Iterated revision by epistemic states: axioms, semantics and syntax

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Abstract. We propose a very general syntactical notion of epistemic state and a compact axiomatization for iterated revision when the new information is an epistemic state. We set representation theorems and give two semantical representations of operators: by polynomials and by weighted belief bases. These representations will result to be equivalent.

1 Introduction

Most of the time, intelligent agents face incomplete, uncertain, inaccurate information and need a revision mechanism to handle their beliefs change in presence of a new piece of information. Belief revision is the study of rational means an agent uses in order to modify his epistemic state in view of new information. The epistemic state has to be changed in order to restore consistency, keeping the new item of information and removing the least possible previous information. Most of the belief revision approaches consider a knowledge base (generally represented by a set of formulas which often is deductively closed) revised by a simple formula expressed in a logical language. In this context one step belief revision has been successfully characterized by the AGM postulates [1, 8]. But, although very elegant, AGM characterization do not capture adequately iterated belief revision.

Thus, iterated belief revision has been bought into focus during the last ten years, and new additional postulates have been proposed in order to characterize iterated belief revision (see e.g. [6, 11, 13, 3]). In all these approaches an epistemic state is something more complex than a simple knowledge base, it not only has to represent the agent's current beliefs, but also the strategy the agent uses in order to modify his beliefs in presence of a new item of information.

Yet all these approaches only consider the revision of an epistemic state by a formula. However some applications require a more general approach, that is the revision of an epistemic state by an epistemic state. This approach is one of our main concerns in this work. This corresponds, for example, to the revision of an agent's beliefs by another, more reliable, agent's beliefs, or to the revision of an agent's beliefs by a stratified set of information. Such line of research has been previously considered by Nayak et al. [12, 14].

We propose a generalization of previously proposed revision operations which stem from the principle of strong primacy of new information [3, 10, 15]. In order to do that, the main idea will be to consider that an epistemic state is given by two parts: the *observable* part and an additional information coding the agent confidence (preferences) in alternative possible worlds. Then, the underlying intuition concerning this principle is based upon the fact that the agent considers the new entire epistemic state more reliable than the old one.

We propose a very compact set of postulates capturing the iteration process and give representation theorems. We do not only logically study these operators but we also provide two particular representations. That shows how to really implement those ideas. In the first one, epistemic states are represented by polynomials. This representation allows to compute the revision operations easily. The second representation is more syntactic (but equivalent to the first one). Epistemic states are then represented by weighted belief bases. This is a natural and concise representation of epistemic states.

Section 2 proposes a general set of postulates for iterated revision by an epistemic state and representation theorems. Therein we consider the case where epistemic states are represented by total pre-orders. In section 3 we show that our operators have good iteration properties. Section 4 presents a polynomial representation of epistemic states and a corresponding revision operator, while section 5 presents a syntactical approach.

2 Epistemic states: syntax, axioms for change and representation

We give in this section a very general syntactical notion of epistemic state and a new axiomatic proposal for change. We then give some representation theorems.

Definition 1 *An epistemic space $\mathcal{E}$ is a triple $\langle E, \pi, F\rangle$ where E is a set, F is the set of formulas of the finite*[5] *propositional logic and π is a function from E into F. The elements of the set E will be called epistemic states. The elements of F will be called the* observables *and π is the projection function; so if $\Phi \in E$, $\pi(\Phi)$ is the observable part of Φ.*

Notice that this idea was already implicit in [6, 11] where the projection function is called Bel. From now on the epistemic states will be denoted by upper case Greek letters. The

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[5] *i.e.* the set of propositional variables will be finite

symbol $\vdash$ will denote the logical consequence relation. By $\mathcal{W}$ we denote the set of classical interpretations, and if ϕ is a formula then $Mod(\phi)$ denotes the set of its classical models.

We want to axiomatize a class of change operators which take into account the iteration process. Essentially a change operator will be a function $\circ$ mapping a couple of epistemic states into a new epistemic state, *i.e.* $\circ : E \times E \longrightarrow E$.

The following set of axioms will be called REEA (Revision by epistemic states axioms).

