On Distances between KD45n Kripke Models and their Use for Belief Revision

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Abstract. In this paper, some distances between $KD45_n$ Kripke models are introduced and investigated. We define several distances between Kripke models, based on different criteria, inspired by various concepts such as bisimulation and propositional distances between valuations for different modal degrees. We study the properties of these distances. Such distances are useful for defining belief change operators in multi-agent scenarios. We show that they can be used to define belief revision operators based on the standard AGM framework and suited to $KD45_n$ Kripke models.

1 INTRODUCTION

Distance proves to be a key concept for a number of applications. Especially, in knowledge representation, distances between interpretations (or between formulas) is a central notion on which many belief change operators (belief revision operators, belief merging operators, etc.) are anchored. Such operators are governed by a principle of minimal change, which consists in selecting the most plausible models of a given constraint (the new piece of information in case of belief revision, or the integrity constraints in case of belief merging), given the current beliefs of the agent(s).

In some applications, a plausibility relation can be easily obtained from the input, so that it can be used to rule the change operator. However, in many cases, no such plausibility relation is directly available. In such cases, it makes sense to derive a plausibility relation from a preset distance. Thus, for instance, in (finite) classical propositional logic, the Hamming distance (also called Dalal distance [10, 17]), that is defined as the number of propositional variables two valuations differ on, is often considered. When one has no particular information on the application and on the logical dependencies of the propositional variables, it is a reasonable assumption to consider that the more variable in common in two interpretations, the closer they are. Accordingly, in the classical propositional setting, many revision operators [10, 17, 20, 21], update operators [12, 16], merging operators [19, 18] and other change operators are actually distance-based ones.

Belief change in classical propositional logic has received much attention so far. However, in numerous applications, agents have not only beliefs about the world, but also beliefs about the beliefs of other agents, which makes classical propositional logic inadequate. The typical semantic for multi-agent epistemic (actually, doxastic) frameworks relies on $KD45_n$ Kripke models. On the other hand, though several approaches in epistemic logic settings aim at modeling revision as a dynamic modality (see e.g., [22, 25, 5, 8, 24, 6]), there are quite few works which tackled the problem of defining belief change for epistemic logics in the standard AGM framework (see mainly [3, 9]).

As defining concrete revision operators for Kripke models is nowadays expected (see [14]), our aim in this work is to define such revision operators for finite $KD45_n$ Kripke models. To do so, we first investigate the notion of distance between such models. This turns out to be a key step towards the definition of belief change operators complying the standard AGM framework, and suited to multi-agent scenarios.

As far as we know, only one distance has been pointed out so far for measuring the extent to which Kripke models are different. This distance was reported in [2] and concerned the revision of subjective epistemic models. Subjective epistemic models represent the beliefs of one agent about the world and about the beliefs of the other agents, whereas $KD45_n$ Kripke models represent the beliefs of an external observer about the world and about the beliefs of the agents. To be more precise, Aucher [2] presents a similarity degree between subjective epistemic models, that can be straightforwardly translated into a distance between $KD45_n$ Kripke models.

In the following, we point out distances between $KD45_n$ Kripke models which are alternatives to this one. Such distances can also be easily adapted to Aucher's subjective models, and therefore be used to define new revision operators in this setting as well [3]. Five new distances between $KD45_n$ Kripke models are investigated. Three of them are based on a weakening of the standard bisimulation relation between Kripke models. The other two rely on an aggregation of the propositional distances between the set of valuations for different modal depths in the two models.

Beyond standard distance properties (indistinguishability, symmetry, subadditivity and nonnegativity), three additional properties, that are sensible for distances between $KD45_n$ Kripke models, are introduced. In a nutshell, the first one expresses the fact that the higher the modal depth of discordance (i.e., the higher the modal degree of the formulas that are not satisfied in both models), the lower is the distance between the two models. The second property expresses that all discordances at a given modal depth should not be considered as equivalent. This means that one has to go beyond the drastic dichotomous distance (same/different) between (the valuations of) the worlds. The third property refines the second one by asking the distance between Kripke models to be based on a non-drastic distance between the valuations of the worlds. When considering the application of these distances for belief revision, we introduce a last property, called boundedness property, that ensures that there are only finitely many models to consider for computing the revision.

For each distance introduced, the properties of interest it satisfies are identified. We show that three of them satisfy all the properties under consideration and can be used as such for characterizing belief

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revision operators based on the standard AGM framework, yet suited to KD45$_n$ Kripke models.

2 PRELIMINARIES

We are interested here in modeling the beliefs of several agents, each of them having her own beliefs about the state of the world. Hence we use a multi-agent epistemic logic. Let $\mathbb{P}$ be a finite, non-empty set of propositional variables and $\mathbb{A}$ a finite, non-empty set of agents. We consider the language $\mathcal{L}$ containing the classical propositional language augmented by a belief modal operator B_a for each agent $a \in \mathbb{A}$. Formally, $\mathcal{L}$ is defined as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathrm{B}_a\varphi$$

A formula of the form $\mathrm{B}_a\varphi$ is read "agent a believes that φ is true". The modal degree $\deg(\varphi)$ is defined as usual [7]:

$$\deg(p) = 0 \qquad \deg(\varphi \wedge \psi) = \max(\deg(\varphi), \deg(\psi))$$
$$\deg(\neg\varphi) = \deg(\varphi) \qquad \deg(\mathrm{B}_a\varphi) = 1 + \deg(\varphi)$$

