Around Propositional Base Merging

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Merging

- Contradictory beliefs/goals coming from different sources
  - Propositional Logic
    - no priority (same reliability, hierarchical importance, ...)

φ_1	φ_2	φ_3
$a, b \longrightarrow c$	a, b	$\neg a$

$$\triangle(\varphi_1 \sqcup \varphi_2 \sqcup \varphi_3) = b \rightarrow c, b, a$$

Plan

- ▷ Propositional Base Merging
  - ▷ Logical Properties
  - ▷ Model-Based Operators
  - ▷ Formula-Based Operators
  - ▷ DA^2 Operators
- ▷ Merging and . . .
  - ▷ . . . Belief Revision
  - ▷ . . . Judgment Aggregation
  - ▷ . . . Social Choice
- ▷ More on Merging
  - ▷ Strategy-Proofness
  - ▷ Negotiation/Conciliation

Definitions

- ▷ A belief base φ is a finite set of propositional formulae.
- ▷ A belief profile Ψ is a multi-set of belief bases: $\Psi = \{\varphi_1, \ldots, \varphi_n\}$.
- ▷ $\bigwedge \Psi$ denotes the conjunction of the belief bases of Ψ.
- ▷ A belief profile Ψ is *consistent* if and only if $\bigwedge \Psi$ is consistent. We will note $Mod(\Psi)$ instead of $Mod(\bigwedge \Psi)$.

Equivalence between belief profiles :

- ▷ Let Ψ_1, Ψ_2 be two belief profiles. Ψ_1 and Ψ_2 are equivalent, noted $\Psi_1 \leftrightarrow \Psi_2$, iff there exists a bijection f from $\Psi_1 = \{\varphi_1^1, \ldots, \varphi_n^1\}$ to $\Psi_2 = \{\varphi_1^2, \ldots, \varphi_n^2\}$ such that $\vdash f(\varphi) \leftrightarrow \varphi$.

Logical Characterization

$\triangle$ is a merging with integrity constraints operator (IC merging operator) if and only if it satisfies the following properties :

(IC0) $\triangle_\mu(\Psi) \vdash \mu$

(IC1) If μ is consistent, then $\triangle_\mu(\Psi)$ is consistent

(IC2) If $\bigwedge \Psi$ is consistent with μ, then $\triangle_\mu(\Psi) = \bigwedge \Psi \wedge \mu$

(IC3) If $\Psi_1 \leftrightarrow \Psi_2$ and $\mu_1 \leftrightarrow \mu_2$, then $\triangle_{\mu_1}(\Psi_1) \leftrightarrow \triangle_{\mu_2}(\Psi_2)$

(IC4) If $\varphi \vdash \mu$ and $\varphi' \vdash \mu$, then $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi \nvdash \bot \Rightarrow \triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$

(IC5) $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2) \vdash \triangle_\mu(\Psi_1 \sqcup \Psi_2)$

(IC6) If $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent, then $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$

(IC7) $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(\Psi)$

(IC8) If $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(\Psi) \vdash \triangle_{\mu_1}(\Psi)$

Majority vs Arbitration

Ally, Brian and Charles have to decide what they will do this night. Brian and Ally want to go to the restaurant and to the cinema. Charles does not want to go out this night and so he does not want to go nor to the restaurant nor to the cinema.

Majority	restaurant and cinema	*Arbitration*	restaurant xor cinema
Ally	:-))	Ally	:-)
Brian	:-))	Brian	:-)
Charles	:-((	Charles	:-)

Majority - Arbitration

(Maj) $\exists n \ \triangle_{\mu}(\Psi_1 \sqcup {\Psi_2}^n) \vdash \triangle_{\mu}(\Psi_2)$

- An IC merging operator is a majority operator if it satisfies (Maj).