(REE*1) $\pi(\Phi \circ \Psi) \vdash \pi(\Psi)$

(REE*2) If $\pi(\Phi) \wedge \pi(\Psi)$ is consistent, then $\pi(\Phi \circ \Psi) \leftrightarrow \pi(\Phi) \wedge \pi(\Psi)$

(REE*3) If $\pi(\Psi)$ is consistent, then $\pi(\Phi \circ \Psi)$ is consistent

(REE*4) If $\pi(\Psi_1) \leftrightarrow \pi(\Psi_2)$, then $\pi(\Phi \circ \Psi_1) \leftrightarrow \pi(\Phi \circ \Psi_2)$

(REE*It) $\pi((\Phi \circ \Theta) \circ \Gamma) \leftrightarrow \pi(\Phi \circ (\Theta \circ \Gamma))$

Notice that the axioms (REE*1)-(REE*4) are the natural generalization of AGM postulates (R1)-(R4) when the new information is an epistemic state. The axiom (REE*It), called iteration axiom is a generalization of postulates corresponding to AGM's (R5) and (R6) [9]. Moreover this axiom, that expresses a sort of associativity of $\circ$ (at observable level), will catch the strong priority of the new information and will guarantee a good behaviour with respect to the iteration process. The name of iteration axiom is due to the fact that it captures the main postulates for iteration proposed by Darwiche and Pearl [6]. We will detail this in section 3.

Proposition 1 *The axiom (REE*It) together with (REE*1)-(REE*4) entail the following axioms:*

(REE*5) $\pi(\Phi \circ \Psi) \wedge \pi(\Theta) \vdash \pi(\Phi \circ \Gamma)$ *with* $\pi(\Gamma) \leftrightarrow \pi(\Psi \circ \Theta)$

(REE*6) *If* $\pi(\Phi \circ \Psi) \wedge \pi(\Theta)$ *is consistent, then* $\pi(\Phi \circ \Gamma) \vdash \pi(\Phi \circ \Psi) \wedge \pi(\Theta)$ *with* $\pi(\Gamma) \leftrightarrow \pi(\Psi \circ \Theta)$.

(REE*Conj) *If* $\pi(\Theta) \wedge \pi(\Gamma)$ *is consistent, then for any epistemic state* Θ' *such that* $\pi(\Theta') \leftrightarrow \pi(\Theta) \wedge \pi(\Gamma)$ *we have* $\pi((\Phi \circ \Theta) \circ \Gamma) \leftrightarrow \pi(\Phi \circ \Theta')$

The axiom (REE*Conj) says that if we sequentially revise by two epistemic states having observables mutually consistent then the resulting observable is the same if we revise by an epistemic state which observable is the conjunction of observables of epistemic states. This property is close to the *Conjunction* postulate proposed in [13] (see also [18]). But (REE*Conj) is weaker than *Conjunction* because the former imposes only the equivalence between observables whereas the later imposes the equality of the epistemic states.

Definition 2 *Consider an epistemic space* $\langle E, \pi, F\rangle$. *A function* $\circ : E \times E \longrightarrow E$ *is said to be a* revision operator by epistemic states *if it satisfies* REEA.

In order to establish a first representation theorem we need the following definition:

Definition 3 *Let* $\leq_1$ *and* $\leq_2$ *be two total pre-orders on* $\mathcal{W}$. *We define the total pre-order* $\leq_{lex(\leq_1,\leq_2)}$ *by putting* $\omega_1 \leq_{lex(\leq_1,\leq_2)} \omega_2$ *iff* $\omega_1 <_1 \omega_2$ *or* $\omega_1 =_1 \omega_2$ *and* $\omega_1 \leq_2 \omega_2$.

Theorem 1 *Let* $\langle E, \pi, F\rangle$ *be an epistemic space. Let* $\circ : E \times E \longrightarrow E$ *be a function.* $\circ$ *is a revision operator by epistemic state iff for any epistemic state* Ψ *we can associate a total pre-order* $\leq'_\Psi$ *such that:*

(i) $Mod(\pi(\Psi)) = min(\mathcal{W}, \leq'_\Psi)$ *if* $\pi(\Psi)$ *is consistent.*

(ii) $Mod(\pi(\Psi \circ \Phi)) = min(\mathcal{W}, \leq_{lex(\leq'_\phi,\leq'_\Psi)})$ *if* $\pi(\Phi)$ *is consistent.*

The *if part* of this theorem is a quite simple verification. For the *only if part*, the key point is the definition of the mapping $\Psi \mapsto \leq'_\Psi$. This is done in the following definition:

Definition 4 *Let* $\langle E, \pi, F\rangle$ *be an epistemic space and let* $\circ$ *be a revision operator by epistemic state. For any epistemic state* Ψ *we define a total pre-order over interpretations* $\leq'_\Psi$ *in the following way:*

$$\omega_1 \leq'_\Psi \omega_2 \;\Leftrightarrow\; \omega_1 \models \pi(\Psi \circ \Theta)$$

with $Mod(\pi(\Theta)) = \{\omega_1, \omega_2\}$.