In order to give meaning to our formulas, and especially to operators B_a, we use the standard KD45$_n$ system for n agents [11]. Such a system consists of the set of formulas in $\mathcal{L}$ that can be derived using the following axioms and inference rules:

(TAU) All instantiations of propositional tautologies
(K) $(\mathrm{B}_i\varphi \wedge \mathrm{B}_i(\varphi \Rightarrow \psi)) \Rightarrow \mathrm{B}_i\psi$ (Belief Distribution)
(D) $\neg\mathrm{B}_i\bot$ (Belief Consistency)
(4) $\mathrm{B}_i\varphi \Rightarrow \mathrm{B}_i\mathrm{B}_i\varphi$ (Positive Introspection)
(5) $\neg\mathrm{B}_i\varphi \Rightarrow \mathrm{B}_i\neg\mathrm{B}_i\varphi$ (Negative Introspection)
(RM) From $\models \varphi \Rightarrow \psi$ and $\models \varphi$ infer $\models \psi$ (Modus Ponens)
(RN) From $\models \varphi$ infer $\models \mathrm{B}_i\varphi$ (Belief Generalization)

The same set of validities can be captured using a semantic approach. The most common representation in this system is based on Kripke models, defined as follows:

Definition 1 (**Finite Pointed Kripke Model**). *A* finite pointed Kripke model *is a tuple $\langle W, R, V, w\rangle$ where W is a finite, non-empty set of possible worlds, $R = \{R_a \mid a \in \mathbb{A}\}$, where $R_a \subseteq W \times W$ is the binary accessibility relation on W for agent a, $V = \{V_v \mid v \in W\}$, where $V_v : \mathbb{P} \to \{0, 1\}$ is a valuation function that defines the truth value of each propositional variable at the world v, and $w \in W$ is the pointed world.*

We sometimes use $R_a(w)$ to denote the set of possible worlds which are accessible from w for agent a, namely, $R_a(w) = \{w' \mid (w, w') \in R_a\}$.

Let M be a finite pointed Kripke model. We denote by $M \models \varphi$ the fact that the formula φ is satisfied in M. This is defined as usual for the propositional connectives, and as follows for the operators B_a: $\langle W, R, V, w\rangle \models \mathrm{B}_a\varphi$ if and only if $\forall w' \in W$ if $w' \in R_a(w)$ then $\langle W, R, V, w'\rangle \models \varphi$

Two finite pointed Kripke models are equivalent if and only if they are bisimilar, in the following sense:

Definition 2 (**Bisimilarity**). *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two finite pointed Kripke models. M and M' are* bisimilar*, noted $M \leftrightarrows M'$, if and only if there is a bisimulation $Z \subseteq W \times W'$.*

Definition 3 (**Bisimulation**). *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two finite pointed Kripke models. Let $Z \subseteq W \times W'$. Z is a* bisimulation *if and only if $(w, w') \in Z$ and for all $(v, v') \in Z$:*

1. *$V_v = V'_{v'}$, and*
2. *if $\exists u \in W$ such that $(v, u) \in R_a$, then $\exists u' \in W'$ such that $(v', u') \in R'_a$ and $(u, u') \in Z$, and*
3. *if $\exists u' \in W'$ such that $(v', u') \in R'_a$, then $\exists u \in W$ such that $(v, u) \in R_a$ and $(u, u') \in Z$.*

Let $\mathcal{K}$ be the set of KD45$_n$ finite pointed Kripke models. In what follows, we refer to Kripke models as a short for models of $\mathcal{K}$.

A formula $\varphi \in \mathcal{L}$ is valid (noted $\models \varphi$) if and only if $M \models \varphi$, for every finite pointed Kripke model $M \in \mathcal{K}$.

Two bisimilar models may have different number of worlds. This means that, depending on how distances between two bisimilar models are computed, one may end up with a non-null distance. However, we need to look at the very information conveyed by each model, and not to be distracted by a particular representation. So we need to use some kind of normal form. We take the corresponding minimal models, defined in the sequel, as "normal forms".

With each KD45$_n$ Kripke model, a minimal model which corresponds to its bisimulation contraction can be associated [7]:

Definition 4 (**Minimal Finite Pointed Kripke Model**). *Let $M = \langle W, R, V, w\rangle$ be a finite pointed Kripke model. M is a* minimal finite pointed Kripke model *if and only if there is no model $M' = \langle W', R', V', w'\rangle$ such that $M \leftrightarrows M'$ and $|W| > |W'|$.*

Finite pointed Kripke models are similar to nondeterministic automata. But the latter can be transformed into deterministic ones easily. The resulting model is sometimes exponentially larger, though. Given a finite pointed Kripke model M, the problem of finding a minimal model associated with it is similar to the problem of minimizing the number of states in a deterministic finite automaton. An algorithm for it can be easily adapted from the one given in [15]. We note that, as in the deterministic finite automata case, the minimal model is unique. We denote by $\mu(M)$ the minimal finite pointed Kripke model corresponding to M. We clearly have $M \leftrightarrows \mu(M)$.