(Arb)

$$\left.\begin{array}{l} \triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2) \\ \triangle_{\mu_1 \leftrightarrow \neg \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg \mu_2) \\ \mu_1 \nvdash \mu_2 \\ \mu_2 \nvdash \mu_1 \end{array}\right\} \Rightarrow \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle_{\mu_1}(\varphi_1)$$

- An IC merging operator is an arbitration operator if it satifies (Arb).

Syncretic Assignment

A syncretic assignment is a function mapping each belief profile Ψ to a total pre-order $\leq_\Psi$ over interpretations such that:

1) If $\omega \models \Psi$ and $\omega' \models \Psi$, then $\omega \simeq_\Psi \omega'$

2) If $\omega \models \Psi$ and $\omega' \not\models \Psi$, then $\omega <_\Psi \omega'$

3) If $\Psi_1 \equiv \Psi_2$, then $\leq_{\Psi_1} = \leq_{\Psi_2}$

4) $\forall \omega \models \varphi_1 \; \exists \omega' \models \varphi_2 \; \omega' \leq_{\varphi_1 \sqcup \varphi_2} \omega$

5) If $\omega \leq_{\Psi_1} \omega'$ and $\omega \leq_{\Psi_2} \omega'$, then $\omega \leq_{\Psi_1 \sqcup \Psi_2} \omega'$

6) If $\omega <_{\Psi_1} \omega'$ and $\omega \leq_{\Psi_2} \omega'$, then $\omega <_{\Psi_1 \sqcup \Psi_2} \omega'$

A majority syncretic assignment is a syncretic assignment which satisfies:

7) If $\omega <_{\Psi_2} \omega'$, then $\exists n \; \omega <_{\Psi_1 \sqcup {\Psi_2}^n} \omega'$

A fair syncretic assignment is a syncretic assignment which satisfies:

8)
$$\left.\begin{array}{l} \omega <_{\varphi_1} \omega' \\ \omega <_{\varphi_2} \omega'' \\ \omega' \simeq_{\varphi_1 \sqcup \varphi_2} \omega'' \end{array}\right\} \Rightarrow \omega <_{\varphi_1 \sqcup \varphi_2} \omega'$$

Representation Theorem

Theorem An operator is an IC merging operator (respectively IC majority merging operator or IC arbitration operator) if and only if there exists a syncretic assignment (respectively majority syncretic assignment or fair syncretic assignment) that maps each belief profile Ψ to a total pre-order $\leq_\Psi$ such that

$$Mod(\triangle_\mu(\Psi))) = \min(Mod(\mu), \leq_\Psi).$$

Model-Based Merging

Idea: Select the interpretations that are the most plausible for a given profile.

$$\omega \leq_{\Psi}^{d_x} \omega' \text{ iff } d_x(\omega, \Psi) \leq d_x(\omega', \Psi)$$

d_x can be computed using:
- a distance between interpretations d
- an aggregation function f

▷ Distance between interpretations
  - ▷ $d(\omega, \omega') = d(\omega', \omega)$
  - ▷ $d(\omega, \omega') = 0$ iff $\omega = \omega'$

▷ Distance between an interpretation and a belief base
  - ▷ $d(\omega, \varphi) = \min_{\omega' \models \varphi} d(\omega, \omega')$

▷ Distance between an interpretation and a belief profile
  - ▷ $d_{d,f}(\omega, \Psi) = f(d(\omega, \varphi_1), \ldots d(\omega, \varphi_n))$

Model-Based Merging

Examples of aggregation function: Σ, $\max$, $leximax$

- Let d be a distance between interpretations.
  - $\triangle^{d,\max}$ operators satisfy (IC1-IC5), (IC7), (IC8) and (Arb).
  - $\triangle^{d,\mathit{GMax}}$ operators are arbitration operators.
  - $\triangle^{d,\Sigma}$ operators are majority operators.