Notice that it is thanks to (REE*4) that $\leq'_\Psi$ is well defined.

Unlike the well known representation theorem for classical revision operators AGM [9], the previous theorem does not allow to build the operator $\circ$ from the pre-orders. In this sense it is a weak representation theorem. But what is quite interesting is that for any representation of epistemic states the theorem gives a concrete representation of observables via the pre-orders. Another important aspect of this theorem is the computation process of the observables of the new epistemic state via the *lex* pre-order. Below we will see that this computation is a key point in our constructions.

One could reproach to this representation the fact that we do not have necessarily $\Psi \circ \Phi = \leq_{lex(\leq'_\phi,\leq'_\Psi)}$ even when the interpretation of epistemic states are total pre-orders. Nevertheless if the operator satisfies a little bit more of rationality that equality holds.

Definition 5 *Let* $\langle E, \pi, F\rangle$ *be an epistemic space such that the elements of* E *are total pre-orders, and if for each* $\leq_\Phi \in E$, $\pi(\leq_\Phi) = \varphi$ *with* $Mod(\varphi) = min(\mathcal{W}, \leq_\Phi)$ *if* $\pi(\Phi)$ *is consistent, and* $\pi(\leq_\Phi) = \perp$ *otherwise. Let* $\circ : E \times E \longrightarrow E$ *be an operator.* $\circ$ *is said to be minimal-model preserving if the following equality holds*

$$Mod(\pi(\leq_\Psi \circ \leq_\Phi)) = min(Mod(\pi(\leq_\Phi)), \leq_\Psi)$$

For this kind of operators we have the following representation theorem:

Theorem 2 *Let* $\circ$ *be a minimal-model preserving operator and* $\leq_\Psi$, $\leq_\Phi$ *the total pre-oders associated to* Ψ *and* Φ. *Then* $\circ$ *is a revision operator by epistemic state iff*

$$\leq_\Psi \circ \leq_\Phi = \leq_{lex(\leq_\Phi,\leq_\Psi)}$$

Note that this representation theorem is simpler than the one presented in Nayak et al [12, 14] which have defined a revision by epistemic states, which are represented by means epistemic entrenchment relations, using the lexicographical ordering.

3 Iterative behaviour of REEA

Now we study the behaviour of these operators with respect to iteration. We begin adapting the postulates of Darwiche and Pearl [5, 6] to framework in which the new information is also an epistemic state. The postulates for the iterated revision of epistemic states are the following ones:

(C1*) If $\pi(\Phi) \vdash \pi(\Gamma)$ then $\pi((\Psi \circ \Gamma) \circ \Phi) \leftrightarrow \pi(\Psi \circ \Phi)$.

(C2*) If $\pi(\Phi) \vdash \neg\pi(\Gamma)$ then $\pi((\Psi \circ \Gamma) \circ \Phi) \leftrightarrow \pi(\Psi \circ \Phi)$.

(C3*) If $\pi(\Psi \circ \Phi) \vdash \pi(\Gamma)$ then $\pi((\Psi \circ \Gamma) \circ \Phi) \vdash \pi(\Gamma)$.

(C4*) If $\pi(\Psi \circ \Phi) \nvdash \neg\pi(\Gamma)$ then $\pi((\Psi \circ \Gamma) \circ \Phi) \nvdash \neg\pi(\Gamma)$.

Let $\langle E, \pi, F\rangle$ and $\circ$ be an epistemic space and a minimal-model preserving operator as in Definition 5. Let $\mathcal{C}$ be a subset of E. Define $\circ_{\mathcal{C}}$ as the restriction of $\circ$ to $E \times \mathcal{C}$. For instance if $\mathcal{C}$ is the class of pre-orders with at most two levels $\circ_{\mathcal{C}}$ is the basic operator defined in [10] (see also [3]). For these operators the following result holds:

Theorem 3 *Suppose that $\circ_{\mathcal{C}}$ is a minimal-model preserving revision operator by epistemic states. Then $\circ_{\mathcal{C}}$ satisfies (C1*-C4*) iff the elements of $\mathcal{C}$ are the pre-orders having at most two levels.*

In the general case we have the following theorem:

Theorem 4 *A revision operator by epistemic states satisfies the postulates (C1*), (C3*) and (C4*). But in general (C2*) does not hold.*

Notice that by Theorems 3, 4 and the fact that we have pre-orders with more than two levels the minimal-model preserving revision operator by epistemic states build over E satisfies (C1*), (C3*) and (C4*) but does not satisfy (C2*).

