An axiomatization and a tableau calculus for the logic of comparative concept similarity

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Résumé : La logique de *similarité comparative des concepts* $\mathcal{CSL}$ a été introduite en 2005 par Shremet, Tishkowsky, Wolter et Zakharyaschev pour représenter des informations qualitatives sur la similarité entre des concepts, du type "A est plus similaire à B qu'à C". La sémantique utilise des espaces de distances afin de représenter le degré de similarité entre objets du domaine. Dans cet article, nous étudions $\mathcal{CSL}$ sur les *minspaces*, i.e des espaces de distances dans lesquels tout ensemble de distances possède un minimum, et donnons la première axiomatisation directe de cette logique dans ce contexte, ainsi qu'une méthode de preuve à tableaux. A notre connaissance, notre calcul est la première méthode de preuve pratique pour $\mathcal{CSL}$.

Mots-clés : logiques modales, procédures à tableaux, logiques de description, similarité comparative des concepts

1 Introduction

The logics of comparative concept similarity $\mathcal{CSL}$ have been recently proposed by Sheremet, Tishkovsky, Wolter et Zakharyaschev in (Sheremet *et al.*, 2005) to capture a form of qualitative comparison between concept instances. In these logics we can express assertions or judgments of the form : "‘Peugeot 207 is more similar to Renault Clio than to Porche Cayenne"’, or "‘Tuscan order is more similar to Doric order than to Ionic order"’, as we might like to express in a KB about archeology. These logics could find a natural application in ontology languages, whose logical base is provided by Description Logics.

The language of $\mathcal{CSL}$ is obtained by the addition of a binary modal connective $\leftleftarrows$ to an underlying language, so that the above examples can be encoded (using a description logic notation) by :

$$Peugeot207 \sqsubseteq (Clio \leftleftarrows PorcheCayenne)$$

$$TuscanOrder \sqsubseteq (DoricOrder \leftleftarrows IonicOrder)$$

The semantics of $\mathcal{CSL}$ is defined in terms of distance spaces, that is to say structures equipped by a distance function d, whose properties may vary according to the logic under consideration. In this setting, the evaluation of $A \leftleftarrows B$ can be informally stated as follows : $x \in (A \leftleftarrows B)$ iff $d(x, A) < d(x, B)$ meaning that the object x is an instance of the concept $A \leftleftarrows B$ (i.e. it belongs to things that are more similar to A than to B) if x is strictly closer to A-objects than to B-objects according to distance function d, where the distance of an object to a set of objects is defined as the *infimum* of the distances to each object in the set.

In a series of papers (Sheremet *et al.*, 2005, 2008; Kurucz *et al.*, 2005; Sheremet *et al.*, 2007), the authors have investigated the logic $\mathcal{CSL}$ with respect to different classes of distance models, see (Sheremet *et al.*, 2008) for a survey of results about decidability, complexity, expressivity, and axiomatisation. Remarkably it is shown that $\mathcal{CSL}$ is undecidable over subspaces of the reals. Moreover $\mathcal{CSL}$ can be seen as a fragment, indeed a powerful one (including for instance the logic $\mathbf{S4}_u$ of topological spaces), of a general logic for spatial reasoning comprising different modal operators defined by (bounded) quantified distance expressions.

The authors have pointed out that in case the distance spaces are assumed to be *minspaces*, that is spaces where the infimum of a set of distances is actually their *minimum*, the logic $\mathcal{CSL}$ is naturally related to some conditional logics. The semantics of the latter is often expressed in terms of preferential structures, that is to say possible-world structures equipped by a family of strict partial (pre)-orders $\prec_x$ parametrised on objects. The intended meaning of the relation $y \prec_x z$ is namely that x is more similar to y than to z.

In this paper we contribute to the study of $\mathcal{CSL}$ on minspaces. The minspace property is essentially equivalent to the restriction to spaces where the distance function is discrete. This requirement does not seem in contrast with the purpose of representing qualitative comparisons of similarity.

