Propositional belief base merging or how to merge beliefs/goals coming from several sources and some links with social choice theory

Sébastien Konieczny [a] Ramón Pino Pérez [b]

[a] *IRIT - CNRS*
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse - France
konieczny@irit.fr

[b] *Departamento de Matemáticas*
Facultad de Ciencias
Universidad de los Andes
Mérida - Venezuela
pino@ula.ve

Abstract

We give in this paper new results on merging operators. Those operators aim to define the goals (or beliefs) of an agents' group after the individuals' goals (beliefs). Using the logical framework of [17] we study the relationships between two important sub-families of merging operators: majority operators and arbitration operators. An open question was to know if those two families were disjoint or not. We show that there are operators that belong simultaneously to the two families. Furthermore, the new family introduced allows the user to choose the "consensual level" he wants for his majority operator. We show at the end of this work some relationships between logical belief merging operators and social choice rules.

Key words: Artificial intelligence, Group decisions and negotiations, Knowledge-based systems, Belief merging

1 Introduction

When several agents interact in order to achieve a common task, they have to agree from time to time on what are the beliefs (or the goals) of the group.

When some agents disagree on these common beliefs (goals), then one has to enter in a *negotiation* process. The problem is that sometimes the negotiation step does not rule out all the conflicts. Even in this case, the group has to make a decision on what are its beliefs (goals) in order to carry on. So, in such cases, an *aggregation* step is needed following the agents' wishes.

So, formally, when a decision has to be made about beliefs (goals) of the group, we can consider this as a two step process. First, a *negotiation* step allows agents to try to convince undecided or opponents. Then, when all agents have fixed opinions, an *aggregation* step states what are the common beliefs (goals) of the group.

While the first step of this process has been extensively studied in multi-agents works, the second one is usually only quickly quoted. Indeed, in most of these works, when some conflict is not solved after the negotiation step, one uses expeditious means to solve the conflicts. For example, by supposing the existence of some oracle that decides what is the good solution, or by using a preference relation among agents denoting the relative reliability of each source. Even if these solutions often allow to rule out the conflicts, the basic problem is not solved and there still are problems in some cases. For example, it is not realistic to suppose that an oracle exists and always knows the good answer. In the case of using a reliability ordering, there are cases where some equally reliable agents disagree and we are back with our basic problem.

Let us illustrate the kind of problem addressed with an example :

Example 1 *Consider three agents φ_1, φ_2, and φ_3 with the following beliefs (this example is stated in terms of belief, but it can also be phrased in terms of goals) :*

$$\varphi_1 = \{a, b \rightarrow c\}, \qquad \varphi_2 = \{a, b\}, \qquad \varphi_3 = \{\neg a\}.$$

What are the beliefs of the group $\triangle(\varphi_1 \sqcup \varphi_2 \sqcup \varphi_3)$?

We can see that taking the conjunction of all the belief bases is not adequate, since it leads to an inconsistent belief base. Nevertheless, we can remark that the inconsistency here is caused by a conflict on the truth of the variable a. So it can prove sensible to listen to φ_1 and to φ_2 about b and c, and so to take the formulas $\{b \rightarrow c, b\}$, that have nothing to do with the contradiction, in the result. We can then remark that the resulting belief base will know that c is true, whereas none of the initial belief bases knows it. This kind of belief was called implicit belief in [11].

A very cautious agent can stop here and keep those beliefs as the merged belief base, since there is no agreement for the other formulas. But it can also prove fully sensible to remark that two bases think that a is true and only one think that a is false, and then to take a in the resulting belief base, giving $\{a, b \rightarrow c, b\}$

as merged belief base.

The formal framework for performing this belief (goals) *aggregation* step, is the use of belief merging operators [6,5,4,20,24,15]. In some related works, different sets of logical properties which have to be satisfied by belief merging operators, have been proposed [24,19,20,16,17]. Those logical characterizations are used to define a taxonomy of merging operators, that allows to compare different merging methods and to choose the method corresponding to the behaviour desired in a particular application.

We will focus on the *merging with integrity constraints* characterization [17]. This characterization allows to make a distinction between two major sub-classes of merging operators: majority operators and arbitration operators. Majority operators solve conflicts using majority wishes, that is, they try to satisfy the group as a whole. Whereas arbitration operators have a more consensual behaviour, trying to satisfy each agent as far as possible.

So, these two sub-classes have very different conflict resolution policies. An open question was to know if these two sub-classes were disjoint or not. Though it seems natural to bet on a strict partition, we show in this paper that it is not the case. That is, there exist operators that belong simultaneously to the two sub-classes. We first give a trivial operator that straightforwardly satisfies this condition. But the real question was to know if more complex operators can satisfy it too. We show that, in the finite case, a whole family of (non-trivial) operators are both arbitration and majority operators when the number of sources is bounded (see Theorem 5). The new family of operators introduced, generalization of a well known majority merging method [24,20,17], allows to choose the "consensual level" that best fits the application needs.

The paper is organized as follows. After some preliminaries in section 2, we give the definition of merging with integrity constraints operators in section 3, arbitration and majority operators are also defined. Then, we give in section 4 some concrete operators in order to illustrate the differences of behaviour between arbitration and majority operators. In section 5, we show that it is possible for an operator to be both a majority and an arbitration operator. In section 6 we show some links between logical merging operators and social choice rules. We conclude in section 7 with some remarks and open questions.

2 Preliminaries

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional atoms. An interpretation is a function from $\mathcal{P}$ to $\{0, 1\}$. The set of all the interpretations is denoted $\mathcal{W}$. For example, if $\mathcal{P} = \{a, b, c\}$, then we will note $\{001\}$ the interpretation (world) that maps both a and b to 0 and c to 1. An interpretation I is a model of a formula if and only if it makes it true in the usual classical truth functional way.

Let φ be a formula, $mod(\varphi)$ denotes the set of models of φ, *i.e.* $mod(\varphi) = \{I \in \mathcal{W} \mid \ I \models \varphi\}$. Conversely, let M be a set of models, $\varphi_{\{M\}}$ denotes the formula (up to logical equivalence) whose models are M.

A *belief base* φ is a finite set of propositional formulae, which are considered conjunctively, *i.e.* $\varphi = \{\alpha_1, \ldots, \alpha_n\}$, where all α_i are propositional formulae, is logically equivalent to $\varphi' = \{\alpha_1 \wedge \ldots \wedge \alpha_n\}$. A belief base φ_i denotes the beliefs of the agent i. We will consider that all the belief bases are consistent, *i.e.* that all agents have non-self-contradictory beliefs.

Let $\varphi_1, \ldots, \varphi_n$ be n belief bases (not necessarily different), we call *belief set* the multi-set Ψ consisting of those n belief bases: $\Psi = \{\varphi_1, \ldots, \varphi_n\}$. Let $\Psi = \{\varphi_1, \ldots, \varphi_n\}$ and $\Psi' = \{\varphi'_1, \ldots, \varphi'_m\}$ be two multi-sets, the union of the two multi-sets is the set $\Psi \sqcup \Psi' = \{\varphi_1, \ldots, \varphi_n, \varphi'_1, \ldots \varphi'_m\}$. We denote by $\bigwedge \varphi$ the conjunction of formulae of φ, *i.e.* if $\varphi = \{\alpha_1, \ldots, \alpha_n\}$, then $\bigwedge \varphi = \alpha_1 \wedge \ldots \wedge \alpha_n$. As the belief bases are always considered conjunctively we will note φ for $\bigwedge \varphi$ in the following. We denote by $\bigwedge \Psi$ the conjunction of the belief bases of Ψ, i.e. $\bigwedge \Psi = \varphi_1 \wedge \cdots \wedge \varphi_n$. Note that, conversely to $\bigwedge \varphi$, $\bigwedge \Psi$ can be (and is often) inconsistent. By abuse if φ is a belief base, φ will also denote the belief set $\Psi = \{\varphi\}$. For a positive integer n we will denote Ψ^n the multi-set when Ψ appears n times.

Whereas the notion of logical equivalence for belief bases is obvious, there is not one straightforward definition of equivalence of belief sets (multi-sets of belief bases). We will consider the following one (see [23,24] for another one):

Definition 1 *Let Ψ_1, Ψ_2 be two belief sets. Ψ_1 and Ψ_2 are equivalent, noted $\Psi_1 \leftrightarrow \Psi_2$, iff there exists a bijection f from $\Psi_1 = \{\varphi^1_1, \ldots, \varphi^1_n\}$ to $\Psi_2 = \{\varphi^2_1, \ldots, \varphi^2_n\}$ such that $\vdash f(\varphi) \leftrightarrow \varphi$.*

A pre-order $\leq$ on a set A is a reflexive and transitive relation on A. A pre-order is total if $\forall I, J \in A, \ \ I \leq J$ or $J \leq I$. Let $\leq$ be a pre-order, we define the corresponding strict order $<$ as follows: $I < J$ iff $I \leq J$ and $J \not\leq I$, and the corresponding equivalence relation $\simeq$ is defined as $I \simeq J$ iff $I \leq J$ and $J \leq I$. We write $I \in min(A, \leq)$ iff $I \in A$ and $\forall J \in A, \ \ I \leq J$.

3 Merging with Integrity Constraints

Once these definitions are stated, we can define merging operators. A belief base φ will denote the beliefs [1] of an agent. A belief set Ψ will denote a group of agents.

The aim of merging operators is to define what are the beliefs of the group after the individuals' beliefs and the constraints imposed by the system (physical constraints, laws, etc.). Those constraints will be encoded in a belief base μ.

So, a merging operator $\triangle$ is a function that maps a belief set Ψ and a belief base μ that denotes the integrity constraints of the system, to a belief base $\triangle_\mu(\Psi)$. Intuitively, $\triangle_\mu(\Psi)$ denotes the beliefs of the group Ψ under the integrity constraints of the system.