In example 1, we give a counterexample for (C2*), where we consider that E is the set of pre-orders over interpretations given by the Hamming distance whose first level is $Mod(\pi(\leq_\Psi))$. Remember that the Hamming (or Dalal [4]) distance between two interpretations is the number of propositional variables on which they differ.

Example 1 *Consider an electric circuit with an adder and a multiplier. There is no initial information about the state of the circuit. Let Φ be such that $\pi(\Phi) = \top$. After, we learn that the adder and the multiplier work well,* i.e. $\pi(\Psi) = adder\_ok \wedge multiplier\_ok$. *But after that we learn that the adder does not work,* i.e. $\pi(\Gamma) = \neg adder\_ok$. *Thus we have $\pi(\Gamma) \vdash \neg\pi(\Psi)$ but $\pi((\Phi\circ\Psi)\circ\Gamma) \leftrightarrow \neg adder\_ok \wedge multiplier\_ok$ whereas $\pi(\Phi\circ\Gamma) \leftrightarrow \neg adder\_ok$. The application of (C2*) would lead to $\pi((\Phi\circ\Psi)\circ\Gamma) \equiv \neg adder\_ok$, that is to "forget" that the multiplier works!*

Below the pre-orders corresponding to the epistemic states are represented. The interpretation 01 denotes adder_ok is false and multiplier_ok is true, etc. Two interpretations are equivalent if they are at the same level. An interpretation ω_1 is better than ω_2 if ω_1 is in a level below the level of ω_2.

$\leq_\Phi =$ 00 01 10 11

$\leq_\Psi =$
00
01 10
11

$\leq_\Gamma =$
10 11
00 01

$\leq_{[\Phi\circ\Gamma]} =$
10 11
00 01

$\leq_{[\Phi\circ\Psi\circ\Gamma]} =$
01
00
11
10

4 Semantic representation by means of polynomials

We propose in this section a suitable representation of epistemic states based on polynomials [15, 2]. This representation allows to formalize the change of epistemic states by simple operations on polynomials, to keep track of the sequence of revisions and hence to come back to previous epistemic states, which is not possible with the other representations (see [15] for more arguments for the use of polynomials).

Let's denote by B, the set of polynomials which coefficients belong to $\{0,1\}$ that is $p \in B$ [6] is of the form $p = \sum_{i=0}^{m} p_i x^i$. Polynomials allow to represent shift operations easily (a right shift is a multiplication by x). We define an order on polynomials of B, denoted by $<_B$ which represents the lexicographical order:

Definition 6 *Let $p, p' \in B$. $p <_B p'$ iff $\exists i \in \mathbb{N}$ such that $\forall j,\ j < i,\ p_j = p'_j$ and $p_i < p'_i$. (The reflexive closure of $<_B$ is denoted by $\leq_B$).*

Definition 7 *A weight distribution is a function which associates with each interpretation $\omega \in \mathcal{W}$ a polynomial of B.*

Semantically, an epistemic state Ψ will then be represented by a weight distribution denoted by $p(\Psi)$ and we will denote by $p^\omega(\Psi)$ the weight of ω in $p(\Psi)$. The ordering $\leq_\Psi$ associated to Ψ is defined by $\omega_1 \leq_\Psi \omega_2$ iff $p_1^\omega(\Psi) \leq p_2^\omega(\Psi)$. The function π is defined by:
$\pi(\Psi) = \psi$ iff $Mod(\psi) = \{\omega \in \mathcal{W} : \nexists \omega'\ p^{\omega'}(\Psi) <_B p^\omega(\Psi)\}$.

Remark 1 *It is always possible to represent ordinal ranking on W by polynomials where we associate with each ordinal n a polynomial, as follows. Let $(n_0, n_1, \ldots, n_j)$ be the binary decomposition of n, that is $n = \sum_{k=0}^{j} n_k 2^k$ where $n_k \in \{0,1\}$ and where j is such that $2^j \leq n < 2^{j+1}$. We assign n the polynomial $\sum_{i=0}^{j} p'_i\, x^i$ where $p'_i = n_{j-i}$ corresponding to binary decomposition of n read in the reverse order. Thus, there is an injection from total pre-orders into weight distributions. For example, let $\mathcal{W}$ be the set of interpretations, $\mathcal{W} = \{\omega_1 = \neg a\neg b, \omega_2 = \neg ab, \omega_3 = a\neg b, \omega_4 = ab\}$, the ordinals corresponding to ω_1, ω_2, ω_3 et ω_4 are respectively 3, 2, 1 et 0. In this example, $j = 1$ and the weight distribution is: $p^{\omega_1}(\Psi) = 1 + x$, $p^{\omega_2}(\Psi) = 1$, $p^{\omega_3}(\Psi) = x$, $p^{\omega_4}(\Psi) = 0$. It is worthy to note that $p^\omega(\Psi) <_B p^{\omega'}(\Psi)$ iff the ordinal corresponding to ω is lower than the ordinal corresponding to ω'.*