Example

$\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I$

$\varphi_1 = \varphi_2 = S \wedge T \wedge P$ $\quad Mod(\varphi_1) = \{(1,1,1,1),(1,1,1,0)\}$

$\varphi_3 = \neg S \wedge \neg T \wedge \neg P \wedge \neg I$ $\quad Mod(\varphi_3) = \{(0,0,0,0)\}$

$\varphi_4 = T \wedge P \wedge \neg I$ $\quad Mod(\varphi_4) = \{(1,1,1,0),(0,1,1,0)\}$

	φ_1	φ_2	φ_3	φ_4	$d_{d_H,\mathrm{Max}}$	$d_{d_H,\Sigma}$	d_{d_H,Σ^2}	$d_{d_H,\mathrm{GMax}}$
$(0,0,0,0)$	3	3	0	2	3	8	22	(3,3,2,0)
$(0,0,0,1)$	3	3	1	3	3	10	28	(3,3,3,1)
$(0,0,1,0)$	2	2	1	1	**2**	6	**10**	**(2,2,1,1)**
$(0,0,1,1)$	2	2	2	2	**2**	8	16	(2,2,2,2)
$(0,1,0,0)$	2	2	1	1	**2**	6	**10**	**(2,2,1,1)**
$(0,1,0,1)$	2	2	2	2	**2**	8	16	(2,2,2,2)
$(0,1,1,1)$	1	1	3	1	3	6	12	(3,1,1,1)
$(1,0,0,0)$	2	2	1	2	**2**	7	13	(2,2,2,1)
$(1,0,0,1)$	2	2	2	3	3	9	21	(3,2,2,2)
$(1,0,1,1)$	1	1	3	2	**2**	7	15	(3,2,1,1)
$(1,1,0,1)$	1	1	3	2	3	7	15	(3,2,1,1)
$(1,1,1,1)$	0	0	4	1	4	**5**	17	(4,1,0,0)

Formula-Based Merging [BKM91,BKMS92]

Idea: Select some formulae from the union of the bases of the profile

$$\textsc{maxcons}(\Psi,\mu) = \{M \subseteq \bigcup \Psi \cup \mu \text{ s.t. } \begin{array}{l} -\ M \nvdash \bot \\ -\ \mu \subseteq M \\ -\ \forall M \subset M' \subseteq \bigcup \Psi \cup \mu \quad M' \vdash \bot\} \end{array}$$

$$\triangle^{C1}{}_{\mu}(\Psi) = \textsc{maxcons}(\Psi,\mu)$$

$$\triangle^{C3}{}_{\mu}(\Psi) = \{M : M \in \textsc{maxcons}(\Psi,\top) \text{ and } M \wedge \mu \text{ consistent}\}$$

$$\triangle^{C4}{}_{\mu}(\Psi) = \textsc{maxcons}_{card}(\Psi,\mu)$$

$$\triangle^{C5}{}_{\mu}(\Psi) = \{M \wedge \mu \ : \ M \in \textsc{maxcons}(\Psi,\top) \text{ and } M \wedge \mu \text{ consistent}\}$$

if this set is nonempty and μ otherwise.

	IC0	IC1	IC2	IC3	IC4	IC5	IC6	IC7	IC8	MI	Maj
$\triangle^{C1}$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$		$\surd$		$\surd$	
$\triangle^{C3}$					$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	
$\triangle^{C4}$	$\surd$	$\surd$	$\surd$					$\surd$	$\surd$	$\surd$	
$\triangle^{C5}$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	

Formula-Based Merging: Selection Functions

Idea: Use a selection function to choose only the best maxcons.