Similarly to conditional logics, we consider a propositional language extended by the $\leftleftarrows$ connective. In this setting (as opposed to concept/subset interpretation), the formula $A \leftleftarrows B$ may be naturally read as "'A is (strictly) more plausible than B"'. We first show that the semantics of $\mathcal{CSL}$ on minspaces can be equivalently restated in terms of preferential models satisfying some additional conditions, namely modularity, centering, and limit assumption. We then give the first sound, complete and direct axiomatisation of this logic ; this problem was left open in the recent (Sheremet *et al.*, 2008). Furthermore, we define a tableaux calculus for checking satisfiability of $\mathcal{CSL}$ formulas. Our tableaux procedure makes use of labelled formulas and pseudo-modalities indexed on worlds $\Box_x$, similarly to the calculi for conditional logics defined in (Giordano *et al.*, 2003, 2009). To the best of our knowledge our calculus provides the first known practical decision procedure for this logic.

2 The logic of *Comparative Concept Similarity* $\mathcal{CSL}$

The langage $\mathcal{L}_{\mathcal{CSL}}$ of $\mathcal{CSL}$ is generated from a set of propositional variables V_i by the following grammar :

$$A, B \ ::= \ V_i \mid \neg A \mid A \wedge B \mid A \leftleftarrows B.$$

The others propositional connectives are defined as usual.

The semantic of $\mathcal{CSL}$ introduced in (Sheremet *et al.*, 2005) makes use of *distance spaces* in order to represent the similarity degree between possible worlds. A distance space is a pair (Δ, d) where Δ is a non-empty set, and $d : \Delta \rightarrow \mathbb{R}^{\geq 0}$ is a *distance function* satisfying the following condition :

(ID) $$\forall x, y \in \Delta, \ \ d(x, y) = 0 \text{ iff } x = y$$

Two further properties are usually assumed : symmetry and triangle inequality. We briefly discuss them at the end of this section.

The distance between an object w and a non-empty subset X of Δ is defined by $d(w, X) = \inf\{d(w, x) \mid x \in X\}$. If $X = \emptyset$, then $d(w, X) = \infty$. If for every onject w and for every (non-empty) subset X we have the following property

(MIN) $$\inf \{d(w, x) \mid x \in X\} = \min \{d(w, x) \mid x \in X\},$$

we will say that (Δ, d) is a *min-space*.

We next define $\mathcal{CSL}$-distance models as Kripke models based on distance spaces :

Definition 2.1 ($\mathcal{CSL}$-distance model)
A $\mathcal{CSL}$-distance model is a triple $\mathcal{M} = (\Delta, d, .^{\mathcal{M}})$ where :
- Δ *is a non-empty set of* objects *(or* possible worlds*).*
- d *is a distance on $\Delta^{\mathcal{M}}$ (so that (Δ, d) is a distance space).*
- $.^{\mathcal{M}} : \mathcal{V}_p \rightarrow 2^{\Delta}$ *is the* evaluation function *which assigns to each propositional variable V_i a set $V_i^{\mathcal{M}} \subseteq \Delta$. $V_i^{\mathcal{M}}$ can be seen as the set of possible worlds where V_i is true. We also stipulate $\perp^{\mathcal{M}} = \emptyset$. For complex formulas, $.^{\mathcal{M}}$ is defined inductively as follows :*
$$(\neg C)^{\mathcal{M}} = \Delta - C^{\mathcal{M}}$$
$$(C \wedge D)^{\mathcal{M}} = C^{\mathcal{M}} \cap D^{\mathcal{M}}$$
$$(C \leftleftarrows D)^{\mathcal{M}} = \left\{ w \in \Delta \,\middle|\, d(w, C^{\mathcal{M}}) < d(w, D^{\mathcal{M}}) \right\}.$$

If (Δ, d) is a min-space, $\mathcal{M}$ is called a $\mathcal{CSL}$-distance minmodel.

We say that a formula A is valid in a model $\mathcal{M}$ *if $A^{\mathcal{M}} = \Delta$. We say that a formula A is* valid *if A is valid in every $\mathcal{CSL}$-distance model.*

As mentioned above, the distance function might be required to satisfy the further conditions of symmetry (SYM) ($d(x, y) = d(y, x)$) and triangular inequality (TR) ($d(x, z) \leq d(x, y) + d(y, z)$). It turns out that $\mathcal{CSL}$ cannot distinguish between minmodels which satisfy (TR) from models which do not. In contrast, $\mathcal{CSL}$ has enough expressive power in order to distinguish between symmetric and non-symmetric minmodels[1]. We intend to consider symmetric models in future research.