Since we represent epistemic states by weight distributions, we need to describe the construction of the revised weight distribution. We want to define a revision by epistemic state operator $\circ_p$, which revise an epistemic state $p(\Psi)$ by a new epistemic state $p(\Phi)$ such that the resulting epistemic state, denoted $p(\Psi \circ_p \Phi)$, satisfies: $p^{\omega_1}(\Psi \circ_p \Phi) <_B p^{\omega_2}(\Psi \circ_p \Phi)$ iff $p^{\omega_1}(\Phi) <_B p^{\omega_2}(\Phi)$ or ($p^{\omega_1}(\Phi) =_B p^{\omega_2}(\Phi)$ and $p^{\omega_1}(\Psi) <_B p^{\omega_2}(\Psi)$). In order to do that we set the following notation. Let $max(\Phi)$ be the highest degree of the polynomials of $p(\Phi)$. More formally, $max(\Phi) = max\{deg(p^\omega(\Phi)),\ \omega \in \mathcal{W}\}$ where $deg(a)$ refers to the degree of the polynomial a. We note $\mathcal{M}_\Phi = max(\Phi) + 1$.

Definition 8 *The revision of the weight distribution $p(\Psi)$ by the weight distribution $p(\Phi)$, denoted $p(\Psi \circ_p \Phi)$ (*i.e. *$p(\Psi) \circ_p p(\Phi) = p(\Psi \circ_p \Phi)$), is defined by the following*

$$\forall \omega \in \mathcal{W},\ p^\omega(\Psi \circ_p \Phi) = x^{\mathcal{M}_\Phi} p^\omega(\Psi) + p^\omega(\Phi)$$

A priority is given to the total pre-order corresponding to the new epistemic state Φ with respect to the total pre-order corresponding to the initial epistemic state Ψ.

With this encoding the following result is easy to see

[6] In order to simplify notation, we denote a polynomial by p instead of the standard notation $p(x)$.

Proposition 2 *$\circ_p$ is a revision operator by epistemic states.*

Let us illustrate all this with the following example.

Example 2 *Let Ψ be an initial epistemic state with total pre-order interpretation $\leq_\Psi$, defined by $\omega_4 <_\Psi \omega_1 <_\Psi \omega_3 =_\Psi \omega_2$. Let Φ be an epistemic state with total pre-order interpretation $\leq_\Phi$ defined by $\omega_2 <_\Phi \omega_4 =_\Phi \omega_1 <_\Phi \omega_3$. The following array shows the behaviour of $\circ_p$ when the weight distributions of Ψ and Φ are defined as in the previous remark:*

$\mathcal{W}$	$p(\Psi)$		$p(\Phi)$		$p(\Psi \circ_p \Phi)$	$p_0p_1p_2p_3$
ω_1	x	01	x	01	x^3+x	0101
ω_2	1	10	0	00	x^2	0010
ω_3	1	10	1	10	x^2+1	1010
ω_4	0	00	x	00	x	0100

After the revision of $p(\Psi)$ by $p(\Phi)$, the column of $p(\Psi \circ_p \Phi)$ describes the new weight distribution. It is easy to see that it corresponds to the total pre-order $\leq_{\Psi\circ_p\Phi}$ where: $\omega_2 <_{\Psi\circ_p\Phi} \omega_4 <_{\Psi\circ_p\Phi} \omega_1 <_{\Psi\circ_p\Phi} \omega_3$. Actually the column $p_0\,p_1\,p_2\,p_3$ gives the total pre-order corresponding to the current epistemic state $\Psi\circ_p\Phi$, the column $p_2\,p_3$ gives the total pre-order corresponding to the initial epistemic state Ψ, and the column $p_0\,p_1$ gives the total pre-order corresponding to the new epistemic state Φ.

It is interesting to notice that if at each iteration of the revision process, we keep the weight distribution corresponding to the epistemic state with which revision is performed, the use of polynomial allows to come back to previous weight distribution (and so to previous total pre-orders). Let $p(\Psi \circ_p \Phi)$ be the weight distribution obtained after revising $p(\Psi)$ by $p(\Phi)$. The distribution corresponding to $p(\Psi)$ can be obtained from $p(\Psi \circ_p \Phi)$ by putting $p^\omega(\Psi) = x^{-\mathcal{M}_\Phi}(p^\omega(\Psi \circ_p \Phi) - p^\omega(\Phi))$.