The height of a possible world v in a finite pointed Kripke model M, noted $\mathrm{height_M}(v)$, is the length of a shortest path between the pointed world of M and v. The height of a model M (noted $\mathrm{height}(M)$) is, as usual [7], the largest n such that there is a world of height n in M.

3 DISTANCES BETWEEN FINITE KRIPKE MODELS

We start with the notion of distance:

Definition 5 (**Distance**). *A* distance *between two Kripke models is a mapping d from $\mathcal{K}^2$ to $\mathbb{R}$ which satisfies the following properties:*

(D1) $d(M, M') = 0$ *iff* $M \leftrightarrows M'$ ***(indistinguishability)***
(D2) $d(M, M') = d(M', M)$ ***(symmetry)***
(D3) $d(M, M'') \leq d(M, M') + d(M', M'')$ ***(subadditivity)***
(D4) $d(M, M') \geq 0$ ***(nonnegativity)***

The following properties taken from [1] are consequences of properties **(D1) – (D4)**:

Lemma 1. *Let d be a mapping from $\mathcal{K}^2$ to $\mathbb{R}$. If d satisfies the properties **(D1) – (D4)**, then d satisfies:*

(DK1) If $M = M'$ then $d(M, M') = 0$
(DK2) If $M \leftrightarrows M'$ then $d(M, M') = 0$
(DK3) If $M' \leftrightarrows M''$ then $d(M, M') = d(M, M'')$
(DK4) If $M' \leftrightarrows M''$ then $d(M', M) = d(M'', M)$

Proof. Let $M, M', M'' \in \mathcal{K}$ and d be a distance between Kripke models (d satisfies **(D1) – (D4)**).

Let us show that d satisfies **(DK1)**. Suppose that $M = M'$, so we have $M \leftrightarrows M'$. **(D1)** allows us to conclude that $d(M, M') = 0$.

Let us show that d satisfies **(DK2)**. The fact that d satisfies **(D1)** allows us to conclude directly.

Let us show that d satisfies **(DK3)**. Suppose that $M' \leftrightarrows M''$. **(D3)** gives us:

$$d(M, M') \leq d(M, M'') + d(M'', M') \quad (1)$$
$$d(M, M'') \leq d(M, M') + d(M', M'') \quad (2)$$

Moreover, **(D2)** and **(D1)** give us:

$$d(M', M'') = d(M'', M') = 0 \quad (3)$$

(1) and (3) lead us to:

$$d(M, M') \leq d(M, M'') \quad \text{(I)}$$

Similarly, (2) and (3) lead us to:

$$d(M, M'') \leq d(M, M') \quad \text{(II)}$$

(I) and (II) allow us to conclude that $d(M, M') = d(M, M'')$.

Let us show that d satisfies **(DK4)**. The fact that d satisfies **(D2)** and **(DK3)** allows us to conclude directly. □

Lemma 2. *Let* d *be a mapping from* $\mathcal{K}^2$ *to* $\mathbb{R}$. *If* d *satisfies* ***(D1)-(D4)****, then* d *cannot satisfy:*

(DK5) $d(M, M'') \geq d(M, M') + d(M', M'')$
(DK6) $d(M, M'') = d(M, M') + d(M', M'')$

Proof. Let d be a distance (d satisfies **(D1)-(D4)**). Let us show that d does not satisfy **(DK5)**.
Suppose that **(DK5)** is satisfied.
Let M, M', M'' three models not pairwise bisimilar.
So, by **(D1)**, we have $d(M, M') \neq 0$, $d(M, M'') \neq 0$ and $d(M', M'') \neq 0$.
(DK5) and **(D3)** give us, $d(M, M'') = d(M, M') + d(M', M'')$, $d(M, M') = d(M, M'') + d(M'', M')$ and $d(M', M'') = d(M', M) + d(M, M'')$.
So we have $d(M, M'') = d(M, M'') + d(M'', M') + d(M', M) + d(M, M'')$ which leads to $d(M'', M') + d(M', M) + d(M, M'') = 0$. From **(D4)**, we should have $d(M'', M') = d(M', M) = d(M, M'') = 0$, contradiction.

Let us show that d does not satisfy **(DK6)**.
Suppose that **(DK6)** is satisfied. So d satisfies both **(D3)** and **(DK5)**. This contradicts the fact that **(DK5)** is not satisfied. □

To define distances on KD45$_n$ Kripke models we consider some additional expected properties. First, we must introduce a modification function that is used to change the valuation of a world w' in a model M to match the valuation ϑ.

Definition 6 (Modification Function). *Let* $M = \langle W, R, V, w \rangle$, $w' \in W$, *and* ϑ *a valuation. We denote by* $M(\vartheta \rightarrow w')$ *the model obtained by changing the valuation of* w' *by* ϑ, *defined as follows:*

$$M(\vartheta \rightarrow w') = \langle W, V', R, w \rangle \text{ where}$$

$$V' = \{V_v | v \neq w'\} \cup \{V_{w'} | \forall p \in \mathbb{P}, V_{w'}(p) = \vartheta(p)\}$$

We can now define the additional properties:

