ then by condition 1 we directly obtain $I \simeq_{\langle\varphi\rangle} J$. Now, let $I, J$ be two interpretations and assume that $I <_{\langle\varphi\rangle} J$. We need to show that $I \models \varphi$ and $J \not\models \varphi$. Assume towards a contradiction that $I \not\models \varphi$. If $J \models \varphi$, then by condition 2 we get $J <_{\langle\varphi\rangle} I$; if $J \not\models \varphi$, then by Equation 11 we get $I \simeq_{\langle\varphi\rangle} J$. Both cases contradict the assumption $I <_{\langle\varphi\rangle} J$. Hence, we have $I \models \varphi$. Now, assume towards a contradiction that $J \models \varphi$. By condition 1 we get that $I \simeq_{\langle\varphi\rangle} J$, which contradicts the assumption $I <_{\langle\varphi\rangle} J$. Hence, $J \not\models \varphi$. This concludes the proof. □

PROPOSITION A.19
Let $\leq_{\mathcal{K}}$ be the preorder over worlds associated with a profile $\mathcal{K}$ by a filtering syncretic assignment. We have

$$I <_{\mathcal{K}} J \text{ iff } |I(\mathcal{K})| > |J(\mathcal{K})|.$$

We first prove the following lemma.

LEMMA A.1
Every assignment satisfying conditions 5 and F satisfies the following condition:

$$\text{If } I <_{\mathcal{K}} J, \text{ then } I \leq_{\mathcal{K}\sqcup\langle\varphi\rangle} J.$$

PROOF. We prove it by recursion on the size $n$ of $\mathcal{K}$:

• Base case ($n = 1$): let $I <_{\langle\varphi\rangle} J$. If $I \leq_{\langle\varphi'\rangle} J$, then by condition 5 we get $I \leq_{\langle\varphi,\varphi'\rangle} J$. If $J <_{\langle\varphi'\rangle} I$, then by condition F we get $I \simeq_{\langle\varphi,\varphi'\rangle} J$.

• Let $n \geq 1$ and assume that for every $\mathcal{K}$ such that $|\mathcal{K}| = n$, we have $I <_{\mathcal{K}} J \Rightarrow I \leq_{\mathcal{K}\sqcup\langle\varphi\rangle} J$ or equivalently $I <_{\mathcal{K}\sqcup\langle\varphi\rangle} J \Rightarrow I \leq_{\mathcal{K}} J$. Let $\mathcal{K}$ be a profile such that $|\mathcal{K}| = n + 1$ and assume $I <_{\mathcal{K}} J$. If $I \leq_{\langle\varphi\rangle} J$, then by condition 5 we get $I \leq_{\mathcal{K}\sqcup\langle\varphi\rangle} J$. Then assume $J <_{\langle\varphi\rangle} I$. Since $I <_{\mathcal{K}} J$, by condition 5 there exists a belief base $\varphi' \in \mathcal{K}$ such that $I <_{\langle\varphi'\rangle} J$, and by the recursion hypothesis, we also get $I \leq_{\mathcal{K}\setminus\langle\varphi'\rangle} J$. Since $J <_{\langle\varphi\rangle} I$ and $I <_{\langle\varphi'\rangle} J$, by condition F we get $I \simeq_{\langle\varphi,\varphi'\rangle} J$. Hence, by condition 5, we get $I \leq_{\mathcal{K}\sqcup\langle\varphi\rangle} J$. □

We now prove Proposition 5.3.

PROOF. We prove the equivalence stated in Proposition 5.3 by recursion on the size $n$ of $\mathcal{K}$:

• Base case ($n = 1$) : by Proposition 5.2, $I <_{\langle\varphi\rangle} J$ iff $I \models \varphi$ and $J \not\models \varphi$, this leads directly to $|I(\mathcal{K})| > |J(\mathcal{K})|$.

• Base case ($n = 2$) : let us show that $I <_{\langle\varphi,\varphi'\rangle} J$ iff ($I \leq_{\langle\varphi\rangle} J$ and $I <_{\langle\varphi'\rangle} J$) or ($I <_{\langle\varphi\rangle} J$ and $I \leq_{\langle\varphi'\rangle} J$). *(Only If)* Direct from condition 6. *(If)* Let $I <_{\langle\varphi,\varphi'\rangle} J$. By contradiction, assume $J <_{\langle\varphi\rangle} I$. Assuming $J \leq_{\langle\varphi'\rangle} I$ (respectively $I <_{\langle\varphi'\rangle} J$) contradicts $I <_{\langle\varphi,\varphi'\rangle} J$ by condition 5 (respectively by condition F). Hence, $I \leq_{\langle\varphi\rangle} J$. In a symmetric way, we can show that $I \leq_{\langle\varphi'\rangle} J$. Now, assuming $I \simeq_{\langle\varphi\rangle} J$ and $I \simeq_{\langle\varphi'\rangle} J$ contradicts $I <_{\langle\varphi,\varphi'\rangle} J$ by condition 5. Hence, ($I \leq_{\langle\varphi\rangle} J$ and $I <_{\langle\varphi'\rangle} J$) or ($I <_{\langle\varphi\rangle} J$ and $I \leq_{\langle\varphi'\rangle} J$). Since we deal with singleton profiles ($|\langle\varphi\rangle| = |\langle\varphi'\rangle| = 1$), we can apply Proposition 5.2 and directly get $|I(\langle\varphi,\varphi'\rangle)| > |J(\langle\varphi,\varphi'\rangle)|$.