In the special case where revision is performed by a formula $\mu \in F$, *i.e.* $\pi(p(\Phi)) = \mu$ and $p(\Phi)$ is defined by if $\omega \in Mod(\mu)$ then $p^\omega(\Phi) = 0$, else $p^\omega(\Phi) = 1$. Revising $p(\Psi)$ by a $\mu \in F$, leads to: if $\omega \in Mod(\mu)$ then $p^\omega(\Psi \circ_p \Phi) = xp^\omega(\Psi))$, else $p^\omega(\Psi \circ_p \Phi) = xp^\omega(\Psi) + 1$. The weights of models of μ are right shifted, while the weights of counter-models of μ are right shifted and translated by 1.

5 Syntactical representation

In the previous section we have characterized the iterated revision of epistemic state by weight distributions. In this section, we give an alternative (but equivalent) syntactical representation of an epistemic state Ψ. Instead of explicitly specifying the weight distribution on all $\mathcal{W}$, the agent specifies a set of weighted formulas, called a weighted (or stratified) belief base and denoted by Σ_Ψ. We then define a function κ which allows to recover $\leq_\Psi$ from Σ_Ψ by also associating to each interpretation ω a polynomial of B, that we denote by $\kappa_{\Sigma_\Psi}(\omega)$. When $\kappa_{\Sigma_\Psi}(\omega) = p^\omega(\Psi)$ for each ω, we say that Σ_Ψ is a compact (or syntactic) representation of Ψ.

Given this compact representation, we are interested in defining a syntactic counterpart of $\circ_p$, which syntactically transforms two weighted belief bases Σ_Ψ and Σ_Φ respectively associated with the epistemic states Ψ and Φ, to a new weighted base, denoted by $\Sigma_{\Psi\circ_s\Phi}$, corresponding to the new epistemic state $\Sigma_\Psi \circ_s \Sigma_\Phi$. This new weighted base should be such that: $\forall\omega$, $\mathrm{p}^\omega(\Psi \circ_p \Phi) = \kappa_{\Sigma_{\Psi\circ_s\Phi}}(\omega)$.

Definition 9 *A weighted belief base Σ_Ψ is a set of pairs $\{(\phi_i,\ p^{\phi_i}(\Psi)) : i = 1, ..., n\}$ where ϕ_i is a propositional formula, and $p^{\phi_i}(\Psi)$ is a non-null polynomial of B (i.e., different from the polynomial 0).*

Polynomials associated with formulas are compared according to Definition 6. When $\mathrm{p}^\phi(\Psi) >_B p^\psi(\Psi)$, we say that ϕ is more important (or has a higher priority, etc) than the formula ψ. A weighted base Σ_Ψ is said to be consistent (resp. to entail ϕ) if its classical base (obtained by forgetting the weights) is also consistent (resp. entails ϕ). Note that Σ_Ψ is not necessarily deductively closed. Moreover, nothing prevents Σ_Ψ from containing two weighted formulas $(\phi, p^\phi(\Psi))$ and $(\psi, p^\psi(\Psi))$ such that ϕ and ψ are classically equivalent, but having different weights $p^\phi(\Psi) \neq p^\psi(\Psi)$. In this case, we will see later that the least important formula (called a subsumed formula) can be removed from the weighted belief base.

Definition 10 *With each weighted belief base Σ_Ψ is associated a weighted distribution, denoted by κ_{Σ_Ψ}, defined by: $\forall\omega, \kappa_{\Sigma_\Psi}(\omega) = max\{p^{\phi_i}(\Psi) : (\phi_i,\ p^{\phi_i}(\Psi)) \in \Sigma_\Psi$ and $\omega \not\models \phi_i\}$, where by convention max($\emptyset$)=0.*

We now interpret an epistemic state Ψ in a weighted base Σ_Ψ and the observable part, $\pi(\Sigma_\Psi)$, is defined by $\pi(\Sigma_\Psi) = \psi$ such that $Mod(\psi) = \{\omega : \nexists\omega'\ s.\ t.\ \kappa_{\Sigma_\Psi}(\omega') < \kappa_{\Sigma_\Psi}(\omega)\}$.

This semantics is basically the same as the one used in possibilistic logic [7], in System Z [16] and for generating a complete epistemic entrenchment relation from a partial one [17]. Indeed, all these approaches share the same idea, where they associate with each interpretation ω the weight of the most important formula falsified by this interpretation. The lowest is the weight of an interpretation, the preferred it is. In particular, models of Σ_Ψ (namely those having a weight equal to 0) are the most preferred ones. Next, we give the operator $\circ_s$ defined over weighted knowledge bases.