- ▷ Partial-meet contraction/revision operators
- ▷ Take into account the distribution of the information among the sources

Example : Consider a belief profile Ψ and a maxcons M :

- ▷ $dist_{\cap}(M, \varphi) = |\varphi \cap M|$
- ▷ $dist_{\cap,\Sigma}(M, \Psi) = \sum_{\varphi \in \Psi} dist_{\cap}(M, \varphi)$

	IC0	IC1	IC2	IC3	IC4	IC5	IC6	IC7	IC8	MI	Maj
$\triangle^{C1}$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$		$\surd$		$\surd$	
$\triangle^{d}$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$		$\surd$			$\surd$
$\triangle^{S,\Sigma}$	$\surd$	$\surd$	$\surd$		$\surd$			$\surd$	$\surd$		$\surd$
$\triangle^{\cap,\Sigma}$	$\surd$	$\surd$	$\surd$			$\surd$	$\surd$	$\surd$	$\surd$		$\surd$

Merging

- ▷ Formula-based Merging

→ Selection of maximal consistent subsets of formulas in the union of belief bases.

– Distribution of information

– Bad logical properties

+ Inconsistent belief bases

- ▷ Model-based Merging

→ Selection of preferred models for the belief bases.

+ Distribution of information

+ Good logical properties

– Inconsistent belief bases

- ▷ DA^2 Operators

+ Distribution of information

+ Good logical properties

+ Inconsistent belief bases

DA^2 Operators

Let d be a distance between interpretations and f and g be two aggregation functions. The DA^2 merging operator $\triangle_\mu^{d,f,g}(\Psi)$ is defined by :
For each $\varphi_i = \{\alpha_{i,1}, \ldots, \alpha_{i,n_i}\}$

$$d(\omega, \varphi_i) = f(d(\omega, \alpha_{i,1}), \ldots, d(\omega, \alpha_{i,n_i}))$$

Let $\Psi = \{\varphi_1, \ldots, \varphi_n\}$

$$d(\omega, \Psi) = g(d(\omega, \varphi_1), \ldots, d(\omega, \varphi_m))$$

$$mod(\triangle_\mu^{d,f,g}(\Psi))) = \{\omega \in mod(\mu) \mid d(\omega, \Psi) \text{ is minimal}\}$$

Example

φ_1	φ_2	φ_3	φ_4
$a,\ b,\ c,\ a \wedge \neg b$	$a,\ b$	$\neg a, \neg b$	$a, a \rightarrow b$

$$\text{MAXCONS} = c \qquad \triangle^{\Sigma} = a \wedge b$$

$$\text{MAXCONS}_{\text{card}} = c \qquad \triangle^{GMax} = (a \wedge \neg b) \vee (\neg a \wedge b)$$

$$\triangle^{d_D,\Sigma,\Sigma} = a \wedge b \wedge c$$

Merging and Belief Revision

The operator $*$ is an *AGM revision operator* if and only if it satisfies the following properties:

(R1) $\varphi * \mu$ implies μ

(R2) If $\varphi \wedge \mu$ is consistent then $\varphi * \mu \equiv \varphi \wedge \mu$

(R3) If μ is consistent then $\varphi * \mu$ is consistent

(R4) If $\varphi_1 \equiv \varphi_2$ and $\mu_1 \equiv \mu_2$ then $\varphi_1 * \mu_1 \equiv \varphi_2 * \mu_2$

(R5) $(\varphi * \mu) \wedge \psi$ implies $\varphi * (\mu \wedge \psi)$

(R6) If $(\varphi * \mu) \wedge \psi$ is consistent then $\varphi * (\mu \wedge \psi)$ implies $(\varphi * \mu) \wedge \psi$

- ▷ If $\triangle$ is an IC merging operator (it satisfies **(IC0-IC8)**), then the operator $*_\triangle$, defined as $\varphi *_\triangle \mu = \triangle_\mu(\varphi)$, is an AGM revision operator (it satisfies **(R1-R6)**).
- ▷ Links between prioritized merging and iterated revision:
  [J. Delgrande, D. Dubois, J. Lang. Iterated Revision as Prioritized Merging. KR'06.]