• Let $n \geq 2$ and assume that for every profile $\mathcal{K}$, $|\mathcal{K}| = k$, $k \leq n$, we have $I <_{\mathcal{K}} J$ iff $|I(\mathcal{K})| > |J(\mathcal{K})|$. Let $\mathcal{K}$ be a profile such that $|\mathcal{K}| = n+1$. We need to show that $I <_{\mathcal{K}} J$ iff $|I(\mathcal{K})| > |J(\mathcal{K})|$. Assume first that $I <_{\mathcal{K}} J$ and let us show that $|I(\mathcal{K})| > |J(\mathcal{K})|$. Let $\varphi \in \mathcal{K}$:

- Assume $I <_{\langle\varphi\rangle} J$. Since $I <_{\mathcal{K}} J$, then by Lemma A.1 we have $I \leq_{\mathcal{K}\setminus\langle\varphi\rangle} J$. By recursion hypothesis, we have $|I(\langle\varphi\rangle)| > |J(\langle\varphi\rangle)|$ and $|I(\mathcal{K} \setminus \langle\varphi\rangle)| \geq |J(\mathcal{K} \setminus \langle\varphi\rangle)|$. Hence, $|I(\mathcal{K})| > |J(\mathcal{K})|$.
- Assume $I \simeq_{\langle\varphi\rangle} J$. Since $I <_{\mathcal{K}} J$, then by condition 5, we have $I <_{\mathcal{K}\setminus\langle\varphi\rangle} J$. By recursion hypothesis, we have $|I(\langle\varphi\rangle)| = |J(\langle\varphi\rangle)|$ and $|I(\mathcal{K} \setminus \langle\varphi\rangle)| > |J(\mathcal{K} \setminus \langle\varphi\rangle)|$. Hence, $|I(\mathcal{K})| > |J(\mathcal{K})|$.
- Assume $J <_{\langle\varphi\rangle} I$. Since $I <_{\mathcal{K}} J$, then by condition 5 there exists a belief base $\varphi' \in \mathcal{K}$ such that $I <_{\langle\varphi'\rangle} J$. By condition F, we get $I \simeq_{\langle\varphi,\varphi'\rangle} J$. And by condition 5, we have necessarily $I <_{\mathcal{K}\setminus\langle\varphi,\varphi'\rangle} J$. By recursion hypothesis, we have $|I(\langle\varphi,\varphi'\rangle)| = |J(\langle\varphi,\varphi'\rangle)|$ and $|I(\mathcal{K} \setminus \langle\varphi,\varphi'\rangle)| > |J(\mathcal{K} \setminus \langle\varphi,\varphi'\rangle)|$. Hence, $|I(\mathcal{K})| > |J(\mathcal{K})|$.

Assume now that $|I(\mathcal{K})| > |J(\mathcal{K})|$ and let us show that $I <_{\mathcal{K}} J$. By hypothesis, there exists a belief base $\varphi \in \mathcal{K}$ such that $I \models \varphi$ and $J \not\models \varphi$, so by condition 2 $I <_{\langle\varphi\rangle} J$. We get $|I(\mathcal{K} \setminus \langle\varphi\rangle)| \geq |J(\mathcal{K}\setminus\langle\varphi\rangle)|$, and by recursion hypothesis, $I \leq_{\mathcal{K}\setminus\langle\varphi\rangle} J$. Hence, by condition 6, $I <_{\mathcal{K}} J$. This concludes the proof. □

PROPOSITION A.20
An EIC merging operator $\Delta$ satisfies (Ind) iff there exists a filtering syncretic assignment associating every profile $\mathcal{K}$ with a total preorder $\leq_{\mathcal{K}}$ such that for every formula $\mu$, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$.