Definition 11 *Let Σ_Ψ and Σ_Φ be the knowledge bases associated with the epistemic states Ψ and Φ. Let $\mathcal{M}_\Phi = max\{deg(p^\phi(\Phi)) : (\phi, p^\phi(\Phi)) \in \Sigma_\Phi\} + 1$. The weighted base $\Sigma_{\Psi\circ_s\Phi}$ (i.e. $\Sigma_\Psi \circ_s \Sigma_\Phi = \Sigma_{\Psi\circ_s\Phi}$) is composed of:*
- *all the formulas ψ of Σ_Ψ with the weight: $p^\psi(\Psi \circ_s \Phi) = x_\Phi^\mathcal{M} p^\psi(\Psi)$,*
- *all the formulas ϕ of Σ_Φ with the weight: $p^\phi(\Psi\circ_s\Phi) = p^\phi(\Phi)$,*
- *all the possible disjunctions between formulas ψ of Σ_Ψ and formulas ϕ of Σ_Φ, different from from tautologies, with the weights: $p^{\phi\vee\psi}(\Psi \circ_s \Phi) = x_\Phi^\mathcal{M} p^\psi(\Psi) + p^\phi(\Phi)$.*

The following result shows that $\Sigma_{\Psi\circ_s\Phi}$ allows us to recover the distribution $p(\Psi \circ_p \Phi)$ syntactically.

Theorem 5 *Let Σ_Ψ and Σ_Φ be two weighted bases associated with the epistemic states Ψ and Φ such that $p^\omega(\Psi) = \kappa_{\Sigma_\Psi}(\omega)$, and $p^\omega(\Phi) = \kappa_{\Sigma_\Phi}(\omega)$. Then: $p(\Psi) \circ_p p(\Phi) = \kappa_{\Sigma_{\Psi\circ_s\Phi}}$*

This theorem together with the Proposition 2 give:

Proposition 3 *$\circ_s$ is a revision operator by epistemic states.*

Once $\Sigma_{\Psi\circ_s\Phi}$ is computed, we propose to compute $\pi(\Sigma_{\Psi\circ_s\Phi})$ directly from $\Sigma_{\Psi\circ_s\Phi}$. But we first proceed to a pre-processing step which makes the computation easier. This pre-processing step consists in removing useless (or redundant) formulas, called subsumed formulas.

Definition 12 *Let $(\phi,\ p^{\phi}(\Psi \circ_s \Phi))$ be a formula in $\Sigma_{\Psi\circ_s\Phi}$, and A_ϕ be a subbase of $\Sigma_{\Psi\circ_s\Phi}$ composed of formulas having a weight greater than $p^{\phi}(\Psi\circ_s\Phi)$, namely: $A_\phi = \{\psi : (\psi,\ p^{\psi}(\Psi\circ_s \Phi)) \in \Sigma_{\Psi\circ_s\Phi}$ and $p^{\psi}(\Psi\circ_s\Phi) > p^{\phi}(\Psi\circ_s\Phi)\}$. Then, $(\phi,\ p^{\phi}(\Psi\circ_s \Phi))$ is said to be subsumed by $\Sigma_{\Psi\circ_s\Phi}$ if it is classically entailed from A_ϕ. We denote by $\Sigma^*_{\Psi\circ_s\Phi}$ the weighted subbase obtained by removing subsumed formulas from $\Sigma_{\Psi\circ_s\Phi}$.*

Theorem 6 *Let $\Sigma_{\Psi\circ_s\Phi}$ be a weighted base. Then $\Sigma_{\Psi\circ_s\Phi}$ and $\Sigma^*_{\Psi\circ_s\Phi}$ are equivalent, in the sense that $\forall\omega$ we have: $\kappa_{\Sigma_{\Psi\circ_s\Phi}}(\omega) = \kappa_{\Sigma^*_{\Psi\circ_s\Phi}}(\omega)$.*

The removing of subsumed formulas allows us a direct computation of $\pi(\Sigma_\Psi \circ_s \Sigma_\Phi)$.

Theorem 7 *If $\Sigma^*_{\Psi\circ_s\Phi}$ is consistent, then $\pi(\Sigma_\Psi \circ_s \Sigma_\Phi)$ is the classical base (i.e., without weights) associated with $\Sigma_{\Psi\circ_s\Phi}$. If $\Sigma^*_{\Psi\circ_s\Phi}$ is not consistent, then let Minweight be the set of formulas in Σ^* having minimal weights. Then $\pi(\Sigma_\Psi \circ_s \Sigma_\Phi)$ is the classical base of $\Sigma^*_{\Psi\circ_s\Phi}$ - Minweight.*

Next example illustrates the concepts and results of this section.