(D5) $\forall M = \langle W, V, R, w \rangle$, $\forall w', w'' \in W$, $\forall \vartheta, \vartheta'$, if $\text{height}_M(w') < \text{height}_M(w'')$ and $M' = M(\vartheta \rightarrow w')$ and $M'' = M(\vartheta \rightarrow w'')$ with $V_{w'} \neq \vartheta \neq V_{w''}$, then $d(M, M') > d(M, M'')$.
(D6) $\exists M = \langle W, V, R, w \rangle$, $\exists w' \in W$, $\exists \vartheta, \vartheta'$ such that $M' = M(\vartheta \rightarrow w')$ and $M'' = M(\vartheta' \rightarrow w')$ with $\vartheta \neq V_{w'} \neq \vartheta'$, and $d(M, M') \neq d(M, M'')$
(D7) There is a non-drastic propositional distance [2] d_V such that $\forall M = \langle W, V, R, w \rangle$, $\forall w' \in W$, $\forall \vartheta, \vartheta'$, if $M' = M(\vartheta \rightarrow w')$ and $M'' = M(\vartheta' \rightarrow w')$ and $d_V(\vartheta, V_{w'}) < d_V(\vartheta', V_{w'})$ then $d(M, M') < d(M, M'')$.

(D5) expresses the fact that the higher the modal depth of discordance (i.e., the higher the modal degree of the formulas that are not satisfied in both models), the lower is the distance between the two models. Basically, this property has to be evaluated by considering the use of epistemic models for making strategic decisions. As a matter of illustration, consider a card game, or any imperfect information game (like Cluedo, for instance). Then it is more harmful for a player A to make a mistake about the beliefs of another player B (since this piece of beliefs is used for making many strategic decisions), rather than to be mistaken with the beliefs of B about the beliefs of A about the beliefs of B.

(D6) expresses that not all discordances at modal degree k are equivalent, which means that one has to do better than the drastic dichotomous distance (same/different) between (the two valuations of) two worlds. This distance, noted D, is define by: for all ϑ, ϑ' valuations over $\mathbb{P}$, we have $D(\vartheta, \vartheta') = 0$ if $\vartheta = \vartheta'$ and $D(\vartheta, \vartheta') = 1$ otherwise.

(D7) stipulates that the distance between two models must be based on a non-drastic propositional distance between valuations. Clearly, **(D7)** is more demanding that **(D6)**:

Proposition 3. *Let* d *be a distance between Kripke models. If* d *satisfies* ***(D7)****, then* d *satisfies* ***(D6)****.*

Proof. Let d be a distance between Kripke models. Let $M = \langle W, V, R, w \rangle$, M' and M'' be three Kripke models, $w' \in W$ such that ϑ_1, ϑ_2 two valuations distinct from $V_{w'}$, $M' = M(\vartheta_1 \rightarrow w')$ and $M'' = M(\vartheta_2 \rightarrow w')$.
Assume that d satisfies **(D7)**. Let d_V be the underlying propositional distance. Therefore, we have that, if $d_V(\vartheta_1, V_{w'}) < d_V(\vartheta_2, V_{w'})$, then $d(M, M') < d(M, M'')$. □

4 PREVIOUS DISTANCES BETWEEN KRIPKE MODELS

In [1], some measures between Kripke models have been pointed out. Those measures have not been primarily defined for KD45$_n$ models but can be adapted for this purpose, as follows.

Definition 7 (Kripke Distance [1]). *Let* $M = \langle W, R, V, w_0 \rangle$ *and* $M' = \langle W, R', V, w_0 \rangle$ *be two Kripke models.* $\delta_{min}(M, M') = \delta_{\mathcal{K}}(\mu(M), \mu(M'))$, *with* $\delta_{\mathcal{K}}(M, M') = \sum_{a \in \mathbb{A}} |R_a \setminus R'_a|$.

Apart from not being symmetric, the fact that only the relations of the two models M and M' can be different for the distance to be defined is too restrictive for our purpose. Indeed, δ_{min} does not satisfy **(D5)**, because it does not look at the depth of the worlds causing

[2] A drastic propositional distance d is a distance between valuations such that $\exists \alpha \in \mathbb{N}^*$ for which $d(\vartheta, \vartheta') = 0$ if $\vartheta = \vartheta'$ and $d(\vartheta, \vartheta') = \alpha$ otherwise (the usual drastic distance d_D is recovered for $\alpha = 1$).

the discordance between the models. And, as it does not check the valuations of the worlds, it does not satisfy **(D6)** nor **(D7)**.

In [2], a notion of similarity degree between Kripke models, based on the notion of $n-$bisimulation, is proposed. This similarity degree can be directly transformed into a distance. The notion of $n-$bisimulation we use is slightly different from the one from [7, 2, 4]. We do not impose any condition for the 0-bisimulation, thus allowing any model to be 0-bisimilar to another one, even if their pointed worlds are different. Intuitively, two models are n-bisimilar (with $n \geq 1$), noted $M \leftrightarrows_n M'$, if they are equivalent until a height of $n-1$. Consequently, two n-bisimilar models satisfy the same formulas with a modal degree at most $n-1$.