The logical properties that one could expect from a belief merging operator are [17]:

Definition 2 $\triangle$ *is a merging with integrity constraints operator (IC merging operator in short) if and only if it satisfies the following properties:*

(IC0) $\triangle_\mu(\Psi) \vdash \mu$
(IC1) *If* μ *is consistent, then* $\triangle_\mu(\Psi)$ *is consistent*
(IC2) *If* $\bigwedge \Psi$ *is consistent with* μ*, then* $\triangle_\mu(\Psi) = \bigwedge \Psi \wedge \mu$
(IC3) *If* $\Psi_1 \leftrightarrow \Psi_2$ *and* $\mu_1 \leftrightarrow \mu_2$*, then* $\triangle_{\mu_1}(\Psi_1) \leftrightarrow \triangle_{\mu_2}(\Psi_2)$
(IC4) *If* $\varphi \vdash \mu$ *and* $\varphi' \vdash \mu$*, then* $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi \nvdash \bot \Rightarrow \triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$
(IC5) $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2) \vdash \triangle_\mu(\Psi_1 \sqcup \Psi_2)$
(IC6) *If* $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2) \nvdash \bot$*, then* $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$
(IC7) $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(\Psi)$
(IC8) *If* $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \nvdash \bot$*, then* $\triangle_{\mu_1 \wedge \mu_2}(\Psi) \vdash \triangle_{\mu_1}(\Psi)$

The intuitive meaning of the properties is the following: (IC0) assures that the result of the merging satisfies the integrity constraints. (IC1) states that, provided that the integrity constraints are consistent, the result of the merging is always consistent. (IC2) states that if possible, the result of the merging is simply the conjunction of the belief bases with the integrity constraints. (IC3) is the principle of irrelevance of syntax, expressing the fact that the result of the merging has to depend only on the expressed opinions and not on their syntactical presentation. (IC4) is the fairness postulate; the point is that when we merge two belief bases, merging operators must not give preference to one of them. (IC5) expresses the following idea: if a group Ψ_1 compromises on a set of alternatives to which A belongs, and another group Ψ_2 compromises on another set of alternatives which contains A too, than A has to be in the chosen

[1] in the following, we will call "beliefs" the beliefs or the goals of an agent

alternatives if we join the two groups. (IC5) and (IC6) together state that if you could find two subgroups which agree on at least one alternative, then the result of the global merging will be exactly those alternatives the two groups agree on. (IC7) and (IC8) state that the notion of closeness is well-behaved, *i.e.* that an alternative that is preferred among the possible alternatives (μ_1), will remain preferred if we restrict the possible choices ($\mu_1 \wedge \mu_2$).

We will now define the two major sub-classes of merging operators: majority and arbitration operators. An IC merging operator is a majority operator if it satisfies the following property:

(Maj) $\exists n \ \triangle_\mu(\Psi_1 \sqcup \Psi_2^n) \vdash \triangle_\mu(\Psi_2)$

This postulate expresses the fact that if an opinion has a large audience, it will be the opinion of the group. So, majority operators try to satisfy the group as a whole. On the other hand, arbitration operators try to satisfy each agent as far as possible. An IC merging operator is an arbitration operator if it satisfies the following property:

$$\textbf{(Arb)} \quad \left.\begin{array}{l} \triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2) \\ \triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2) \\ \mu_1 \nvdash \mu_2 \\ \mu_2 \nvdash \mu_1 \end{array}\right\} \Rightarrow \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle_{\mu_1}(\varphi_1)$$

This postulate ensures that this is the median possible choices that are preferred. It is much more intuitive when it is expressed in terms of syncretic assignment (*cf* condition 8 below). We will illustrate this on the following scenario:

Example 2 *Tom and David missed the soccer match yesterday between reds and yellows. So they don't know the result of the match. Tom listened in the morning that reds made a very good match. So he thinks that a win of reds is more plausible than a draw and that a draw is more reliable than a win of yellows. David was told that after that match yellows have now a lot of chances of winning the championship. From this information he infers that yellows won the match, or otherwise at least took a draw. Confronting their points of view, Tom and David agree on the fact that the two teams are of the same strength, and that they had the same chances of winning the match. What* arbitration *demand is that, with those informations, Tom and David have to agree that a draw between the two teams is the more plausible result.*

Now we will give a representation theorem for these operators in terms of pre-orders on interpretations. It provides a more constructive definition for these operators.

Definition 3 *A* syncretic assignment *is a function mapping each belief set* Ψ *to a total pre-order* $\leq_\Psi$ *over interpretations such that for any belief sets* Ψ, Ψ_1, Ψ_2 *and for any belief bases* φ_1, φ_2*:*

1. *If* $I \models \Psi$ *and* $J \models \Psi$*, then* $I \simeq_\Psi J$
2. *If* $I \models \Psi$ *and* $J \not\models \Psi$*, then* $I <_\Psi J$
3. *If* $\Psi_1 \equiv \Psi_2$*, then* $\leq_{\Psi_1} = \leq_{\Psi_2}$
4. $\forall I \models \varphi_1 \; \exists J \models \varphi_2 \; J \leq_{\varphi_1 \sqcup \varphi_2} I$
5. *If* $I \leq_{\Psi_1} J$ *and* $I \leq_{\Psi_2} J$*, then* $I \leq_{\Psi_1 \sqcup \Psi_2} J$
6. *If* $I <_{\Psi_1} J$ *and* $I \leq_{\Psi_2} J$*, then* $I <_{\Psi_1 \sqcup \Psi_2} J$

A majority syncretic assignment *is a syncretic assignment which satisfies the following:*

7. *If* $I <_{\Psi_2} J$*, then* $\exists n \; I <_{\Psi_1 \sqcup \Psi_2{}^n} J$

A fair syncretic assignment *is a syncretic assignment which satisfies the following:*

8. $$\left.\begin{array}{l} I <_{\varphi_1} J \\ I <_{\varphi_2} J' \\ J \simeq_{\varphi_1 \sqcup \varphi_2} J' \end{array}\right\} \Rightarrow I <_{\varphi_1 \sqcup \varphi_2} J$$

One can note that, conversely to most domains, where people try to maximize something (something is often a utility), here, and in all the belief revision area (see e.g. [1,9,10,12]) people try to minimize the differences, so $I \leq J$ will mean that I is at least as good as J. Let us see what is the meaning of the previous conditions:

The two first conditions ensure that the models of the belief set (if any) are the more plausible interpretations for the pre-order associated to the belief set. The third condition states that two equivalent belief sets have the same associated pre-orders. Those three conditions are very close to the ones existing in belief revision for faithful assignments [12]. The fourth condition states that, when merging two belief bases, for each model of the first one, there is a model of the second one that is at least as good as the first one. It ensures that the two belief bases are given the same consideration. The fifth condition says that if an interpretation I is at least as plausible as an interpretation J for a belief set Ψ_1 and if I is at least as plausible as J for a belief set Ψ_2, then if one joins the two belief sets, I will still be at least as plausible as J. The sixth condition strengthens the previous condition by saying that if an interpretation I is at least as plausible as an interpretation J for a belief set Ψ_1 and if I is strictly more plausible than J for a belief set Ψ_2, then if one joins the two belief sets, then I will be strictly more plausible than J. These two previous conditions are closely related to Pareto conditions in Social Choice Theory [2,14] (see section 6). Condition 7 says that if an interpretation I is strictly more plausible than an interpretation J for a belief set Ψ_2, then there is a quorum n of repetitions of the belief set from which I will be more plausible than J for the larger belief set $\Psi_1 \sqcup {\Psi_2}^n$. This condition seems to be the weakest form of "majority" condition one could state. Condition 8 states that if an interpretation I is more plausible than an interpretation J for a belief base φ_1, if I is more plausible than J' for another base φ_2, and if J and J' are equally plausible for the belief set $\varphi_1 \sqcup \varphi_2$, then I has to be more plausible than J and J' for $\varphi_1 \sqcup \varphi_2$ (see Example 2 for an intuitive explanation).

Now the following representation theorem shows that defining a merging operator satisfying all the wanted properties is the same as considering some kind of preference relation over interpretations induced by the group (belief set).

Theorem 1 *An operator is an IC merging operator (respectively IC majority merging operator or IC arbitration operator) if and only if there exists a syncretic assignment (respectively majority syncretic assignment or fair syncretic assignment) that maps each belief set* Ψ *to a total pre-order* $\leq_\Psi$ *such that* $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$.

The proof is in the Appendix.

This theorem shows that a merging operator corresponds to a family of pre-orders. In fact, a lot of operators are defined directly from those pre-orders, using a function that maps each belief set to a pre-order. It is the case with all distance based operators. We give some of them in the following section.

4 Some IC merging operators

We give in this section the definitions of three families of operators. All these operators are based on a distance between interpretations that induces the pre-order associated to each belief set. We define also a new family of operators, that generalizes the $\triangle^{d,\Sigma}$ family.

Let d be a distance [2] between interpretations, that is, a function $d : \mathcal{W} \times \mathcal{W} \mapsto I\!N$ such that:

- $d(I, J) = d(J, I)$, and
- $d(I, J) = 0$ iff $I = J$.

[2] Remark that the triangular inequality $d(I, J) \leq d(I, J') + d(J', J)$ is not required, hence, strictly speaking, d is only a pseudo-distance.

For example, one can use the Dalal distance [7], noted d_H, that is the Hamming distance between two interpretations (the number of propositional letters on which the two interpretations differ). We will use this distance in the examples because it is a well known, easy to define, distance, but one has to keep in mind that it is not the sole possible choice and that the logical properties do not depend on the chosen distance.

This distance between interpretations induces naturally a distance between an interpretation and a belief base as follows:

$$d(I,\varphi) = \min_{J\models\varphi} d(I,J)$$

The difference between the four families of operators we define next lie in the way that this distance between an interpretation and a belief base is used in order to define the distance between an interpretation and the belief set. So, it is this aggregation step of the individual preferences (distances) in a global one that makes behaviour differences between the families.