[1] See (Sheremet *et al.*, 2005).

3 A preferential semantics for $\mathcal{CSL}$

$\mathcal{CSL}$ is a logic of pure qualitative comparisons. This motivates an alternative semantics where the distance function is replaced by a family of comparisons relations, one for each object. We call this semantics *preferential* semantic, similarly to conditional logics. Technically, preferential structures are equipped by a family of strict preorders. We may interprete this relations as expressing a similarity information between objects. For three worlds/objects, $x \prec_w y$ states that w is more similar to x than to y.

The preferential semantic in itself is more general than distance model semantic. However, if we assume the additional conditions of the definition 3.1, it turns out that these two are equivalent (theorem 3.3).

Definition 3.1
We will say that a preferential relation $\prec_w$ over Δ :

(i) is modular *iff* $\forall x, y, z \in \Delta$, $(x \prec_w y) \rightarrow (z \prec_w y \ \vee \ x \prec_w z)$.

(ii) is centered *iff* $\forall x \in \Delta$, $x = w \ \vee \ w \prec_w x$.

(iii) satisfies the limit assumption *iff* $\forall X \subseteq \Delta$, $X \neq \emptyset \ \rightarrow \ \min_{\prec_w}(X) \neq \emptyset$.

Modularity is strongly related to the fact that the preferential relations represents distance comparisons. This is the key property to enforce the equivalence with distance models. Centering states that w is the *unique* minimal element for its preferential relation $\prec_w$, and can be seen as the preferential counterpart of (ID). The limit assumption states that each non-empty set has at least one minimal element wrt. a preferential relation (i.e it does not contain an infinitely descending chain), and corresponds to (MIN).

Definition 3.2 ($\mathcal{CSL}$-preferential model)
A $\mathcal{CSL}$-preferential model is a triple $\mathcal{M} = (\Delta, (\prec_w)_{w \in \Delta}, .^{\mathcal{M}})$ where :
- $\Delta^{\mathcal{M}}$ *is a non-empty set of* objects *(or* possible worlds*).*
- $(\prec_w)_{w \in \Delta}$ *is a family of* preferential relation*, each one being transitive, irreflexive, asymmetric, modular, centered, and satisfying the* limit assumption.
- $.^{\mathcal{M}}$ *is the evaluation function defined as in definition 2.1, except for* $\leftleftarrows$ *:*

$$(A \leftleftarrows B)^{\mathcal{M}} = \left\{ w \in \Delta \middle| \exists x \in A^{\mathcal{M}} \text{ such that } \forall y \in B^{\mathcal{M}},\ x \prec_w y \right\}$$

Validity is defined as in definition 2.1.

We now show the equivalence between preferential models and distance minmodels. We say that a $\mathcal{CSL}$-preferential model $\mathcal{I}$ and a $\mathcal{CSL}$-distance minmodel $\mathcal{J}$ are *equivalent* iff they are based on the same set Δ, and for all formulas $A \in \mathcal{L}_{\mathcal{CSL}}$, $A^{\mathcal{I}} = A^{\mathcal{J}}$.

Theorem 3.3 (Equivalence between $\mathcal{CSL}$-preferential models and $\mathcal{CSL}$-distance models)

1. *For each $\mathcal{CSL}$-distance min-model, there is an equivalent $\mathcal{CSL}$-preferential model.*
2. *For each $\mathcal{CSL}$-preferential model, there is an equivalent $\mathcal{CSL}$-distance min-model.*

Sketch of the proof

1. Contained in (Sheremet *et al.*, 2005).
2. Since the relation $\prec_w$ is modular, we can assume that there exists a *ranking function* r_w such that $x \prec_w y$ iff $r_w(x) < r_w(y)$. Therefore, given a $\mathcal{CSL}$-preferential model $\mathcal{J} = (\Delta^{\mathcal{J}}, (\prec_w)_{w \in \Delta^{\mathcal{J}}}, .^{\mathcal{J}})$, we can define a $\mathcal{CSL}$-distance min-model $\mathcal{I} = (\Delta^{\mathcal{J}}, d, .^{\mathcal{J}})$, where the distance function d is defined as follow : if $w = x$ then $d(w,x) = 0$, and $d(w,x) = r_w(x)$ otherwise. We can easily check that I is a min-space (by the limit assumption), and that $\mathcal{I}$ and $\mathcal{J}$ are equivalent.