Merging and Judgment Aggregation

	Merging	Judgment Aggregation
Input	A profile of belief bases	A profile of individual judgments
$\longrightarrow$	Fully informed process	Partially informed process
Computation	Global	Local
Consequences	– computational complexity + logical properties	+ computational complexity – logical properties
	Ideal Process	Practical Process

Merging and Social Choice

- Merging as social choice function
  - Social choice function $(\leq_1, \ldots, \leq_n) \rightarrow \leq$
  - Belief Merging $(\varphi_1, \ldots, \varphi_n) \rightarrow \varphi$
- Arrow's impossibility theorem
  - There is no social choice function that satisfies all of:
    - Universality
    - Pareto Efficiency
    - Independence of Irrelevant Alternatives
    - Non-dictatorship
- Gibbard-Satterthwaite theorem
  - There is no social choice function that satisfies all of:
    - Surjectivity
    - Strategy-proofness
    - Non-Dictatorship

Strategy-Proof Merging

Intuitively, a merging operator is strategy-proof if and only if, given the beliefs/goals of the other agents, reporting untruthful beliefs/goals does not enable an agent to improve her satisfaction.

- A merging operator Δ is *strategy-proof for a satisfaction index* i if and only if there is no integrity constraint μ, no profile $\Psi = \{\varphi_1, \ldots, \varphi_n\}$, no base φ and no base φ' such that

$$i(\varphi, \Delta_\mu(\Psi \sqcup \{\varphi'\})) > i(\varphi, \Delta_\mu(\Psi \sqcup \{\varphi\}))$$

Clearly, there are numerous different ways to define the satisfaction of an agent given a merged base.

Strategy-Proof Merging: Satisfaction Indexes

- ▷ Weak drastic index: the agent is considered satisfied if her beliefs/goals are consistent with the merged base.

$$i_{d_w}(\varphi, \varphi_\Delta) = \begin{cases} 1 \text{ if } \varphi \wedge \varphi_\Delta \text{ is consistent} \\ 0 \text{ otherwise.} \end{cases}$$

- ▷ Strong drastic index: in order to be satisfied, the agent must impose her beliefs/goals to the whole group.

$$i_{d_s}(\varphi, \varphi_\Delta) = \begin{cases} 1 \text{ if } \varphi_\Delta \models \varphi \\ 0 \text{ otherwise.} \end{cases}$$

- ▷ Probabilistic index: the more compatible the merged base with the agent's base the more satisfied the agent.