The three families stated next are well known, the first one, the $\triangle^{d,Max}$ family has been defined in [23,24,18]:

Definition 4 *Let Ψ be a belief set, I be an interpretation and d be a distance between interpretations. The* max *distance is defined by:*

$$d_{d,Max}(I,\Psi) = \max_{\varphi\in\Psi} d(I,\varphi)$$

This distance induces a pre-order on interpretations:

$$I \leq_{\Psi}^{d,max} J \textit{ iff } d_{d,max}(I,\Psi) \leq d_{d,max}(J,\Psi)$$

And the corresponding merging operator is defined by:

$$mod(\triangle_{\mu}^{d,Max}(\Psi)) = \min(mod(\mu), \leq_{\Psi}^{d,max})$$

The $\triangle^{d,\Sigma}$ family has been defined in [24,20,17]:

Definition 5 *Let Ψ be a belief set, I be an interpretation and d be a distance between interpretations. The Σ distance is defined by:*

$$d_{d,\Sigma}(I,\Psi) = \sum_{\varphi\in\Psi} d(I,\varphi)$$

This distance induces a pre-order on interpretations:

$$I \leq_{\Psi}^{d,\Sigma} J \textit{ iff } d_{d,\Sigma}(I,\Psi) \leq d_{d,\Sigma}(J,\Psi)$$

And the corresponding merging operator is defined by:

$$mod(\triangle_{\mu}^{d,\Sigma}(\Psi)) = \min(mod(\mu), \leq_{\Psi}^{d,\Sigma})$$

The $\triangle^{d,GMax}$ family has been defined in [16,17]:

Definition 6 *Let Ψ be a belief set, I be an interpretation and d be a distance between interpretations. The GMax distance is defined by:*

Suppose $\Psi = \{\varphi_1 \dots \varphi_n\}$. For each interpretation I we build the list $(d_1^I \dots d_n^I)$ of distances between this interpretation and the n belief bases in Ψ, i.e. $d_j^I = d(I, \varphi_j)$. Let $d_{d,GMax}(I, \Psi)$ be the list obtained from $(d_1^I \dots d_n^I)$ by sorting it in descending order [3].

This distance induces a pre-order on interpretations:

$$I \leq_{\Psi}^{d,GMax} J \text{ iff } d_{d,GMax}(I, \Psi) \leq d_{d,GMax}(J, \Psi)$$

And the corresponding merging operator is defined by:

$$mod(\triangle_{\mu}^{d,GMax}(\Psi)) = \min(mod(\mu), \leq_{\Psi}^{d,GMax})$$

We will give an example showing the behaviour of these families of operators at the end of this section. For the moment, let us see what are the logical properties of those operators.

Theorem 2 *$\triangle^{d,Max}$ operators satisfy (IC1-IC5), (IC7), (IC8) and (Arb). $\triangle^{d,GMax}$ operators are arbitration operators. $\triangle^{d,\Sigma}$ operators are majority operators.*

See [18] for the proofs. It is possible to generalize the $\triangle^{d,\Sigma}$ family in the following $\triangle^{d,\Sigma^n}$ operators:

Definition 7 $d_{d,\Sigma^n}(I, \Psi) = \sqrt[n]{\sum_{\varphi \in \Psi} d(I, \varphi)^n}$. *Then the corresponding pre-order is:*

$$I \leq_{\Psi}^{d,\Sigma^n} J \text{ iff } d_{d,\Sigma^n}(I, \Psi) \leq d_{d,\Sigma^n}(J, \Psi)$$

And the $\triangle^{d,\Sigma^n}$ operator is defined by:

$$mod(\triangle_{\mu}^{d,\Sigma^n}(\Psi)) = \min(mod(\mu), \leq_{\Psi}^{d,\Sigma^n})$$

[3] The d_{GMax} distance do not strictly obey to the requirements of a distance, since it does not give numbers. In fact there is a natural mapping: choose a sufficiently big number N (where sufficiently means strictly bigger than all possible distances $d(I, \varphi_i)$, it is always possible since we work in the finite case), and then define $d_{d,GMax} = \Sigma_{j=1\dots n}(d_{i_j}^I * N^{n-j+1})$, where i_j denotes the jth element in the sorted list.

It is easy to show then that:

Theorem 3 $\triangle^{d,\Sigma^n}$ *operators are majority operators.*

Now we illustrate the behaviour of these families on an example:

Example 3 *At a meeting of a block of flats co-owners, the chairman proposes for the coming year the construction of a swimming-pool, a tennis-court and a private-car-park. But if two of these three items are build, the rent will increase significantly. We will denote by S, T, P respectively the construction of the swimming-pool, the tennis-court and the private-car-park. We will denote I the rent increase. The chairman outlines that building two items or more will have an important impact on the rent: $\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I$*

There are four co-owners $\Psi = \{\varphi_1 \sqcup \varphi_2 \sqcup \varphi_3 \sqcup \varphi_4\}$. Two of the co-owners want to build the three items and don't care about the rent increase: $\varphi_1 = \varphi_2 = S \wedge T \wedge P$. The third one thinks that building any item will cause at some time an increase of the rent and wants to pay the lowest rent so he is opposed to any construction: $\varphi_3 = \neg S \wedge \neg T \wedge \neg P \wedge \neg I$. The last one thinks that the flat really needs a tennis-court and a private-car-park but doesn't want a high rent increase: $\varphi_4 = T \wedge P \wedge \neg I$.

The propositional letters S, T, P, I will be considered in that order for the valuations: $mod(\mu) = \mathcal{W} \setminus \{(0,1,1,0), (1,0,1,0), (1,1,0,0), (1,1,1,0)\}$

$mod(\varphi_1) = \{(1,1,1,1), (1,1,1,0)\}$ $\quad$ $mod(\varphi_2) = \{(1,1,1,1), (1,1,1,0)\}$

$mod(\varphi_3) = \{(0,0,0,0)\}$ $\quad$ $mod(\varphi_4) = \{(1,1,1,0), (0,1,1,0)\}$

We sum up the calculations in table 1. The lines shadowed correspond to the interpretations rejected by the integrity constraints. Thus the result has to be found among the interpretations that are not shadowed.

With the $\triangle^{d_H,Max}$ operator, the minimum distance is 2 and the chosen interpretations are $mod(\triangle_\mu^{d_H,Max}(\Psi)) = \{(0,0,1,0), (0,0,1,1), (0,1,0,0), (0,1,0,1), (1,0,0,0)\}$. So, the decision that best fits the group wishes is then not to increase the rent and to build one of the three items, or to increase the rent and build either the tennis court or the private car-park.

We can see on that example why $\triangle^{d,Max}$ operators are not IC merging operators. For example, the two interpretations $(0,0,1,0)$ and $(0,0,1,1)$ are chosen by $\triangle^{d_H,Max}$, although $(0,0,1,0)$ is better for φ_3 and φ_4 than $(0,0,1,1)$, whereas these two interpretations are equally preferred by φ_1 and φ_2. It seems then natural to globally prefer $(0,0,1,0)$ to $(0,0,1,1)$.

The $\triangle^{d,GMax}$ family has been built with that idea of being more selective than the $\triangle^{d,Max}$ family by taking this kind of requirements into account. With the $\triangle^{d_H,GMax}$ operator the result is $mod(\triangle_\mu^{d_H,GMax}(\Psi)) = \{(0,0,1,0), (0,1,0,0)\}$, so the decision in this case will be to build either the tennis court or the car-

	φ_1	φ_2	φ_3	φ_4	$\mathbf{dist_{Max}}$	$\mathbf{dist_{\Sigma}}$	$\mathbf{dist_{GMax}}$	$\mathbf{dist_{\Sigma^2}}$
$(0,0,0,0)$	3	3	0	2	3	8	(3,3,2,0)	22
$(0,0,0,1)$	3	3	1	3	3	10	(3,3,3,1)	28
$(0,0,1,0)$	2	2	1	1	**2**	6	**(2,2,1,1)**	**10**
$(0,0,1,1)$	2	2	2	2	**2**	8	(2,2,2,2)	16
$(0,1,0,0)$	2	2	1	1	**2**	6	**(2,2,1,1)**	**10**
$(0,1,0,1)$	2	2	2	2	**2**	8	(2,2,2,2)	16
$(0,1,1,0)$	1	1	2	0	2	4	(2,1,1,0)	6
$(0,1,1,1)$	1	1	3	1	3	6	(3,1,1,1)	12
$(1,0,0,0)$	2	2	1	2	**2**	7	(2,2,2,1)	13
$(1,0,0,1)$	2	2	2	3	3	9	(3,2,2,2)	21
$(1,0,1,0)$	1	1	2	1	2	5	(2,1,1,1)	7
$(1,0,1,1)$	1	1	3	2	3	7	(3,2,1,1)	15
$(1,1,0,0)$	1	1	2	1	2	5	(2,1,1,1)	7
$(1,1,0,1)$	1	1	3	2	3	7	(3,2,1,1)	15
$(1,1,1,0)$	0	0	3	0	3	3	(3,0,0,0)	9
$(1,1,1,1)$	0	0	4	1	4	**5**	(4,1,0,0)	17

Table 1
Distances

park but without increasing the rent.

But if one chooses $\triangle^{d_H,\Sigma}$ *for solving the conflict according to majority wishes, the result is then* $mod(\triangle_{\mu}^{d_H,\Sigma}(\Psi)) = \{(1,1,1,1)\}$*, and the decision will be to build the three items and to increase the rent.*

Majority voting, *à la* $\triangle^{d,\Sigma}$, often seems more democratic than the other methods but, for example in this case, this only works if φ_3 accepts to obey to this decision that is strictly opposed to his opinion. If φ_3 decides not to pay the rent increase, the works will perhaps not carry on because of a lack of money. So if a decision requires the approval of all the members a more consensual, arbitration like, method seems more adequate. These kinds of issues are highly related with social choice theory [2,14,21].

On this example, one can illustrate the use of the $\triangle^{d,\Sigma^n}$ family, since with the operator $\triangle^{d_H,\Sigma^2}$ we can see that the result is the same as with the $\triangle^{d_H,GMax}$ operator. The reason is that the power used in the definition of the operator allows to be more consensual while keeping the majority behaviour.

5 Arbitration versus Majority

We show in this section that some operators are both majority and arbitration operators. We first show this with an (over)simple operator. Then, we show that a whole family of full sense operators (the $\triangle^{d,\Sigma^n}$ operators) satisfies also this condition when the cardinality of the belief set is bounded.