4 An axiomatization of $\mathcal{CSL}$ over minspaces

We now give an axiomatisation of $\mathcal{CSL}$ over minspaces. An axiomatisation of $\mathcal{CSL}$ in various classes of models can be found in (Sheremet *et al.*, 2008), but it makes use of an extended language. Moreover, the case of minspaces has not been studied yet, and it does not seem that our axioms can be easily derived from (Sheremet *et al.*, 2008).

$(Ax\bot)$	$\neg(\bot \leftleftarrows A)$	$(AxTR)$	$(A \leftleftarrows B) \wedge (B \leftleftarrows C) \rightarrow (A \leftleftarrows C)$
$(AxAS)$	$\neg(A \leftleftarrows B) \vee \neg(B \leftleftarrows A)$	$(Ax\neg B)$	$(A \leftleftarrows B) \rightarrow \neg B$
$(AxMN)$	$A \wedge \neg B \rightarrow (A \leftleftarrows B)$	$(AxMD)$	$(A \leftleftarrows B) \rightarrow (A \leftleftarrows C) \vee (C \leftleftarrows B)$
$(Ax\vee)$	$(A \leftleftarrows B) \wedge (A \leftleftarrows C) \rightarrow (A \leftleftarrows (B \vee C))$	$(Ax\wedge)$	$(A \leftleftarrows B) \rightarrow ((A \wedge \neg B) \leftleftarrows B)$
$(AxU1)$	$\neg(A \leftleftarrows \bot) \rightarrow \neg((A \leftleftarrows \bot) \leftleftarrows \bot)$	$(AxU2)$	$(A \leftleftarrows \bot) \rightarrow \neg(\neg(A \leftleftarrows \bot) \leftleftarrows \bot)$
(Rr)	$\dfrac{\vdash (A \rightarrow B)}{\vdash (A \leftleftarrows C) \rightarrow (B \leftleftarrows C)}$	(Rl)	$\dfrac{\vdash (A \rightarrow B)}{\vdash (C \leftleftarrows B) \rightarrow (C \leftleftarrows A)}$
$(Taut)$	Classical tautologies and rules.		

FIG. 1 – **CSMS** axioms.

Our axiomatisation, named **CSMS** is presented in figure 1. If we interpret $A \leftleftarrows B$ as "A is more plausible than B", we can give the following intuitive meanings to the axioms : $(Ax\bot)$ states that a contradiction cannot be more plausible than any formula. $(Ax\vee)$ states that if a formula A is more plausible than two others, A is more plausible than their disjunction. $(Ax\wedge)$ states that if a formula A is more plausible than a formula B, then $A \wedge \neg B$ is more plausible than A. $(AxTR)$, $(AxAS)$ and $(AxMD)$ represent respectively transitivity, asymmetry and modularity of the preferential relation. $(Ax\neg B)$ and $(AxMN)$ are needed to express the centering and limit assumption conditions.

$(AxU1)$ and $(AxU2)$ were introduced for a technical purpose. They are the translations in $\mathcal{CSL}$ of the modal **S5** axioms $\Box A \rightarrow \Box\Box A$ and $\Diamond A \rightarrow \Box\Diamond A$[2], meaning that all preference relations have the same range. The rules (Rr) and (Rl) express the monotony of $\leftleftarrows$ in the first argument, and the anti-monotony in the second one.

We can show that our axiomatisation is sound and complete with respect to the preferential semantics introduced in section 3.

[2]Note that we can define $\Diamond A$ in $\mathcal{CSL}$ by $(A \leftleftarrows \bot)$.