Definition 8 (**n-Bisimilarity).** *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two pointed Kripke models. M and M' are n-*bisimilar*, noted $M \leftrightarrows_n M'$, if and only if there is a n-bisimulation $Z \subseteq W \times W'$.*

Definition 9 (**n-Bisimulation).** *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two pointed Kripke models. Let $Z \subseteq W \times W'$:*

- *Z is a* 0-bisimulation.
- *Z is a* 1-bisimulation *if and only if $(w, w') \in Z$ and $V_w = V'_{w'}$.*
- *Z is a $(n+1)$*-bisimulation *($n \geq 1$) if and only if $(w, w') \in Z$ and for each $(v, v') \in Z$:*
  1. *$V_v = V'_{v'}$ and*
  2. *if $\exists u \in W$ such that $(v, u) \in R_a$, then $\exists u' \in W'$ such that $(v', u') \in R'_a$ and $(u, u') \in Z$, and*
  3. *if $\exists u' \in W'$ such that $(v', u') \in R'_a$, then $\exists u \in W$ such that $(v, u) \in R_a$ and $(u, u') \in Z$,*

  and Z is a n-bisimulation.

The next lemma follows immediately from Definitions 2 and 8.

Lemma 4. *$(\forall n \in \mathbb{N}, M \leftrightarrows_n M')$ if and only if $M \leftrightarrows M'$.*

Let us now recall the similarity degree proposed in [2]:

Definition 10 (**Similarity Degree [2]).** *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models, $v \in W$ and $v' \in W'$, S and S' two finite sets of possible worlds. Let $n = |W| \cdot |W'| + 1$ and $k \in \mathbb{N}$. We define the similarity degree $s^k(M, M')$ between M and M' by:*

- $\sigma(v, v') = \max(\frac{i}{n} | \langle W, R, V, v\rangle \leftrightarrows_i \langle W', R', V', v'\rangle$ *and* $i \in [\![0; n]\!])$
- $\sigma(S, S') = \frac{1}{2}(avg\{\sigma(s, S') | s \in S\} + avg\{\sigma(S, s') | s' \in S'\})$ *where* $\sigma(s, S') = \max\{\sigma(s, s') | s' \in S'\}$ *and* $\sigma(S, s') = \max\{\sigma(s, s') | s \in S\}$
- $s^k(M, M') = (\sigma(w, w'), avg\{\sigma(R_a(w), R_a(w')) | a \in \mathbb{A}\}, \ldots, avg\{\sigma(R_{a_1} \circ \cdots \circ R_{a_k}(w), R_{a_1} \circ \cdots \circ R_{a_k}(w')) | \forall i, a_i \in \mathbb{A}$ *and* $a_i \neq a_{i+1}\})$.[3]

$\sigma(v, v')$ measures a degree of similarity between the worlds v and v'. Likewise, $\sigma(S, S')$ measures a degree of similarity between the sets of worlds S and S'. To be more precise, $\sigma(v, S')$ is the degree of similarity of a world v with S'. The degree of similarity between S and S' is just the average of such degrees. $s^k(M, M')$ is a tuple which represents how much two Kripke models are similar relatively to their respective modal depth. See [2] for more details and justifications on this similarity degree.

[3] avg denotes here the average mapping and $\circ$ is used here for denoting relation composition.

Based on this similarity degree, we can define a distance between Kripke models. We add the distances between $\mu(M)$ and $\mu(M')$ relative to their modal depth, by recovering the element of the tuple $s^k(\mu(M), \mu(M'))$ corresponding to each depth. The distance between these two models at a depth $p \leq k$ is given by the difference between 1 (the maximal degree) and the (p+1)th element of $s^k(\mu(M), \mu(M'))$.

Definition 11 (**Similarity Distance).** *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models. Let $n = |W| \cdot |W'| + 1$.*

$$d\mathcal{A}(M, M') = \sum_{i=0}^{n} (1 - s_i^n(\mu(M), \mu(M')))$$

where $s_i^n(\mu(M), \mu(M'))$ is the (i+1)th element of the tuple $s^n(\mu(M), \mu(M'))$.

The problem with those distances is that none of them satisfies the expected properties introduced in the previous section:

Proposition 5. *δ_{min} and $d\mathcal{A}$ do not satisfy any of **(D5)**, **(D6)** or **(D7)**.*

Proof. Let $M = \langle W, R, V, w\rangle$ be a Kripke model.