5.1 Drastic Distance

The simplest distance between interpretations one can define is the following one:

$$d_D(I,J) = \begin{cases} 0 \text{ if } I = J \\ 1 \text{ otherwise} \end{cases}$$

The induced distance between an interpretation and a belief base is then also 0 or 1 if the interpretation respectively satisfies or not the belief base.

It is then easy to show that the operators given with this distance by the two families $\triangle^{d,GMax}$ and $\triangle^{d,\Sigma}$ are the same. And we have the following result:

Theorem 4 *The operator $\triangle^{d_D,\Sigma} = \triangle^{d_D,GMax}$ satisfies (IC0)-(IC8), (Maj) and (Arb).*

Once stated that $\triangle^{d_D,\Sigma} = \triangle^{d_D,GMax}$, the result is a direct consequence of theorem 2.

This easy to state result is not very surprising. But the real question was to know if more elaborate distances can lead to such *"collision"* between majority and arbitration classes. We answer this question in the next section.

5.2 Graphical study

We show in this section that some $\triangle^{d,\Sigma^n}$ operators are simultaneously majority and arbitration operators. For an easy explanation, we will use a graphical construction showing the behaviour of the operators "at work". In order to have a 2D representation we will restrict ourselves to two belief bases (All the results of this section do not depend on the chosen distance d).

The graphical construction is simple. We put the interpretations in the plane with their distance to the φ_2 base as abscissa and with their distance to φ_1 as ordinate. Then, the aim of the merging is to find the set of interpretations that are the closest to the (0,0) point. The differences between the operators lie in the chosen distance and in this definition of "closeness".

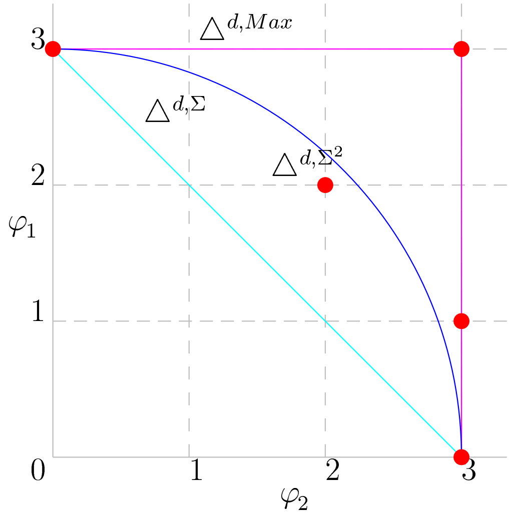


Fig. 1. Merging of two belief bases

On figure 1, the curves represent the interpretations that are at a distance 3 from the belief set $\{\varphi_1, \varphi_2\}$ according to the operators $\triangle^{d,Max}$, $\triangle^{d,\Sigma}$ and $\triangle^{d,\Sigma^2}$. $\triangle^{d,Max}$ is represented by a square of size a, $\triangle^{d,\Sigma}$ by the line $x = a - y$, and $\triangle^{d,\Sigma^2}$ by a circle arc of radius a, where a denotes the distance from the belief set. The $\triangle^{d,GMax}$ operator is hardly representable in this way, but one can figure out a curve that follows the one of $\triangle^{d,Max}$ but prefers the interpretations that are closest to the axes. We will see soon how to approximate graphically the $\triangle^{d,GMax}$ operator. Then the result of the merging, using these three operators, is the set of interpretations which the respective curves meet first when a varies from 0 to ∞.

In particular, on this example, the result for $\triangle^{d,Max}$ and $\triangle^{d,\Sigma^2}$ is the interpretation placed in (2,2). And for $\triangle^{d,\Sigma}$ the result is the interpretations placed in (3,0) and (0,3). In the same way, one can rebuild the pre-orders $\leq_{\Psi}^{d,Max}$, $\leq_{\Psi}^{d,\Sigma}$ and $\leq_{\Psi}^{D,\Sigma^2}$ when one considers the order in which the interpretations are met by the curves (when a varies from 0 to ∞).

On the figure, we can see the problem of $\triangle^{d,Max}$: it does not make any distinction between the (3,0) and (3,3) points for example. It is why $\triangle^{d,Max}$ is not an IC merging operator.

On the other side, $\triangle^{d,\Sigma}$ does not make any distinction on the sources of disagreements. Indeed, the distance from an interpretation to the belief set can be viewed as a measure of the disagreement induced by this interpretation on the belief set. Hence $\triangle^{d,\Sigma}$ is absolutely not consensual, since it allows to choose interpretations that satisfy completely one of the two bases and that dissatisfy completely the other one (the one placed in (0,3) for example), whereas there are more consensual choices (an interpretation placed in (2,1) or in (2,2) for example). This behaviour may seem normal for a majority operator. But it is not systematic. Indeed, the operators $\triangle^{d,\Sigma^n}$ with $n > 1$ prefer more consensual choices, that is the ones closest to the line $x = y$. So, an interpretation placed in (2,2) would be preferred to one placed in (3,0).

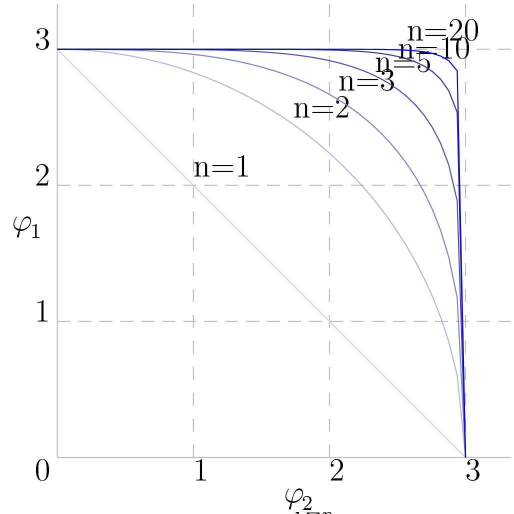


Fig. 2. The $\triangle^{d,\Sigma^n}$ family

We should stress that the operator $\triangle^{d,\Sigma^2}$ is a particular operator of the $\triangle^{d,\Sigma^n}$ class, since it uses the Euclidean distance as distance between an interpretation and the belief set. This gives a spherical distance, that is very natural and obeys to majority wishes but without the excesses of $\triangle^{d,\Sigma}$.

Furthermore, one can see on figure 2 that, when one increases the value of n, the curve of $\triangle^{d,\Sigma^n}$ comes near to the one of $\triangle^{d,Max}$. This fact is the point leading, in a heuristic way, to think that one can take the curve of $\triangle^{d,\Sigma^n}$ as an approximation of the one of $\triangle^{d,GMax}$ for a sufficiently big n. More formally, we have the following result (see the proof in Appendix):

Theorem 5 *Let k be a given positive integer. Then for any belief set Ψ such that the number of belief bases in Ψ is less than k , there exists n_0 such that for any $n > n_0$*

$$\triangle_\mu^{d,\Sigma^n}(\Psi) = \triangle_\mu^{d,GMax}(\Psi)$$

A corollary of the previous theorem's proof is that one can prove that for any integer $n \geq 1$ the operators $\triangle^{d,\Sigma^n}$ and $\triangle^{d,GMax}$ are different. More precisely we have the following result:

Theorem 6 *For all integers $n \geq 1$ there exists a belief set Ψ such that*

$$\triangle_\mu^{d,\Sigma^n}(\Psi) \neq \triangle_\mu^{d,GMax}(\Psi)$$

These results are a partial answer to the partition between majority and arbitration operators. Since, if we restrict the domain of $\triangle^{d,\Sigma^n}$ operators to belief sets of a fixed size, the $\triangle^{d,\Sigma^n}$ operators (for all n greater than a given n_0) have the same behavior as $\triangle^{d,GMax}$, therefore they are both arbitration and majority operators. So, the intersection between these two classes is, in the bounded case, non empty.

However, the theorems 5 and 6 together say that the identification between $\triangle^{d,\Sigma^n}$ and $\triangle^{d,GMax}$ cannot be done when the size of the belief set is not bounded. Thus, the problem of finding a non-trivial operator having the behavior of majority and arbitration operators at the same time remains unsolved.

6 Logic-Based Merging and Social Choice Theory

In economy, the problem of the aggregation of individual preferences into a collective preference is prominent. The comparative study of the different voting systems has been dealt with for a long time [8,22]. It is easy to show that the result of an election depends on the voting system as much as on individual preferences, that is to say that with the same individual preferences the winner of the election can be chosen (or at least changed) by changing the voting system. Therefore it is important to be able to answer a certain number of questions. How can the different voting systems be characterized? What is a good voting system? How can we say that a system is better than another one? These are (some of) the goals of Social Choice Theory [2,14,3].

Thanks to the representation theorem (theorem 1) we can highlight tight links between logic-based merging and preferences aggregation methods. Thus, it will be interesting to see the similarities and the differences between them.

6.1 Arrow's Impossibility Theorem

A very important result in Social Choice Theory is Arrow's Impossibility theorem [2,13]. This theorem states that it is not possible to define a "good" aggregation method. More exactly, Arrow gives a set of five very intuitive properties which all seem necessary for an aggregation method, and then he shows that it is not possible to satisfy all of these properties.

To state Arrow's theorem we first need to give some definitions :

Let X be a non-empty set. Elements of X are called *alternatives*. These alternatives have to be exclusive and we will suppose that they are a complete description of the world.

Usually, when one has to make a choice, all the possible alternatives are not available. Some constraints come and limit the number of those alternatives to a subset v of X. Such a set is called an *agenda*.

An *(individual) preference relation* is a total pre-order $\leq_i$ over X that denotes the preferences of the individual i over the set of alternatives X.

We will call *profile* a (multi-)set of individual preference relations.

A *choice function* C is a function that chooses among the alternatives of an agenda v a set of (best) alternatives $C(v)$ such that $C(v) \neq \emptyset$ and $C(v) \subseteq v$.

A *social choice rule* is a function f such that for each profile u, $f(u) = C$, where C is a choice function (*cf* figure 3). In other words, a social choice rule is a function that associates to each set of individual preferences (a profile), the corresponding choice function.