Theorem 4.1 (Soundness of CSMS)

(Soundness) If a formula is derivable in **CSMS**, *then it is valid in every $\mathcal{CSL}$-preferential model.*

(Completeness) If a formula is valid in every $\mathcal{CSL}$-preferential model, then it is derivable in **CSMS**.

Sketch of the proof

(Soundness) By induction on the derivation length. We check that the axioms are valid in all $\mathcal{CSL}$-preferential models, and that the rules preserve validity.

(Completeness) We show the contrapositive : if a formula is not derivable in **CSMS**, then its negation is satisfiable in some $\mathcal{CSL}$-preferential model. For this, we show how to construct for a given non-derivable formula C a canonical model in which $\neg C$ is satisfiable.

To begin, let U be the set of all *maximal consistent* sets for $\mathcal{L}_{\mathcal{CSL}}$. We define a binary relation R over U by $R(x, y)$ iff $\forall A \in \mathcal{L}_{\mathcal{CSL}},\ A \in y \rightarrow (A \leftleftarrows \bot) \in x$. We can prove that R is an equivalence relation (by virtue of $(AxU1)$ and $(AxU2)$). For all $x \in U$, we let $[x]$ be its equivalence class with respect to R.

Since C is not derivable in **CSMS**, $\neg C$ is consistent, and so there is a maximal consistent set $z \in U$ such that $\neg C \in z$. We can define a model (called canonical model) $\mathcal{M}_C = (\Delta, (\prec_w)_{w \in \Delta}, .^{\mathcal{M}_C})$ as follow :

- $\Delta = [z]$.
- $x \prec_w y$ iff there exists a formula $B \in y$ such that for all formulas $A \in x$, $(A \leftleftarrows B) \in w$.
- $V_i^{\mathcal{M}_C} = \{x \in \Delta \mid V_i \in x\}$, for all propositional variables V_i.

We can check that each preferential relation $\prec_w$ is centered, modular, and satisfies the limit assumption, and that for all formulas A and for all worlds $w \in \Delta$, we have $w \in A^{\mathcal{M}}$ iff $A \in w$. Since $z \in \Delta$ and $\neg C \in z$, we obtain that $(\neg C)^{\mathcal{M}_C} \neq \emptyset$, and thus that C is not valid. Details are in (Alenda, 2008).

By virtue of theorem 3.3, we obtain :

Corollary 4.2

CSMS *is sound and complete wrt. the $\mathcal{CSL}$-distance min-models.*

5 Tableau algorithm

In this section, we sketch a tableau-based decision procedure for $\mathcal{CSL}$. As usual, a tableau is a tree whose branches are sets of formulas. These formulas are either the initial formulas or they are obtained from previous formulas by the application of tableau rules, the rules may produce branching in the tree.

Our calculus makes use of labels to represent possible worlds. In order to check whether a formula A is satisfiable, we initialise a tableau by $x : A$ for an arbitrary label x. Informally, the tableau is construction proceeds as follows : we apply the tableaux rules to each formula in a branch until, either we detect a contradition (the branch is closed), or no new formula is introduced in the branch (the branch is saturated). If every branch is closed A is not satisfiable, otherwise it is satisfiable and each open branch specifies a model of it. In order to check the validity of a formula A, we check whether its negation is satisfiable, so that A is valid if the tableau initialised with $x : \neg A$ is closed.

Tableau rules encode the semantics of the formulas. It is well known how this works for boolean operators. Let us look at the formulas $(A \leftleftarrows B)$ and $\neg(A \leftleftarrows B)$ under preferential semantics. We have :

$$w \in (A \leftleftarrows B)^{\mathcal{M}} \text{ iff } \exists x(x \in A^{\mathcal{M}} \wedge \forall z(z \in B^{\mathcal{M}} \rightarrow x \prec_w z))$$

In minspaces, the right part is equivalent to :

$$\exists u\ u \in A^{\mathcal{M}} \wedge \forall y(y \in B^{\mathcal{M}} \rightarrow \exists x(x \in A^{\mathcal{M}} \wedge x \prec_w y))$$

We introduce a pseudo-modality $\Box_w$ indexed on possible worlds.

$$x \in (\Box_w A)^{\mathcal{M}} \text{ iff } \forall y(y \prec_w x \rightarrow y \in A^{\mathcal{M}})$$

This pseudo-modality is needed for a technical purpose. Its meaning is that $x \in (\Box_w A)^{\mathcal{M}}$ iff A holds in all preferred worlds to x with respect to $\prec_w$. It makes it possible to obtain analytic rules for $A \leftleftarrows B$.