- We show that $d\mathcal{A}$ does not satisfy **(D5)**, **(D6)** nor **(D7)**.
  - First we show that $d\mathcal{A}$ does not satisfy **(D5)**.
    Consider the following example, $\mathbb{P} = \{x, y\}$, $W = \{w, v\}$, $R = \{R_1, R_2\}$ with $R_1 = \{(w, v), (v, v)\}$ and $R_2 = \{(w, v), (v, w)\}$, $V_w(x) = 0$, $V_w(y) = 1$ and $V_v(x) = 1$, $V_v(y) = 0$.
    Let ϑ be a valuation such that $\vartheta(x) = 1$ and $\vartheta(y) = 0$. Let $M_1 = M(\vartheta \to w)$ and $M_2 = M(\vartheta \to v)$. Let $k = |W|^2 + 1 = 5$. Thus, we have $s^5(\mu(M), \mu(M_1)) = (0, 1, 1, 1, 1, 1)$ and $s^5(\mu(M), \mu(M_2)) = (\frac{1}{5}, 0, 0, 0, 0, 0)$. Therefore, we have $d\mathcal{A}(M, M_1) = 1$ and $d\mathcal{A}(M, M_2) = 5.8$. We clearly have here that $\text{height}_\text{M}(w) < \text{height}_\text{M}(v)$ and $d\mathcal{A}(M, M_1) < d\mathcal{A}(M, M_2)$. So $d\mathcal{A}$ does not satisfy **(D5)**.
  - We show that $d\mathcal{A}$ does not satisfy **(D6)**.
    Let M_1 and M_2 be two Kripke models such that $M_1 = M(\vartheta_1 \to v)$ and $M_2 = M(\vartheta_2 \to v)$ with $\vartheta_1 \neq V_v \neq \vartheta_2$ and $\text{height}_\text{M}(v) = p$. We clearly have $\text{height}(M) = \text{height}(M_1) = \text{height}(M_2)$. Let $k = |W|^2 + 1$. We have $s_0^k(\mu(M), \mu(M_1)) = \frac{p}{k} = s_0^k(\mu(M), \mu(M_2))$. Following Definition 10, iterating over s_i^k gives us $s_i^k(\mu(M), \mu(M_1)) = s_i^k(\mu(M), \mu(M_2))$ for all $i \in [\![0; k]\!]$. Thus $d\mathcal{A}(M, M_1) = d\mathcal{A}(M, M_2)$.
  - The fact that $d\mathcal{A}$ does not satisfy **(D6)** allows us to conclude, using Proposition 3, that $d\mathcal{A}$ does not satisfy **(D7)**.
- We show that δ_{min} does not satisfy **(D5)**, **(D6)** nor **(D7)**.
  The fact that the distance δ_{min} only compares the set of relations between the models is sufficient to show that δ_{min} can not satisfy **(D5)**, **(D6)** nor **(D7)**.

□

In the next section we introduce new distances that will be proved to satisfy the expected properties.

5 FAMILIES OF DISTANCES BETWEEN KRIPKE MODELS

In the following, we define several families of distances $\mathcal{D}_m^d$. Each distance of a family $\mathcal{D}_m^d$ is defined for finite Kripke models containing at most m worlds. Whatever the finite set of models under consideration, the existence of an m suited to it is ensured by the fact that all the models it contains are finite. For each family of distances we point out, all distances will be considered between the minimal models associated with the two Kripke models considered at start (note the use of the μ minimisation function in the definitions). This is necessary to ensure that bisimilar models are at a null distance, as expected.

5.1 Bisimulation Distances

Here we exploit the ideas behind bisimulation in order to define distances betweens Kripke models. First, we introduce a useful result, namely that there is a rank k from which a k-bisimulation implies a $(k+1)$-bisimulation since we consider a finite set $\mathbb{P}$ of propositional variables. In a first place, we prove a lemma which states that for two Kripke models $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$, if there is a n-bisimulation ($n > 1$) Z and $(w, w') \in Z$, then the existence of a sequence of $n-1$ worlds w_i such that $wRw_1Rw_2R\cdots Rw_{n-1}$ implies the existence of a sequence of $n-1$ worlds w'_i such that $w'R'w'_1R'w'_2R'\cdots R'w'_{n-1}$, such that Z is a 1-bisimulation and $(w_{n-1}, w'_{n-1}) \in Z$, and conversely.

Lemma 6. *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two finite pointed Kripke models. Let $Z \subseteq W \times W'$, if Z is a n-bisimulation ($n > 1$) and $(w, w') \in Z$ then:*

1. *if $\exists v \in W$ such that $wR_{a_1}\cdots R_{a_{n-1}}v$, then $\exists v' \in W'$ such that $w'R'_{a_1}\cdots R'_{a_{n-1}}v'$ and there exists a 1-bisimulation Z' and $(v, v') \in Z'$.*[4]
2. *if $\exists v' \in W'$ such that $w'R'_{a_1}\cdots R'_{a_{n-1}}v'$, then $\exists v \in W$ such that $wR_{a_1}\cdots R_{a_{n-1}}v$ and there exists a 1-bisimulation Z' and $(v, v') \in Z'$.*

Proof. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two finite pointed Kripke models and $Z \subseteq W \times W'$.
If Z is a 2-bisimulation and $(w, w') \in Z$, then Definition 9 allows us to conclude.
Suppose that $\forall n > 1$, if Z is a n-bisimulation and $(w, w') \in Z$, then:

(1.1) if $\exists v \in W$ such that $w\underbrace{R_{a_i}\cdots R_{a_j}}_{(n-1)\times}v$, then $\exists v' \in W'$ such that $w'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{(n-1)\times}v'$ and there exists a 1-bisimulation Z' and $(v, v') \in Z'$.

(1.2) if $\exists v' \in W'$ such that $w'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{(n-1)\times}v'$, then $\exists v \in W$ such that $w\underbrace{R_{a_i}\cdots R_{a_j}}_{(n-1)\times}v$ and there exists a 1-bisimulation Z' and $(v, v') \in Z'$.

If Z is a $(n+1)$-bisimulation and $(w, w') \in Z$, then from definition 9, we have:

(2.1) if $\exists v \in W$ such that $(w, v) \in R_a$, then $\exists v' \in W'$ such that $(w', v') \in R'_a$ and Z is a n-bisimulation and $(v, v') \in Z$.

(2.2) if $\exists v' \in W'$ such that $(w', v') \in R'_a$, then $\exists u \in W$ such that $(w, v) \in R_a$ and Z is a n-bisimulation and $(v, v') \in Z$.