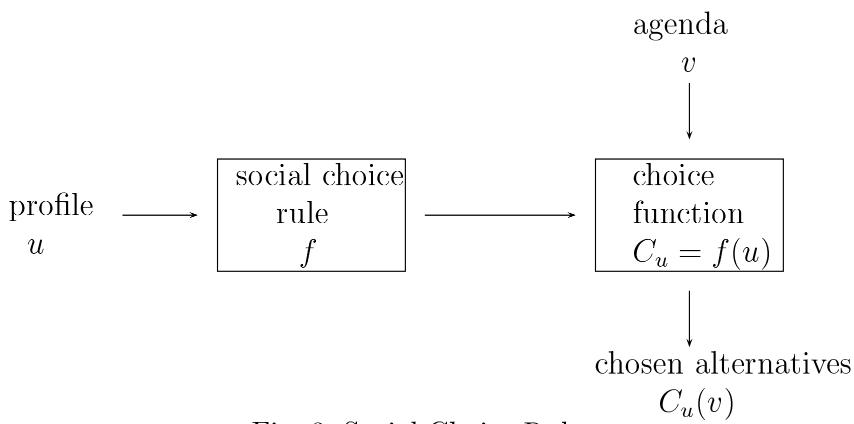


Fig. 3. Social Choice Rule

The aim of Social Choice Theory is to study these social choice rules. An easy combinatorial calculus shows that it is not possible to study social choice rules individually, as shown in example 4 [14]. It is why people rather study classes of rules that satisfy some given properties. The same idea justifies the axiomatic approach for non-monotonic inference relations, belief revision, and knowledge base merging.

Example 4 *Consider a small example with only five individuals and four alternatives. So, there are* 75 *preference relations on those four alternatives. That gives* $75^5 > 10^9$ *possible profiles. On the other hand, four alternatives give* $15 * 7^4 * 3^6 = 26\,254\,935$ *possible choice functions. So the number of possible social choice rules is :*

$$(15 * 7^4 * 3^6)^{75^5} > 10^{10^{10}}$$

One can argue that such a number denotes only the number of social choice rules mathematically conceivable but that the number of "reasonable" ones is much smaller. For example the social choice rule that chooses systematically the alternative preferred for $\leq_1$, without taking the rest of the profile into account (that is the rule that listens only to the first individual), is one of these rules. But it is rarely used in real applications !

So, it is possible to reduce significantly the number of desirable social choice rules by considering only those that fit a set of "intuitive" rationality criteria. We will now enumerate some of these criteria.

The *standard domain constraint* :

- There are at least three alternatives in X
- There are at least three individuals
- A social choice rule has as domain the set of all profiles definable from the preference relations over X.
- Any choice function result of the social choice rule has as domain all possible agendas.

This constraint is very natural. The two first points aim only to drop the simplest cases. So the constraint states simply that the social choice rule has to give a result for all given profile and agenda.

The *strong Pareto condition* :

> Let u be a profile, and let $C_u = f(u)$ be the choice function associated to u by the social choice rule f. If all the individuals of u consider an alternative x at least as good as an alternative y, and if at least one individual prefers strictly x to y, then if x is an available alternative ($x \in v$), then y will not be chosen ($y \notin C_u(v)$).

This condition allows to drop away an alternative y from the final choice if there exists another alternative that no one considers less preferable than y and that at least one individual finds strictly preferable to y.

The *independence of irrelevant alternatives* :

> If the restrictions of two profiles u and u' to an agenda v are identical, then the choices made from this agenda will be the same : $C_u(v) = C_{u'}(v)$

This property ensures that when one has to make a choice among a set of available alternatives v, this choice will be made only from the preferences on those alternatives, and not from non-available (irrelevant) ones. For example, suppose that at an election we have the choice between three candidates : Garry, Anatoly and Bobby. Just before the election Bobby says that he resigns, then it seems reasonable to make the decision by considering only the relative preferences between Garry and Anatoly.

The property of *having transitive explanations* :

> A choice function C has transitive explanations if there is a pre-order $\prec_C$ such that $C(v) = \{x \in v : x \prec_C y \text{ for all } y \in v\}$.
>
> A social choice rule has transitive explanations if all the choice functions that it gives have transitive explanations.

This property asks for some kind of rationality for collective preferences. For example, if when the two available alternatives are $v = \{x, y\}$, then $C(v) = \{x\}$, and if when the two alternatives are $v' = \{y, z\}$, then $C(v') = \{y\}$, then if we can choose among the whole set of alternatives $v'' = \{x, y, z\}$, then x will be chosen, that is, $C(v'') = \{x\}$. This seems reasonable since the profile (*i.e.* the individual preferences) do not change.

The property of *absence of dictator* :

> A social choice rule has no dictator if there is no individual i such that $\forall x, y \in v$ if $x <_i y$ then $y \notin C_u(v)$.

This property says that there is no one that has full power. This rule is hardly debatable for a social choice rule.

We have enumerated a set of properties that characterized "sensible" social choice rules. Then, one could expect that the number of social choice rules has significantly decreased. The "surprising" result stated in the Arrow's impossibility theorem is that there is no social choice rule that satisfies those five properties [2] :

Theorem 7 *There is no social choice rule that satisfies all of the following properties :*

- *the standard domain constraint*
- *the strong Pareto condition*
- *the independence of irrelevant alternatives*
- *the property of having transitive explanations*
- *the absence of dictator*

This result is interesting since, taken individually, all these conditions seem acceptable, and even necessary, for a social choice rule.

This theorem is one of the most important results in social choice theory. Since then, other impossibility theorems have been proved (*cf* e.g. [25]).

We see in this section which of the previous properties are satisfied by merging operators, and why some of them are not.

First we explain the correspondence between merging operators and social choice theory. This correspondence is based on the representation theorem of merging operators in terms of family of pre-orders on interpretations.

An alternative is an interpretation (complete and exclusive description of the world).

The individuals i for merging operators are represented by their knowledge base φ_i. Given a merging operator (*i.e.* a social choice rule), the knowledge base φ_i corresponds to a pre-order (via the representation theorem) $\leq_{\varphi_i}$. So this pre-order will take place of individual preference relation.

A profile is a set of individual preference relations, so it corresponds to a knowledge set.

An agenda is a subset of alternatives, that is, a base μ that represents the integrity constraints for the merging.

The social choice rules we are going to consider are merging operators. They aggregate individual preference relations in a collective one. The chosen alternatives by the choice function are the minimum alternatives for this collective preference relation. More exactly, the corresponding choice functions are the functions $f(u) = \min(v, \leq_u)$ where the pre-order $\leq_u$ is given by the social choice rule (the merging operator).

The correspondence between social choice theory and logic-based merging operators is summarized in the figure 4.

	Belief Merging	**Social Choice Theory**
individual	φ_i	i
individual preferences	$\leq_{\varphi_i}$	$\leq_i$
profile	$\Psi = \{\leq_{\varphi_1}, \ldots, \leq_{\varphi_n}\}$	$u = \{\leq_1, \ldots, \leq_n\}$
agenda	μ	v
choice function	$\leq_\Psi$	C_u
chosen alternatives	$\triangle_\mu(\Psi) = \min(\mu, \leq_\Psi)$	$C_u(v)$

Fig. 4. Merging operators vs Social Choice Theory

We can now investigate the different properties of social choice rules :

The standard domain constraint is not satisfied. Merging operators give a result for any given knowledge set and for any integrity constraints knowledge base. So it seems to fit the requirements. But one has to recall that the pre-order (individual preferences) $\leq_{\varphi_i}$ associated to a belief base (individual) i is not chosen by the individual i, but extrapolated from the belief base φ_i by the merging operator. Even if any pre-order can be obtained as input by using an adequate merging operator, some profiles are forbidden by requirements given by the syncretic assignments. For example, whatever merging operator is used, one can't have two pre-orders in the same profile with the same minimal interpretations, and with differences for the non-minimal ones.

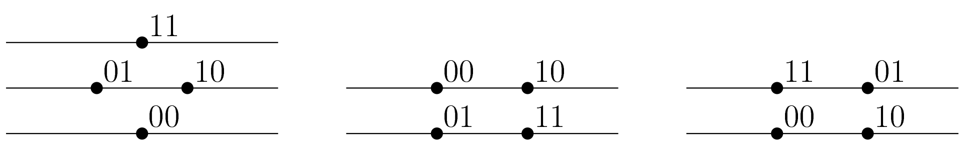
The strong Pareto condition is satisfied. This is mainly ensured by postulates (IC5) and (IC6) of merging operators.

The property of having transitive explanation is trivially satisfied since the "social choice rule" defined by a merging operator gives as result a pre-order representing the collective preferences. This pre-order is a transitive explanation relation for the choice function.

The property of absence of dictator is satisfied. It is mainly ensured by the fairness postulate (IC4).

The independence of irrelevant alternatives is not satisfied. It is easy to see, for example, that the $\triangle^{d,\Sigma}$ family operators do not satisfy this property since the "score" of each interpretation is computed by taking into account all the interpretations (including "irrelevant" ones). We will illustrate this point on the example 5.

Example 5 *Suppose that we work with four interpretations* $I_0 = \{00\}$*,* $I_1 = \{01\}$*,* $I_2 = \{10\}$ *and* $I_3 = \{11\}$*, and consider three knowledge bases* φ_1*,* φ_2 *and* φ_3 *such that* $mod(\varphi_1) = \{I_0\}$*,* $mod(\varphi_2) = \{I_1, I_3\}$*, and* $mod(\varphi_3) = \{I_0, I_2\}$*. With the* $\triangle^{d,\Sigma}$ *operator using the Dalal distance we get the pre-orders of figure 5 (An interpretation* I *is in a lower level than the interpretation* J *iff* $I < J$*).*



(a) φ_1 (b) φ_2 (c) φ_3

Fig. 5. Pre-orders

If we compute the two mergings $\triangle^{d,\Sigma}_{\{I_0,I_3\}}(\varphi_1 \sqcup \varphi_2) = \varphi_{\{I_0\}}$ *and* $\triangle^{d,\Sigma}_{\{I_0,I_3\}}(\varphi_3 \sqcup \varphi_2) = \varphi_{\{I_0,I_3\}}$ *we get two different results whereas the restriction of the two profiles*

$\varphi_1 \sqcup \varphi_2$ and $\varphi_3 \sqcup \varphi_2$ to the agenda $\{I_0, I_3\}$ are identical since $I_0 <_{\varphi_1} I_1$, $I_0 <_{\varphi_3} I_1$, and $I_1 <_{\varphi_2} I_0$.