By means of this modality, we obtain the following equivalence :

$$w \in (A \leftleftarrows B)^{\mathcal{M}} \text{ iff } A^{\mathcal{M}} \neq \emptyset \ \wedge\ \forall y(y \notin B^{\mathcal{M}} \vee y \in (\neg \Box_w \neg A)^{\mathcal{M}}$$

This last equivalence yields the tableaux rules $(T \leftleftarrows)$ and $(F \leftleftarrows)$. Figure 2 shows all tableaux rules for our logic.

Let us point out that rule $(F\Box_x)$ introduces the formula $\Box_x \neg A$. This corresponds to the limit assumption and will prevent the calculus to produce infinit descending chains. We can show that no tableau contains infinite descending chains of labels related by the same relation $<_x$, provided it does not contain an infinite number of positive $\leftleftarrows$-formulas, i.e. formulas of the form $x : \phi \leftleftarrows \psi$ labelled with the same label x. A formal proof can be found in (Alenda, 2008)

Definition 5.1 (Closed branch, closed tableau)
A branch $\mathbf{B}$ *of a* $\mathcal{CSL}$*-tableau is* closed *if one of the three following conditions hold : (i)* $x : A \in \mathbf{B}$ *and* $x : \neg A \in \mathbf{B}$*, for any formula* A*, or* $x : \bot \in \mathbf{B}$*. (ii)* $y <_x y \in \mathbf{B}$*. (iii)* $y <_x x \in \mathbf{B}$.

A $\mathcal{CSL}$*-tableau is closed if every branch is closed.*

In order to prove soundness and completeness of the tableaux rules, we introduce the notion of satisfiability of a branch by a model.

Given a branch B, we denote by W_B the set of labels occurring in B.

Definition 5.2 ($\mathcal{CSL}$-mapping, satisfiable branch)
Let $\mathcal{M} = (\Delta^{\mathcal{M}}, (\prec_w)_{w \in \Delta^{\mathcal{M}}}, .^{\mathcal{M}})$ *a* $\mathcal{CSL}$*-preferential model, and* $\mathbf{B}$ *a branch of a* $\mathcal{CSL}$*-tableau. A* $\mathcal{CSL}$*-mapping from* $\mathbf{B}$ *to* $\mathcal{M}$ *is a function* $f : W_{\mathbf{B}} \longrightarrow \Delta^{\mathcal{M}}$ *satisfying the two following conditions :*

(i) for every $y <_x z$ *in* $\mathbf{B}$*, we have* $f(y) \prec_{f(x)} f(z)$ *in* $\mathcal{M}$*.*

$(T\wedge)$	$\frac{x : A \wedge B}{x : A,\ x : B}$	$(N\wedge)$	$\frac{x : \neg(A \wedge B)}{x : \neg A \mid x : \neg B}$
(NEG)	$\frac{x : \neg\neg A}{x : A}$		
$(T \leftleftarrows)(*)$	$\frac{x : A \leftleftarrows B}{\neg\Box\neg A,\ y : \neg B \mid y : \neg\Box_x\neg A}$	$(F \leftleftarrows)(**)$	$\frac{x : \neg(A \leftleftarrows B)}{\Box\neg A \mid y : B,\ y : \Box_x\neg A}$
$(T\Box_x)(*)$	$\frac{z : \Box_x A, y <_x z}{y : A}$	$(F\Box_x)(**)$	$\frac{z : \neg\Box_x A}{y <_x z,\ y : \neg A,\ y : \Box_x A}$
$(T\Box)(*)$	$\frac{\Box A}{y : A}$	$(F\Box)(**)$	$\frac{\neg\Box A}{y : \neg A}$
$(Trans)$	$\frac{y <_x z, z_x u}{y <_x u}$	$(Mod)(*)$	$\frac{z <_x u}{z <_x y \mid y <_x u}$
$(Cent)$	$\frac{}{x <_x y \mid x = y}$	$(E-R)(*)$	$\frac{}{y = y}$
$(E-S)$	$\frac{x = y}{y = x}$	$(E-T)$	$\frac{x = y,\ y = z}{x = z}$
$(E-<)$	$\frac{x = x', y = y', z = z', y <_x z}{y' <_{x'} z'}$	$(E-A)$	$\frac{x = y,\ x : A}{y : A}$