PROOF. *(Only If)* Let $\Delta$ be an IC merging operator satisfying (Ind) and let us consider the assignment mapping every profile $\mathcal{K}$ to a preorder $\leq_{\mathcal{K}}$ over worlds, defined for all worlds $I, J$ as $I \leq_{\mathcal{K}} J$ iff $I \models \Delta_{\varphi_{\{I,J\}}}(\mathcal{K})$. By Proposition 3.2 this assignment is an extended syncretic assignment, i.e. it satisfies conditions 0–6. It remains to show that it satisfies F. Let $I, J$ such that $I <_{\langle\varphi\rangle} J$ and $J <_{\langle\varphi'\rangle} I$. By definition of the assignment, we have $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi\rangle) \equiv \varphi_{\{I\}}$ and $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi'\rangle) \equiv \varphi_{\{J\}}$. Since $\Delta$ satisfies (Ind), by Corollary 14 it also satisfies (Ind-$\circ$), and by Proposition 4.2 every revision operator involved in (Ind-$\circ$) is the revision operator $\circ_\Delta$ corresponding to $\Delta$ (in the sense of Definition 2.8). Therefore, we have $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi,\varphi'\rangle) \equiv \Delta_{\varphi_{\{I,J\}}}(\langle\varphi \circ_\Delta \varphi_{\{I,J\}}, \varphi' \circ_\Delta \varphi_{\{I,J\}}\rangle) \equiv \Delta_{\varphi_{\{I,J\}}}(\langle\Delta_{\varphi_{\{I,J\}}}(\langle\varphi\rangle), \Delta_{\varphi_{\{I,J\}}}(\langle\varphi'\rangle)\rangle) \equiv \Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I\}}, \varphi_{\{J\}}\rangle)$. Yet by ($IC0$), ($IC1$) and ($IC4$), we have $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi_{\{I\}}, \varphi_{\{J\}}\rangle) \equiv \varphi_{\{I,J\}}$. So $\Delta_{\varphi_{\{I,J\}}}(\langle\varphi,\varphi'\rangle) \equiv \varphi_{\{I,J\}}$. Hence, $I \simeq_{\langle\varphi,\varphi'\rangle} J$, i.e. condition F is satisfied by the assignment. *(If)* Consider a filtering syncretic assignment and define $\Delta$ by $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_{\mathcal{K}})$. From Proposition 3.2, $\Delta$ satisfies (Inc), (IC0–IC8). Let $\mathcal{K} = \langle\varphi_1, \ldots, \varphi_n\rangle$ be a profile, $\mu$ be any formula and $\mathcal{K}' \equiv \langle\varphi_1 \wedge \mu, \ldots, \varphi_m \wedge \mu\rangle$. To prove that $\Delta$ satisfies (Ind), it is enough to show that for all worlds $I, J \models \mu$, $I <_{\mathcal{K}} J$ iff $I <_{\mathcal{K}'} J$. Yet we have that $|I(\mathcal{K})| = |I(\mathcal{K}')|$ and $|J(\mathcal{K})| = |J(\mathcal{K}')|$, so by using Proposition 19 we can state that for all worlds $I, J \models \mu$, $I <_{\mathcal{K}} J$ iff $|I(\mathcal{K})| > |J(\mathcal{K})|$ iff $|I(\mathcal{K}')| > |J(\mathcal{K}')|$ iff $I <_{\mathcal{K}'} J$. This concludes the proof.

□

PROPOSITION A.21
$\Delta^{d_D,\Sigma}$ is the only EIC merging operator satisfying (Ind).

PROOF. By Corollary 3, $\Delta^{d_D,\Sigma}$ is an EIC merging operator satisfying (Ind). Then by Proposition 5.4, $\Delta^{d_D,\Sigma}$ can be associated with a filtering syncretic assignment such that for every profile $\mathcal{K}$ and every formula $\mu$, $mod(\Delta_\mu(\mathcal{K})) = \min(mod(\mu), \leq_\mathcal{K})$. And by Corollary 3.1, there is only one such filtering syncretic assignment. Therefore, $\Delta^{d_D,\Sigma}$ is the only EIC merging operator satisfying (Ind). □

PROPOSITION A.22
Any distance-based merging operator $\Delta^{d,\Sigma}$ satisfies (ESF-I).

PROOF. Let $\Delta^{d,\Sigma}$ be an extended distance-based merging operator. Let $\mathcal{K} = \langle\varphi_1,\ldots,\varphi_n\rangle$, $\mathcal{K}' = \langle\varphi'_1,\ldots,\varphi'_n\rangle$, and let $\mu$ be a formula. Assume that for every formula $\mu' \models \mu$ and for each $i \in \{1,\ldots,n\}$, $\varphi_i \circ_{\Delta^{d,\Sigma}} \mu' \equiv \varphi'_i \circ_{\Delta^{d,\Sigma}} \mu'$, or equivalently, that $\Delta^{d,\Sigma}_{\mu'}(\langle\varphi_i\rangle) \equiv \Delta^{d,\Sigma}_{\mu'}(\langle\varphi'_i\rangle)$. By definition of $\Delta^{d,\Sigma}$, this precisely means that for each $i \in \{1,\ldots,n\}$ and for all interpretations $I,J \models \mu$, $d(I,\varphi_i) \leq d(J,\varphi_i)$ if and only if $d(I,\varphi'_i) \leq d(J,\varphi'_i)$; hence, $\sum_{\varphi_i\in\mathcal{K}}(d(I,\varphi_i)) \leq \sum_{\varphi_i\in\mathcal{K}}(d(J,\varphi_i))$ if and only if $\sum_{\varphi'_i\in\mathcal{K}'}(d(I,\varphi'_i)) \leq \sum_{\varphi'_i\in\mathcal{K}'}(d(J,\varphi'_i))$. Therefore, $\Delta^{d,\Sigma}_\mu(\mathcal{K}) \equiv \Delta^{d,\Sigma}_\mu(\mathcal{K}')$. This shows that $\Delta^{d,\Sigma}$ satisfies (ESF-I). □

Received 12 September 2018