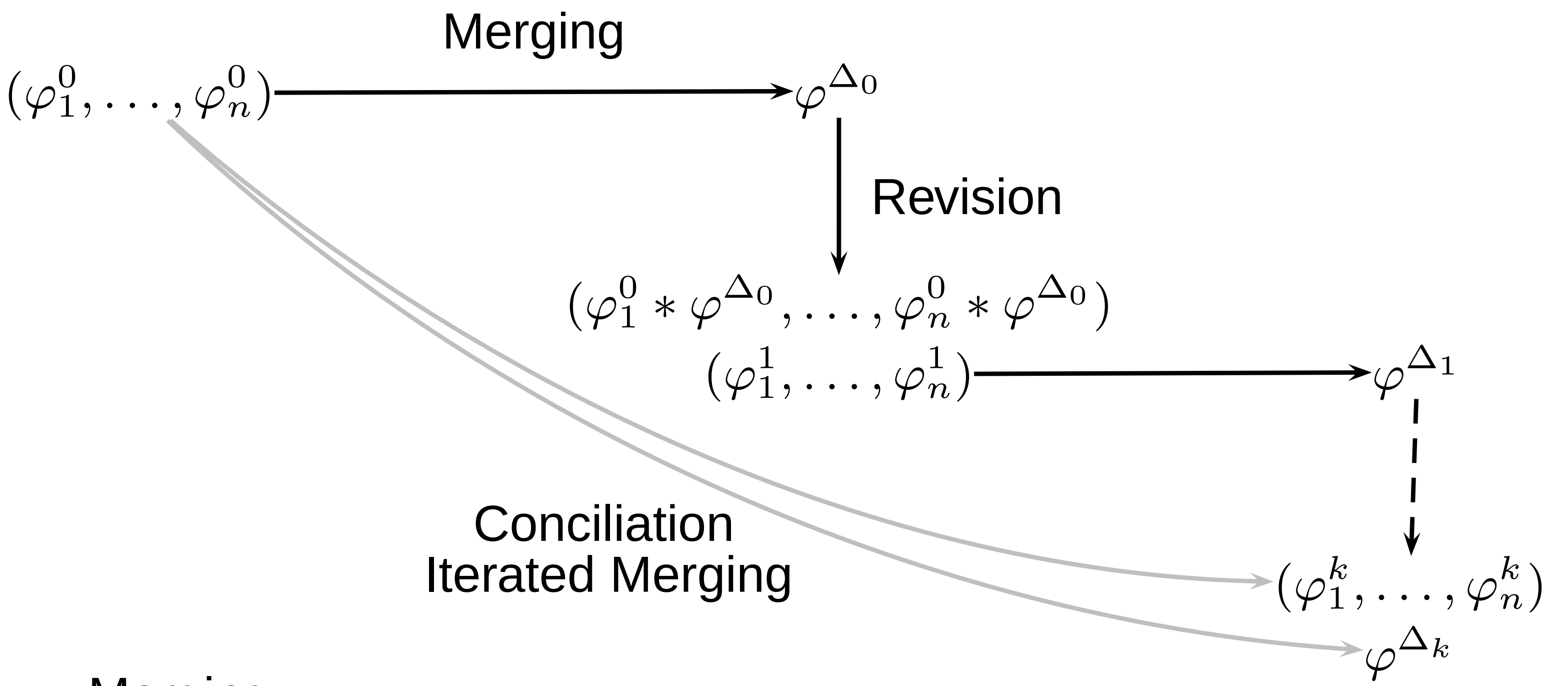
$$i_p(\varphi, \varphi_\Delta) = \frac{\#(Mod(\varphi) \cap Mod(\varphi_\Delta))}{\#(Mod(\varphi_\Delta))}$$

Strategy-Proof Merging: Some Results for i_{d_w}

$\#(\Psi)$	φ	μ	$\Delta^{d_H,\Sigma}$	$\Delta^{d_H,G_{max}}$	Δ^{C1}	Δ^{C3}	Δ^{C4}	Δ^{C5}
2	φ_ω	$\top$	**sp**	$\overline{sp}$	**sp**	**sp**	$\overline{sp}$	**sp**
		μ	**sp**	$\overline{sp}$	**sp**	$\overline{sp}$	$\overline{sp}$	**sp**
	φ	$\top$	**sp**	$\overline{sp}$	**sp**	**sp**	$\overline{sp}$	**sp**
		μ	$\overline{sp}$	$\overline{sp}$	**sp**	$\overline{sp}$	$\overline{sp}$	$\overline{sp}$
> 2	φ_ω	$\top$	**sp**	$\overline{sp}$	**sp**	**sp**	$\overline{sp}$	**sp**
		μ	**sp**	$\overline{sp}$	**sp**	$\overline{sp}$	$\overline{sp}$	**sp**
	φ	$\top$	$\overline{sp}$	$\overline{sp}$	**sp**	**sp**	$\overline{sp}$	**sp**
		μ	$\overline{sp}$	$\overline{sp}$	**sp**	$\overline{sp}$	$\overline{sp}$	$\overline{sp}$

Negotiation - Conciliation

- Iterated Merging Operators



- Merging

$$(\varphi_1, \ldots, \varphi_n) \longrightarrow \varphi_\Delta$$

- Conciliation

$$(\varphi_1, \ldots, \varphi_n) \longrightarrow (\varphi_1^*, \ldots, \varphi_n^*)$$

Thanks to...

Works related to this talk were joint works with:

- ▷ Patricia Everaere
- ▷ Jérôme Lang
- ▷ Pierre Marquis
- ▷ Ramón Pino Pérez