(*) y is a label occuring in the branch
(**) y is a new label not occuring in the branch

FIG. 2 – Tableau rules for $\mathcal{CSL}$.

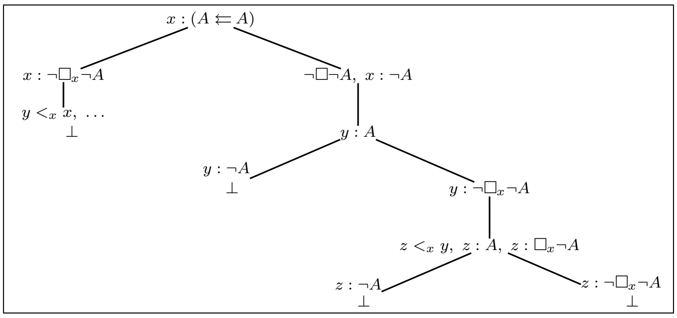

FIG. 3 – An exemple of tableau : provability of $\neg(A \leftleftarrows A)$.

(ii) for every $x = y$ in $\mathbf{B}$, we have $f(x) = f(y)$ in $\mathcal{M}$.

Given a branch $\mathbf{B}$ of a $\mathcal{CSL}$-tableau, a $\mathcal{CSL}$-preferential model $\mathcal{M}$, and a $\mathcal{CSL}$-mapping f from $\mathbf{B}$ to $\mathcal{M}$, we say that $\mathbf{B}$ is satisfiable under f in $\mathcal{M}$ if

$$x : A \in \mathbf{B} \text{ implies } f(x) \in A^{\mathcal{M}}.$$

A branch $\mathbf{B}$ *is satisfiable if it is satisfiable in some* $\mathcal{CSL}$*-preferential model* $\mathcal{M}$ *under some* $\mathcal{CSL}$*-mapping* f*. A* $\mathcal{CSL}$*-tableau is satisfiable if at least one of its branches is satisfiable.*

Theorem 5.3 (Soundness and completeness of the calculus)

(Soundness) If the tableau starting by $x : \neg A$ *is closed, then* A *is* $\mathcal{CSL}$*-valid (wrt. the preferential semantics).*

(Completeness) If a formula is $\mathcal{CSL}$*-valid (wrt. our preferential semantics), then the tableau starting by* $x : \neg A$ *is closed.*

Sketch of the proof

(Soundness) The proof is standard : we show that rule application preserves satisfiability.

(Completeness) The completeness of the tableaux calculus is proved in the standard way : we show by contraposition that if a formula is not provable by the calculus then it is not valid. To this purpose we need a notion of saturated branch (see (Alenda, 2008)). Intuitively a saturated branch is a branch that is closed under the application of every rule of the calculus, that is to say if it contains the premise of the rule, then it contains at least one of its conclusions. Then the proof runs as follows : if a C is not provable, then there exists an open saturated branch B containing $x : \neg C$. We can define a model M_B associated to B which satisfies all formulas occurring in B under a suitable mapping, thus in particular it satisfies $x : \neg C$. Details can be found in (Alenda, 2008).

Corollary 5.4

Our calculus is sound and complete wrt. the $\mathcal{CSL}$*-distance min-models.*

The calculus presented above can lead to non-terminating computations due to the interplay between the rules which generate new labels (the dynamic rules $(F \leftleftarrows)$, $(F\Box)$ and $(F\Box_x)$) and the static rule $(T \leftleftarrows)$ which generates formula $\neg\Box_x A$ to which $(F\Box_x)$ may again be applied. For instance, the tableaux construction for $w : p \leftleftarrows (q \leftleftarrows r)$ can generate an infinite branch containing $x_1 : r$, $x_1 : \Box_x\neg q$, $x_1 : \neg(q \leftleftarrows r)$, $x_2 : r$, $x_2 : \Box_x\neg q$, $x_2 : \neg(q \leftleftarrows r)$, $\ldots$. Our calculus can be made terminating, without loosing completeness, by defining a systematic procedure for applying the rules and by introducing appropriate blocking conditions.