(1.1) and (2.1) give us: if $\exists v \in W$ such that wRv, then $\exists v' \in W'$ such that $w'R'v'$ and:

- if $\exists u \in W$ such that $v\underbrace{R_{a_i}\cdots R_{a_j}}_{(n-1)\times}u$, then $\exists u' \in W'$ such that $v'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{(n-1)\times}u'$ and there exists a 1-bisimulation Z' and $(u, u') \in Z'$.
- if $\exists u' \in W'$ such that $v'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{(n-1)\times}u'$, then $\exists u \in W$ such that $v\underbrace{R_{a_i}\cdots R_{a_j}}_{(n-1)\times}u$ and there exists a 1-bisimulation Z' and $(u, u') \in Z'$.

(1.2) and (2.2) give us a similar result.

So we have, if Z is a $(n+1)$-bisimulation and $(w, w') \in Z$, then:

- if $\exists v \in W$ such that $w\underbrace{R_{a_i}\cdots R_{a_j}}_{n\times}v$, then $\exists v' \in W'$ such that $w'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{n\times}v'$ and there exists a 1-bisimulation Z' and $(v, v') \in Z'$.
- if $\exists v' \in W'$ such that $w'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{n\times}v'$, then $\exists v \in W$ such that $w\underbrace{R_{a_i}\cdots R_{a_j}}_{n\times}v$ and there exists a 1-bisimulation Z' and $(v, v') \in Z'$.

□

The following proposition, based on Lemma 6, shows that for a given rank k, if two models are k-bisimilar, then they are $(k+1)$-bisimilar. k is taken here as the size of the largest set of worlds of the two models, plus one. This result reinforces a similar result due to Balbiani [2].

Proposition 7. *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models containing at most m worlds. If $M \leftrightarroweq_{m+1} M'$, then $M \leftrightarroweq M'$.*

Proof. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models containing at most m worlds such that $\text{height}(M) = n$ and $\text{height}(M') = n - p$ with $0 \leq p \leq n$.
We have $m \geq n + 1$. Let $k = n + 2$ and $Z \subseteq W \times W'$ such that Z is a k-bisimulation and (w, w').
So we have $M \leftrightarroweq_k M'$, thus, from Lemma 6, we also have:

(1) if $\exists v \in W$ such that $w\underbrace{R_{a_i}\cdots R_{a_j}}_{(n+1)\times}v$, then $\exists v' \in W'$ such that $w'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{(n+1)\times}v'$ and Z is a 1-bisimulation and $(v, v') \in Z$.

(2) if $\exists v' \in W'$ such that $w'\underbrace{R'_{a_i}\cdots R'_{a_j}}_{(n+1)\times}v'$, then $\exists v \in W$ such that $w\underbrace{R_{a_i}\cdots R_{a_j}}_{(n+1)\times}v$ and Z is a 1-bisimulation and $(v, v') \in Z$.

---

[4] We use the notation $wR_{a_1}\cdots R_{a_{n-1}}v$ to abbreviate $wR_{a_1}w_1R_{a_2}w_2....w_{n-2}R_{a_{n-1}}v$.

Let us look closely at these two cases:

(1) if such a v exists, as $\text{height}(M) = n$, $\exists \alpha < n+1$ such that $w\underbrace{R_{a_i} \cdots R_{a_j}}_{\alpha\times} v \underbrace{R_{a_{i'}} \cdots R_{a_{j'}}}_{(n-\alpha+1)\times} v$, then there is a v', such that $w'\underbrace{R'_{a_i} \cdots R'_{a_j}}_{\alpha\times} v' \underbrace{R'_{a_{i'}} \cdots R'_{a_{j'}}}_{(n-\alpha+1)\times} v'$ and Z is a α-bisimulation and $(v, v') \in Z$ with $\alpha \geq 1$.
(2) by a similar reasoning, the same conclusion can be drawn.

Thus, Z is (at least) a $(k+1)$-bisimulation and $(w, w') \in Z$. Using a simple induction, we show that $\forall i \geq k$, $M \leftrightarrows_i M'$. Lemma 4 allows us to conclude that $M \leftrightarrows M'$.
The converse is trivially true. □

We now use the notion of n-bisimulation to define a family of distances $\mathcal{D}_m^{d\mathcal{NB}}$. To do so, we look at how deep the two Kripke models under consideration are bisimilar, then we subtract that value to the maximal possible value.

Definition 12 (n-Bisimulation-based Distance). *Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models containing at most m worlds. We denote by $d\mathcal{NB}(M, M')$ the distance between M and M', defined as follows:*

$$d\mathcal{NB}(M, M') = (m+1) - \max(i \mid \mu(M) \leftrightarrows_i \mu(M'),\, i \in [\![0; m+1]\!])$$

An illustration of this distance (and the other distances introduced in the paper) can be found in the forthcoming Example 1.

It is easy to check that $d\mathcal{NB}$ satisfies **(D5)**. Indeed, the purpose of this distance is to look at how deep the two models are bisimilar. Thus, when the modal depth of difference increases, the distance between the models decreases. But since we do not consider valuation of the worlds, $d\mathcal{NB}$ does not satisfy **(D6)** nor **(D7)**.

Proposition 8.