We get two different results for the two mergings because in φ_1 the preference for I_0 over I_1 is stronger than in φ_3.

This independence of irrelevant alternatives is mainly motivated by the fact that in economy, one generally can not compare individual utilities. Utility can be roughly seen as the "score" that an individual gives to an alternative, so it denotes the attractiveness of an alternative. But it is very difficult to work with such a subjective notion. So it seems reasonable to limit oneself to an ordinal scale (a pre-order). Arrow illustrates this idea in [2] by quoting Bentham :

> *" 'Tis in vain to talk of adding quantities which after the addition will continue distinct as they were before, one man's happiness will never be another man's happiness : a gain to one man is no gain to another : you might as well pretend to add 20 apples to 20 pears..."*

But, in the case of merging operators, this individual utility is not given *a priori*, but (objectively) computed by the operator. So, since we have a same scale for all individuals, there is no more a point in forbidding to compare these utilities. So merging operators escape to Arrow's impossibility theorem since they compare individual utilities. This is due to the fact that these "utilities" are computed uniformly by the operator when it associates a pre-order to each belief base (which implies, by the way, that all profiles are not possible as input). So merging operators do not satisfy the condition of independence of irrelevant alternatives and the standard domain constraint since the pre-orders (individual preferences) of each source (individual) is "objectively" given by the merging operator (and not "subjectively" chosen by the individual), which allows to compare the obtained "utilities".

7 Conclusion

We have explored in this paper the frontier between two important subclasses of merging operators: arbitration and majority operators. The former aiming to prefer consensual choices, whereas the later referring to majority wishes.

An open question until now was to now if there was an intersection between these two classes or not. We have shown that it is the case for some trivial operator, and that it is possible for a non trivial operator to be both an arbitration and a majority operator if we bound the size of the belief sets. Those operators seem to be a good compromise between democratic ideas lying in majority operators and consensual behaviour of arbitration operators.

We have introduced, in particular, a new family of operators (the $\triangle^{d,\Sigma^n}$ family), that allows to choose the *"consensual level"* of the majority operator according to the particular application needs.

An open question that remains is to know if there are non trivial operators belonging simultaneously to the two classes and in case of positive answer if it is possible to characterize exactly when the operators are in the two classes.

Finally it might be interesting to exploit the links given in section 6 between logical merging and social choice rules to give compact logical representations for social choice rules.

Acknowledgments

We would like to thank the referees for their observations which have contributed to improve the quality of this work and Professor Olga Porras for her careful revision of the english version of this work.

Appendix : Proofs

Proof of Theorem 1 We first give the proof for IC merging operators, and then the ones for majority and arbitration operators.

(Only if part) Let $\triangle$ be an operator satisfying postulates (IC0-IC8). Let us define a syncretic assignment as follows: for each belief set Ψ we define a total pre-order $\leq_\Psi$ by putting $\forall I, J \in \mathcal{W}$ $I \leq_\Psi J$ if and only if $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi)$.

First we show that $\leq_\Psi$ is a total pre-order:

Totality: $\forall I, J \in \mathcal{W}$, from (IC1) $\triangle_{\varphi_{\{I,J\}}}(\Psi) \neq \emptyset$ and from (IC0) $\triangle_{\varphi_{\{I,J\}}}(\Psi) \vdash \varphi_{\{I,J\}}$, so $I \leq_\Psi J$ or $J \leq_\Psi I$.

Reflexivity: From (IC0) and (IC1) we have that $\triangle_{\varphi_I}(\Psi) = \varphi_I$. So $I \leq_\Psi I$.

Transitivity: Assume that $I \leq_\Psi J$ and $J \leq_\Psi L$ and suppose towards a contradiction that $I \not\leq_\Psi L$. So by definition and from (IC0) and (IC1), $\triangle_{\varphi_{\{I,L\}}}(\Psi) \leftrightarrow \varphi_{\{L\}}$. By (IC7) we find that $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}} \vdash \triangle_{\varphi_{\{I,L\}}}(\Psi)$. We consider two cases:

Case 1: $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}}$ is consistent then $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}} \leftrightarrow \varphi_{\{L\}}$. Thus we have that $I \not\models \triangle_{\varphi_{\{I,J,L\}}}(\Psi)$. But by (IC1) $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \neq \emptyset$, so by (IC0) we have $mod(\triangle_{\varphi_{\{I,J,L\}}}(\Psi)) = \{J, L\}$ or $mod(\triangle_{\varphi_{\{I,J,L\}}}(\Psi)) = \{L\}$. In the first case by (IC7) and (IC8) we conclude that $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,J\}} \leftrightarrow \triangle_{\varphi_{\{I,J\}}}(\Psi)$ and so $I \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi)$. Contradiction. In the second case by (IC7) and (IC8) $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{J,L\}} \leftrightarrow \triangle_{\varphi_{\{J,L\}}}(\Psi)$ but $J \not\models \triangle_{\varphi_{\{I,J,L\}}}(\Psi)$ so $J \not\models \triangle_{\varphi_{\{J,L\}}}(\Psi)$.

Contradiction.

Case 2: $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,L\}}$ is not consistent, so $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) = \varphi_{\{J\}}$. Then $\triangle_{\varphi_{\{I,J,L\}}}(\Psi) \wedge \varphi_{\{I,J\}} = \varphi_{\{J\}}$. By (IC7) and (IC8) it follows that $\triangle_{\varphi_{\{I,J\}}}(\Psi) = \varphi_{\{J\}}$, that is by definition $J <_\Psi I$. Contradiction.

Now we show that $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. For the first inclusion $mod(\triangle_\mu(\Psi)) \subseteq \min(mod(\mu), \leq_\Psi)$ assume that $I \models \triangle_\mu(\Psi)$ and suppose towards a contradiction that I is not in $\min(mod(\mu), \leq_\Psi)$. So we can find a $J \models \mu$ s.t. $J <_\Psi I$, then $I \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi)$. Since $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}}$ is consistent from (IC7) and (IC8) we have $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}} \leftrightarrow \triangle_{\varphi_{\{I,J\}}}(\Psi)$. But $I \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi)$ so $I \not\models \triangle_\mu(\Psi)$. Contradiction.

For the other inclusion $mod(\triangle_\mu(\Psi)) \supseteq \min(mod(\mu), \leq_\Psi)$, suppose that $I \in \min(mod(\mu), \leq_\Psi)$. We want to show that $I \models \triangle_\mu(\Psi)$. Since $I \in \min(mod(\mu), \leq_\Psi)$, then $\forall J \models \mu \; I \leq_\Psi J$ and so $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi)$. Since $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}}$ is consistent from (IC7) and (IC8) we have $\triangle_\mu(\Psi) \wedge \varphi_{\{I,J\}} \leftrightarrow \triangle_{\varphi_{\{I,J\}}}(\Psi)$. But $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi)$ so $I \models \triangle_\mu(\Psi)$.

It remains to verify the conditions of the syncretic assignment:

(1) If $I \models \Psi$ and $J \models \Psi$, then by (IC2) we have $\triangle_{\varphi_{\{I,J\}}}(\Psi) = \varphi_{\{I,J\}}$, so $I \leq_\Psi J$ and $J \leq_\Psi I$ by definition and then $I \simeq_\Psi J$.

(2) If $I \models \Psi$ and $J \not\models \Psi$, then by (IC2) $\triangle_{\varphi_{\{I,J\}}}(\Psi) = \varphi_I$, so $I \leq_\Psi J$ and $J \not\leq_\Psi I$, *i.e.* $I <_\Psi J$.

(3) Straightforward from (IC3)

(4) We want to show that $\forall I \models \varphi \; \exists J \models \varphi' \; J \leq_{\varphi \sqcup \varphi'} I$. First we show that $\exists J \models \triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi'$. If not we have $\triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi' \vdash \bot$, from (IC0) and (IC1) we have that $\triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \vdash \varphi$, now by (IC4) we get that $\triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$. Contradiction.

Let I be a model of φ and take J such that $J \models \triangle_{\varphi \vee \varphi'}(\varphi \sqcup \varphi') \wedge \varphi'$. We get from (IC7) and (IC8) that $J \models \triangle_{\varphi_{\{I,J\}}}(\varphi \sqcup \varphi')$. So $J \leq_{\varphi \sqcup \varphi'} I$.

(5) If $I \leq_{\Psi_1} J$ and $I \leq_{\Psi_2} J$ then $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \wedge \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$. So from (IC5) $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and by definition $I \leq_{\Psi_1 \sqcup \Psi_2} J$.

(6) Suppose that $I <_{\Psi_1} J$ and $I \leq_{\Psi_2} J$. We want to show that $I <_{\Psi_1 \sqcup \Psi_2} J$. By the hypothesis $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \wedge \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$ and $J \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi_1) \wedge \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$. So from (IC5) and (IC6) $\triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2) = \varphi_I$. Then $I \models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and $J \not\models \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup \Psi_2)$ and by definition $I <_{\Psi_1 \sqcup \Psi_2} J$.

(If part) Let's consider a syncretic assignment that maps each belief set Ψ to a total pre-order $\leq_\Psi$ and define an operator $\triangle$ by putting $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. We want to show that $\triangle$ satisfies (IC0-IC8).

(IC0) By definition $mod(\triangle_\mu(\Psi)) \subseteq mod(\mu)$.

(IC1) If μ is consistent, then $mod(\mu) \neq \emptyset$. As there is a finite number of interpretations, there is no infinite descending chains of inequalities, so $\min(mod(\mu), \leq_\Psi) \neq \emptyset$. Then $\triangle_\mu(\Psi)$ is consistent.