Example 3 *Let $\Sigma_\Psi = \{(\neg a \vee b,\ 1), (a \vee \neg b,\ 1), (b,\ x)\}$ and $\Sigma_\Phi = \{(\neg a \vee b,\ 1), (\neg a,\ x), (b,\ x)\}$ be two weighted belief bases. A straightforward verification shows that these two weighted belief bases are the compact representation of the epistemic states $p(\Psi)$ and $p(\Phi)$ given in the Example 2. Applying Definition 11, we have $\mathcal{M}_\Phi = 2$, and we get:*

$\Sigma_{\Psi\circ_s\Phi} = \{(\neg a \vee b,\ x^2), (a \vee \neg b,\ x^2), (b,\ x^3), (\neg a \vee b,\ 1), (\neg a,\ x), (b,\ x), (\neg a \vee b,\ x^2+1), (\neg a \vee b,\ x^2+x), (\neg a \vee b,\ x^3+1), (\neg a \vee b,\ x^3+x), (b,\ x^3+x)\}$.

Let us compute the function κ associated with $\Sigma_{\Psi\circ_s\Phi}$:

$\kappa_{\Sigma_{\Psi\circ_s\Psi}}(ab) = x,\qquad \kappa_{\Sigma_{\Psi\circ_s\Psi}}(a\neg b) = x^2+1,$

$\kappa_{\Sigma_{\Psi\circ_s\Psi}}(\neg ab) = x^2,\qquad \kappa_{\Sigma_{\Psi\circ_s\Psi}}(\neg a\neg b) = x^3+x.$

We can easily check that we got the same weights as in example 2, namely: $\omega \in \mathcal{W} : \kappa_{\Sigma_{\Psi\circ_s\Psi}}(\omega) = p^{\omega}(\Psi \circ_p \Phi)$.

Now we want to compute $\Sigma^*_{\Psi\circ_s\Phi}$. Notice that

- *The formulas $(\neg a \vee b,\ x^2), (\neg a \vee b,\ 1), (\neg a \vee b,\ x^2+x), (\neg a \vee b,\ x^3+1), (\neg a \vee b,\ x^3+x)$ are all subsumed, since they are entailed by $(\neg a \vee b,\ x^2+1)$.*
- *The formulas $\{(b,\ x^3), (b,\ x)\}$ are all subsumed, since they are entailed by $(b,\ x^3+x)$.*

After removing these formulas we get the final subbase:

$\Sigma^*_{\Psi\circ_s\Phi} = \{(\neg a \vee b,\ x^2+1), (a \vee \neg b,\ x^2), (\neg a,\ x), (b,\ x^3+x)\}$.

Now let us compute $\pi(\Sigma_\Psi \circ_s \Sigma_\Phi)$. Since $\Sigma^*_{\Psi\circ_s\Phi}$ is not consistent, then Minweight $= \{(a \vee \neg b,\ x^2)\}$. Thus, $\pi(\Sigma_\Psi \circ_s \Sigma_\Phi)$ is the classical base (by forgetting weights) of $\Sigma^*_{\Psi\circ_s\Phi} -$ Minweight $= \{(\neg a \vee b,\ x^2+1), (\neg a,\ x), (b,\ x^3+x)\}$. Clearly, $\pi(\Sigma_\Psi \circ_s \Sigma_\Phi)$ has exactly one model which is $\neg ab$. Moreover, it is easy to check that $\neg ab$ is the unique minimal model in $\kappa_{\Sigma_{\Psi\circ_s\Phi}}$ (i.e., has the minimal weight) computed previously.

An algorithm that computes $\Sigma^*_{\Psi\circ_s\Phi}$ from $\Sigma_{\Psi\circ_s\Phi}$ and which needs a logarithmic number of satisfiability tests can be easily provided.

6 Conclusion

We have proposed a general notion of epistemic state and illustrated its semantics with three interpretations: total preorders, weighted distributions and weighted bases. We proved that postulates defining our revision operators by epistemic states capture the postulates for iteration of Darwiche and Pearl. But in general the controversial postulate C2 is not valid. This postulate will be valid only if we restrain the new epistemic states to be extremely simple, *i.e.* when they can be identified to formulas. The polynomial representation has an interesting feature: it allows the reversibility of the revision process. The syntactical representation, which is very natural, provides a reasonable way for the computation of the resulting epistemic state. This suggests syntactical restriction over formulas in order to consider simple weighted bases for which the computations might be done in polynomial time.

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