To this aim, we first define a total ordering $\prec$ on the labels of a branch such that $x \prec y$ for all labels x that are already in the tableau when y is introduced. The systematic procedure for constructing the tableau for formula $x : A$ applies first all static rules as far as possible and then applies one dynamic rules to some formula labelled x only if no dynamic rule is applicable to a formula labelled y, such that $y \prec x$.

In order to stop the infinite expansion of a branch, we consider an equivalence relation on labels : given a branch B, we say that two labels x and y are $B{-}equivalent$, denoted by $x \equiv_B y$, if they label the same set of formulas in B, ignoring modal-pseudo formulas of type $(\neg)\Box_u C \in B$ (on which they might differ). Since the tableau is initialised by a finite number of formulas, there can be only a finite number of labels that are not $\equiv_B$-equivalent in any branch B. Thus a loop-checking mechanism can be devised to ensure termination. Moreover, it can be shown, similarly to (Giordano *et al.*, 2009), that idenditying $\equiv_B$ labels preserve the completeness of the method (an open branch will not become close). Details will be given in a full paper.

6 Conclusion

In this paper, we have studied the logic $\mathcal{CSL}$ over minspaces, and we have obtained two main results : first we have provided a direct, sound and complete axiomatisation of this logic. Furthermore, we defined a tableau calculus, which gives a decision procedure for this logic.

There are a number of issues to explore in future research. The decision procedure outlined in the previous section is not guaranteed to have an optimal complexity. To this concern, it is shown in (Sheremet *et al.*, 2005) that $\mathcal{CSL}$ is EXPTIME-complete, so that we can consider how to improve our calculus in order to match this upper bound. Another issue is the extension of our results to symmetric minspaces, and possibly to other classes of models. Finally, since one original motivation of $\mathcal{CSL}$ is to reason about concept similarity in ontologies, and particularly in description logics, we plan to study further its integration with this family of formalisms.

Références

ALENDA R. (2008). Logique et raisonnement automatique pour la comparative concept similarity. Master's thesis, Université de Provence, Aix-Marseille I.

GIORDANO L., GLIOZZI V., OLIVETTI N. & SCHWIND C. (2003). Tableau calculi for preference-based conditional logics. In M. C. MAYER & F. PIRRI, Eds., *TABLEAUX 2003*, volume 2796 of *Lecture Notes in Artificial Intelligence*, p. 81–101 : Springer.

GIORDANO L., GLIOZZI V., OLIVETTI N. & SCHWIND C. (2009). Tableau calculus for preference-based conditional logics : Pcl and its extensions. *ACM Transactions on Computational Logic (TOCL)*. à paraître.

KURUCZ A., WOLTER F. & ZAKHARYASCHEV M. (2005). Modal logics for metric spaces : Open problems. In S. N. ARTËMOV, H. BARRINGER, A. S. D'AVILA GARCEZ, L. C. LAMB & J. WOODS, Eds., *We Will Show Them ! (2)*, p. 193–108 : College Publications.

SHEREMET M., TISHKOVSKY D., WOLTER F. & ZAKHARYASCHEV M. (2005). Comparative similarity, tree automata, and diophantine equations. In G. SUTCLIFFE & A. VORONKOV, Eds., *LPAR 2005*, volume 3835 of *Lecture Notes in Computer Science*, p. 651–665 : Springer.

SHEREMET M., TISHKOVSKY D., WOLTER F. & ZAKHARYASCHEV M. (2007). A logic for concepts and similarity. *J. Log. Comput.*, **17**(3), 415–452.

SHEREMET M., WOLTER F. & ZAKHARYASCHEV M. (2008). A modal logic framework for reasoning about comparative distances and topology. submitted.