(IC2) Assume that $\bigwedge \Psi \wedge \mu$ is consistent. We show that $\min(mod(\mu), \leq_\Psi) = mod(\bigwedge \Psi \wedge \mu)$. First note that if $I \models \Psi$ then from conditions 1 and 2, $I \in \min(\mathcal{W}, \leq_\Psi)$. So if $I \models \Psi \wedge \mu$ then $I \in \min(mod(\mu), \leq_\Psi)$. So

$\min(mod(\mu), \leq_\Psi) \supseteq mod(\bigwedge \Psi \wedge \mu)$. For the other inclusion consider $I \in \min(mod(\mu), \leq_\Psi)$. Suppose towards a contradiction that $I \not\models \Psi \wedge \mu$. Since $I \not\models \Psi$, by condition 2 we have that $\forall J \models \Psi\ J <_\Psi I$. In particular $\forall J \models \Psi \wedge \mu\ J <_\Psi I$. So $I \notin \min(mod(\mu), \leq_\Psi)$. Contradiction.

(IC3) Direct from condition 3 and the definition of $\triangle$.

(IC4) Assume that $\varphi \vdash \mu$, $\varphi' \vdash \mu$, and $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi \nvdash \bot$, we want to show that $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$. Consider $I \models \triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi$. Then $\forall I' \models \mu\ I \leq_{\varphi \sqcup \varphi'} I'$. But from condition 4 we have that $\exists J \models \varphi'$ such that $J \leq_{\varphi \sqcup \varphi'} I$. Then $\forall I' \models \mu\ J \leq_{\varphi \sqcup \varphi'} I'$. Then $J \models \triangle_\mu(\varphi \sqcup \varphi')$ and therefore $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$.

(IC5) If $I \models \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ then $I \in \min(mod(\mu), \leq_{\Psi_1})$ and so $\forall J \models \mu\ I \leq_{\Psi_1} J$. We have in the same way $\forall J \models \mu\ I \leq_{\Psi_2} J$. So by condition 5 we have that $\forall J \models \mu\ I \leq_{\Psi_1 \sqcup \Psi_2} J$. So $I \in \min(mod(\mu), \leq_{\Psi_1 \sqcup \Psi_2})$. So by definition $I \models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$.

(IC6) Assume that $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent. We want to show that $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ holds. Take $I \models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$, so $\forall J \models \mu\ I \leq_{\Psi_1 \sqcup \Psi_2} J$. Suppose towards a contradiction that $I \not\models \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$. So $I \not\models \triangle_\mu(\Psi_1)$ or $I \not\models \triangle_\mu(\Psi_2)$. Suppose that $I \not\models \triangle_\mu(\Psi_1)$ (the other case is symmetrical). As $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent $\exists J \models \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$. So $J <_{\Psi_1} I$ and $J \leq_{\Psi_2} I$ so by condition 6 $J <_{\Psi_1 \sqcup \Psi_2} I$ and then $I \not\models \triangle_\mu(\Psi_1 \sqcup \Psi_2)$. Contradiction.

(IC7) Let's take $I \models \triangle_{\mu_1}(\Psi) \wedge \mu_2$. We have $\forall J \models \mu_1\ I \leq_\Psi J$. So $\forall J \models \mu_1 \wedge \mu_2\ I \leq_\Psi J$, so $I \models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$.

(IC8) Assume that $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, so $\exists J \models \triangle_{\mu_1}(\Psi) \wedge \mu_2$. Consider $I \models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$ and suppose that $I \not\models \triangle_{\mu_1}(\Psi)$. So $J <_\Psi I$. But $J \models \mu_1 \wedge \mu_2$ then $I \notin \min(mod(\mu_1 \wedge \mu_2), \leq_\Psi)$. Thus $I \not\models \triangle_{\mu_1 \wedge \mu_2}(\Psi)$. Contradiction.

Concerning majority operators the proof goes as follows :

(Only if part) Let $\triangle$ be an operator satisfying postulates (IC0-IC8) and (Maj). Define an assignment as in the proof of theorem 1.

By theorem 1 this is a syncretic assignment representing $\triangle$. It remains to prove condition 7. Assume that $I <_{\Psi_2} J$. Then $\triangle_{\varphi_{\{I,J\}}}(\Psi_2) = \varphi_I$. From (Maj) we get that $\exists n$ such that $\triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup {\Psi_2}^n) \vdash \triangle_{\varphi_{\{I,J\}}}(\Psi_2)$, so $\exists n \triangle_{\varphi_{\{I,J\}}}(\Psi_1 \sqcup {\Psi_2}^n) = \varphi_I$, *i.e.* $\exists n\ I <_{\Psi_1 \sqcup {\Psi_2}^n} J$.

(If part) Let's consider a majority syncretic assignment that maps each belief set Ψ to a total pre-order $\leq_\Psi$ and define the operator $\triangle$ by $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. By theorem 1 we know that $\triangle$ satisfies (IC0-IC8). It remains to prove (Maj). From conditions 6 and 7 we get easily the following condition:

$$I <_{\Psi_2} J \ \Rightarrow\ \exists n_0\ \forall n \geq n_0\ \ I <_{\Psi_1 \sqcup {\Psi_2}^n} J$$

Since for each Ψ, $\leq_\Psi$ is total this condition is equivalent to

$$\forall n_0\ \exists n \geq n_0\ I \leq_{\Psi_1 \sqcup {\Psi_2}^n} J \ \ \Rightarrow\ \ I \leq_{\Psi_2} J \qquad (*)$$

Now, suppose towards a contradiction that $\forall n\, \triangle_\mu\left(\Psi_1 \sqcup {\Psi_2}^n\right) \nvdash \triangle_\mu(\Psi_2)$. From this hypothesis we get that $\forall n\ \exists I \models \mu\ \forall J \models \mu\ I \leq_{\Psi_1 \sqcup {\Psi_2}^n} J$ and $\exists J' \models$

μ $J' <_{\Psi_2} I$. Since the number of possible worlds is finite, by a combinatorial argument (pigeon hole principle) there exists I such that $I \leq_{\Psi_1 \sqcup \Psi_2{}^n} J$ for any $J \models \mu$ and an infinity of integers n and such that $\exists J' \models \mu$ $J' <_{\Psi_2} I$. This obviously entails the premises of condition $(*)$, so we have $I \leq_{\Psi_2} J$ for any $J \models \mu$ which is obviously in contradiction with the fact that $\exists J' \models \mu$ $J' <_{\Psi_2} I$.

Finally, for arbitration operators the proof is :

(Only if part) Let $\triangle$ be an operator satisfying postulates (IC0-IC8) and (Arb). Define an assignment as in the proof of theorem 1.

By theorem 1 this assignment is a syncretic assignment, so it remains to show that condition 8 holds. Assume that both $J <_{\varphi_1} I$, $J <_{\varphi_2} J'$ and $I \simeq_{\varphi_1 \sqcup \varphi_2} J'$ hold. First if $I = J'$ then $J <_{\varphi_1 \sqcup \varphi_2} I$ follows from condition 6. Now suppose $I \neq J'$. By hypothesis $\triangle_{\varphi_{\{I,J\}}}(\varphi_1) \leftrightarrow \triangle_{\varphi_{\{J,J'\}}}(\varphi_2) \leftrightarrow \varphi_J$ and $\triangle_{\varphi_{\{I,J'\}}}(\varphi_1 \sqcup \varphi_2) = \varphi_{\{I,J'\}}$. By the assumption $I \neq J'$, we have that both of $\varphi_{\{I,J\}} \wedge \neg\varphi_{\{I,J'\}}$ and $\varphi_{\{I,J'\}} \wedge \neg\varphi_{\{I,J\}}$ are consistent. Then by (Arb) we get that $\triangle_{\varphi_{\{I,J,J'\}}}(\varphi_1 \sqcup \varphi_2) = \varphi_J$. And by (IC7) and (IC8) we conclude that $\triangle_{\varphi_{\{I,J\}}}(\varphi_1 \sqcup \varphi_2) = \varphi_J$, that is $J <_{\varphi_1 \sqcup \varphi_2} I$.

(If part) Let's consider a fair majority syncretic assignment that maps each belief set Ψ to a total pre-order $\leq_\Psi$ and define $\triangle$ by putting $mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$. We know by theorem 1 that $\triangle$ satisfies (IC0-IC8), then it is enough to prove (Arb).

Assume that $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$, $\triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2)$, $\mu_1 \wedge \neg\mu_2 \nvdash \bot$ and $\mu_2 \wedge \neg\mu_1 \nvdash \bot$ hold. We want to show that $\triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle_{\mu_1}(\varphi_1)$. First we prove that $\triangle_{\mu_1}(\varphi_1) \vdash \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Consider $I \models \triangle_{\mu_1}(\varphi_1)$ and suppose towards a contradiction that $I \not\models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Then $\exists J \models \mu_1 \vee \mu_2$ $J <_{\varphi_1 \sqcup \varphi_2} I$.

We consider three cases: $J \models \mu_1 \wedge \mu_2$, $J \models \mu_1 \wedge \neg\mu_2$ or $J \models \neg\mu_1 \wedge \mu_2$.

case 1: $J \models \mu_1 \wedge \mu_2$. Since $I \models \triangle_{\mu_1}(\varphi_1)$, $I \leq_{\varphi_1} J$. By hypothesis $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$. So $I \models \triangle_{\mu_2}(\varphi_2)$ and then $I \leq_{\varphi_2} J$. Then by condition 5 we have that $I \leq_{\varphi_1 \sqcup \varphi_2} J$. Contradiction.

case 2: $J \models \mu_1 \wedge \neg\mu_2$ (the *case 3*, $J \models \neg\mu_1 \wedge \mu_2$, is symmetrical). Since $J \not\models \mu_2$ and $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$ we have $J \not\models \triangle_{\mu_1}(\varphi_1)$, so $I <_{\varphi_1} J$. By hypothesis we can find a $J' \models \mu_2 \wedge \neg\mu_1$ and with an analogous argument $I <_{\varphi_2} J'$. We also know that $\triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2)$, this implies $J \simeq_{\varphi_1 \sqcup \varphi_2} J'$. And then by condition 8 we get that $I <_{\varphi_1 \sqcup \varphi_2} J$. Contradiction.

Now we prove $\triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \vdash \triangle_{\mu_1}(\varphi_1)$. Assume that $I \models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$ and suppose towards a contradiction that $I \not\models \triangle_{\mu_1}(\varphi_1)$. There are three cases:

case 1: $I \models \mu_1 \wedge \mu_2$ then $\exists J \models \triangle_{\mu_1}(\varphi_1)$, so $J <_{\varphi_1} I$. And, as $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$, $J <_{\varphi_2} I$. So by condition 8 we have that $J <_{\varphi_1 \sqcup \varphi_2} I$, so $I \not\models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Contradiction.

case 2: $I \models \mu_1 \wedge \neg\mu_2$ (the *case 3*, where $I \models \neg\mu_1 \wedge \mu_2$, is symmetrical). By hypothesis we know that $\exists I' \models \neg\mu_1 \wedge \mu_2$. Since $\triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2)$ $\exists J \models \triangle_{\mu_1}(\varphi_1)$ such that $J <_{\varphi_1} I$ and $J <_{\varphi_2} I'$. We obtain also from $\triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2)$ that $I \simeq_{\varphi_1 \sqcup \varphi_2} I'$, so by condition 8 we get that $J <_{\varphi_1 \sqcup \varphi_2} I$.

So $I \not\models \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2)$. Contradiction.

□

Proof of Theorem 5 We will show that, given a belief set Ψ consisting of m belief bases with $m \leq k$, there exists n_0 such that $\forall n > n_0$ the two pre-orders $\leq_\Psi^{d,\Sigma^n}$ and $\leq_\Psi^{d,GMax}$ coincide. And we conclude by theorem 1.

As the language is finite, it is easy to see (by a combinatorial argument) that there exists N such that $d(I,J) \leq N$ for any interpretations I, J. Put $n_0 = min\{n : k \cdot N^n < (N+1)^n\}$ (note that n_0 exists because $\lim_{n\to\infty}(\frac{N+1}{N})^n = \infty$). It will be useful to stress that for all integers x, y such that $1 \leq x < y \leq N$ we have $\frac{N+1}{N} < \frac{y}{x}$ and therefore

$$k \cdot x^{n_0} < y^{n_0} \qquad (*)$$

We will show that $\forall n > n_0 \forall I\ \forall J\ \ I \leq_\Psi^{d,GMax} J$ iff $I \leq_\Psi^{d,\Sigma^n} J$.

(only if part) Let I and J be two interpretations such that $I \leq_\Psi^{d,GMax} J$. We show that $I \leq_\Psi^{d,\Sigma^n} J$. Consider the two following cases:

- $I \simeq_\Psi^{d,GMax} J$, hence the two sorted lists $(d^I_{\sigma(1)}, \ldots, d^I_{\sigma(m)})$ and $(d^J_{\sigma(1)}, \ldots, d^J_{\sigma(m)})$ are the same. Therefore for all n,

$$d_{d,\Sigma^n}(I,\Psi) = \sqrt[n]{\sum_{i=1}^{m} (d^I_{\sigma(i)})^n} = d_{d,\Sigma^n}(J,\Psi)$$

  Hence if $I \simeq_\Psi^{d,GMax} J$, then for all n $I \simeq_\Psi^{d,\Sigma^n} J$.

- $I <_\Psi^{d,GMax} J$. This means that for the two sorted lists $(d^I_{\sigma(1)}, \ldots, d^I_{\sigma(m)})$ and $(d^J_{\sigma(1)}, \ldots, d^J_{\sigma(m)})$ there exists $p \leq m$ such that $\forall i < p$ $d^I_{\sigma(i)} = d^J_{\sigma(i)}$ and $d^I_{\sigma(p)} < d^J_{\sigma(p)}$. Consider the worst case, where $p = 1$ and such that $(d^I_{\sigma(1)}, \ldots, d^I_{\sigma(m)}) = (x, \ldots, x)$ and $(d^J_{\sigma(1)}, \ldots, d^J_{\sigma(m)}) = (y, 0, \ldots, 0)$ with $x < y$. The other cases will be directly retrieved by sum properties. Then note that if $x = 0$ we have trivially $0 = \sqrt[n_0]{\sum_{i=1}^m (d^I_{\sigma(i)})^{n_0}} < \sqrt[n_0]{\sum_{i=1}^m (d^J_{\sigma(i)})^{n_0}}$ and therefore $I <_\Psi^{d,\Sigma^{n_0}} J$.

  Now suppose that $x \geq 1$. By the observation $(*)$ we have

$$\sqrt[n_0]{\sum_{i=1}^{m} (d^I_{\sigma(i)})^{n_0}} = \sqrt[n_0]{m \cdot x^{n_0}} \leq \sqrt[n_0]{k \cdot x^{n_0}} < \sqrt[n_0]{y^{n_0}} = \sqrt[n_0]{\sum_{i=1}^{m} (d^J_{\sigma(i)})^{n_0}}$$

  that is $I <_\Psi^{d,\Sigma^{n_0}} J$.

(If part) Assume that $I \leq_\Psi^{d,\Sigma^n} J$, we show then that $I \leq_\Psi^{d,GMax} J$. Simply observe that the contraposition is : if $J <_\Psi^{d,GMax} I$, then $J <_\Psi^{d,\Sigma^n} I$, which has been proved in the only if part.

□

Proof of Theorem 6 By the representation theorem 1 it is enough to see that the two pre-orders $\leq_{\Psi}^{d,\Sigma^n}$ and $\leq_{\Psi}^{d,GMax}$ are different. But this is implicit in the previous proof. Let n be a positive integer. Take Ψ such that $I <_{\Psi}^{d,GMax} J$ with $(d^I_{\sigma(1)}, \ldots, d^I_{\sigma(m)}) = (x, \ldots, x)$ and $(d^J_{\sigma(1)}, \ldots, d^J_{\sigma(m)}) = (y, 0, \ldots, 0)$, $1 \leq x < y$ and m satisfying $m \cdot x^n > y^n$. Then

$$\sqrt[n]{\sum_{i=1}^{m} (d^I_{\sigma(i)})^n} = \sqrt[n]{m \cdot x^n} > \sqrt[n]{y^n} = \sqrt[n]{\sum_{i=1}^{m} (d^J_{\sigma(i)})^n}$$

that is $I >_{\Psi}^{d,\Sigma^{n_0}} J$.

□

References

[1] C. E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50:510–530, 1985.

[2] K. J. Arrow. *Social choice and individual values.* Wiley, New York, second edition, 1963.

[3] K. J. Arrow, A. K. Sen, and K. Suzumura, editors. *Handbook of Social Choice and Welfare.* Elsevier, 2002.

[4] C. Baral, S. Kraus, J. Minker, and V. S. Subrahmanian. Combining knowledge bases consisting of first-order theories. *Computational Intelligence*, 8(1):45–71, 1992.

[5] L. Cholvy. Reasoning about merged information. In D. M. Gabbay and Ph. Smets, editors, *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, volume 3, pages 233–263. Kluwer Academic Publishers, 1998.

[6] L. Cholvy and T. Hunter. Fusion in logic: a brief overview. In *Proceedings of the Fourth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'97)*, Lecture Notes in Computer Science 1244, pages 86–95, 1997.

[7] M. Dalal. Investigations into a theory of knowledge base revision: preliminary report. In *Proceedings of the National Conference on Artificial Intelligence (AAAI'88)*, pages 475–479, 1988.

[8] Marquis de Condorcet. Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix. Paris, 1785.

[9] P. Gärdenfors. *Knowledge in flux.* MIT Press, 1988.

[10] P. Gärdenfors, editor. *Belief Revision.* Cambridge University Press, 1992.

[11] J. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54(3):319–379, 1992.

[12] H. Katsuno and A. O. Mendelzon. Propositional knowledge base revision and minimal change. *Artificial Intelligence*, 52:263–294, 1991.

[13] J. S. Kelly. *Arrow impossibility theorems*. Series in economic theory and mathematical economics. Academic Press, New York, 1978.

[14] J. S. Kelly. *Social Choice Theory : An Introduction*. Springer-Verlag, 1988.

[15] S. Konieczny. On the difference between merging knowledge bases and combining them. In *Proceedings of the Seventh International Conference on Principles of Knowledge Representation and Reasoning (KR'00)*, pages 135–144, 2000.

[16] S. Konieczny and R. Pino Pérez. On the logic of merging. In *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, pages 488–498, 1998.

[17] S. Konieczny and R. Pino Pérez. Merging with integrity constraints. In *Proceedings of the Fifth European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'99)*, Lecture Notes in Artificial Intelligence 1638, pages 233–244, 1999.

[18] S. Konieczny and R. Pino Pérez. Merging information under constraints: a logical framework. *Journal of Logic and computation*, 12(5):773–808, 2002.

[19] P. Liberatore and M. Schaerf. Arbitration (or how to merge knowledge bases). *IEEE Transactions on Knowledge and Data Engineering*, 10(1):76–90, 1998.

[20] J. Lin and A. O. Mendelzon. Knowledge base merging by majority. In *Dynamic Worlds: From the Frame Problem to Knowledge Management*. Kluwer, 1999.

[21] H. Moulin. *Axioms of cooperative decision making*. Monograph of the Econometric Society. Cambridge University Press, 1988.

[22] V. Pareto. *Cours d'économie politique*. Rouge, Lausanne, 1897.

[23] P. Z. Revesz. On the semantics of theory change: arbitration between old and new information. In *Proceedings of the* 12^{th} *ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Databases*, pages 71–92, 1993.

[24] P. Z. Revesz. On the semantics of arbitration. *International Journal of Algebra and Computation*, 7(2):133–160, 1997.

[25] A. K. Sen. *Collective Choice and Social Welfare*. Advanced Textbooks in Economics. Elsevier, 1979.