1. *$d\mathcal{NB}$ satisfies **(D1)-(D4)**.*
2. *$d\mathcal{NB}$ satisfies **(D5)**.*
3. *$d\mathcal{NB}$ satisfies neither **(D6)** nor **(D7)**.*

Proof. Let $M = \langle W, R, V, w\rangle$, $M' = \langle W', R', V', w'\rangle$ and $M'' = \langle W'', R'', V'', w''\rangle$ be three Kripke models.

Let us show that $d\mathcal{NB}$ satisfies **(D1)**.
Suppose that $d\mathcal{NB}(M, M') = 0$. From the definition of $d\mathcal{NB}$, $d\mathcal{NB}(M, M') = 0$ if and only if $\max(i \mid M \leftrightarrows_i M'$ and $i \in [\![0; m+1]\!]) = m+1$. The fact that for each model M of $\mathcal{M}$ we have $\text{height}(M) < m$ allows us, using Proposition 7, to conclude that $d\mathcal{NB}$ satisfies **(D1)**.

Let us show that $d\mathcal{NB}$ satisfies **(D2)**.
Let $n = m+1$. Let $p_1 = \max(i \mid M \leftrightarrows_i M'$ and $i \in [\![0; n]\!])$ and $p_2 = \max(i \mid M' \leftrightarrows_i M$ and $i \in [\![0; n]\!])\}$, the definition 9 allows us to establish that $p_1 = p_2$. Thus, $d\mathcal{NB}(M, M') = n - p_1 = n - p_2 = d\mathcal{NB}(M', M)$.

Let us show that $d\mathcal{NB}$ satisfies **(D3)**.
Let $n = m+1$. Assume that, there are some $x, y, z \in \mathbb{N}$ such that $M \leftrightarrows_x M'$, $M \not\leftrightarrows_{x+1} M'$, $M \leftrightarrows_y M''$, $M \not\leftrightarrows_{y+1} M''$, $M' \leftrightarrows_z M''$ and $M' \not\leftrightarrows_{z+1} M''$.
We want to show that $n - x \leq n - y + n - z$. To do so, we use an inductive reasoning. *Induction Base:* $0 = n - x \leq n - y + n - z$ for $x = y = z = n$. *Induction Hypothesis:* there are some i, j, k such that $n - x \leq n - y + n - z$ for all $i \leq x \leq n$, $j \leq y \leq n$ and $k \leq z \leq n$. *Induction Steps:* We have several cases to consider:

1. Let $x = i-1$, $y = j-1$ and $z = k-1$.
$$n - i \leq n - j + n - k$$
$$n - i + 1 \leq n - j + n - k + 2$$
$$n - (i-1) \leq n - (j-1) + n - (k-1)$$
$$n - x \leq n - y + n - z$$
2. Let $x = i-1$, $y = j-1$ and $z = k$.
$$n - i \leq n - j + n - k$$
$$n - i + 1 \leq n - j + n - k + 1$$
$$n - (i-1) \leq n - (j-1) + n - k$$
$$n - x \leq n - y + n - z$$
3. Let $x = i-1$, $y = j$ and $z = k-1$.
$$n - i \leq n - j + n - k$$
$$n - i + 1 \leq n - j + n - k + 1$$
$$n - (i-1) \leq n - j + n - (k-1)$$
$$n - x \leq n - y + n - z$$
4. Let $x = i$, $y = j-1$ and $z = k-1$.
$$n - i \leq n - j + n - k$$
$$n - i \leq n - j + n - k + 2$$
$$n - i \leq n - (j-1) + n - (k-1)$$
$$n - x \leq n - y + n - z$$
5. Let $x = i$, $y = j-1$ and $z = k$.
$$n - i \leq n - j + n - k$$
$$n - i \leq n - j + n - k + 1$$
$$n - i \leq n - (j-1) + n - k$$
$$n - x \leq n - y + n - z$$
6. Let $x = i$, $y = j$ and $z = k-1$.
$$n - i \leq n - j + n - k$$
$$n - i \leq n - j + n - k + 1$$
$$n - i \leq n - j + n - (k-1)$$
$$n - x \leq n - y + n - z$$
7. Let $x = i-1$, $y = j$ and $z = k$.
   (a) Assume that $i > k$ (and thus, $i - 1 \geq k$). Hence, we have $n - (i-1) \leq n - k$. From this and $n - j \geq 0$ we can deduce that $n - (i-1) \leq n - j + n - k$.
   (b) Assume that $i \leq k$. We have $M \leftrightarrows_{i-1} M'$, but $M \not\leftrightarrows_i M'$. Then $M \leftrightarrows_{i-1} M''$ (because $i \leq k$ and $M' \leftrightarrows_k M''$). We also have $M \leftrightarrows_j M''$ but $M \leftrightarrows_{j+1} M''$. Assume $j \geq i$, we have $M \ / \leftrightarrows_i M''$ (because $M \not\leftrightarrows_i M'$ and $M' \leftrightarrows_i M''$). Thus $M \not\leftrightarrows_j M''$, we have a contradiction. Therefore, we have $j \leq i - 1$. So, from $n - j \geq n - (i-1)$ and $n - k \geq 0$ we can deduce that $n - (i-1) \leq n - j + n - k$.

   Hence, $n - x \leq n - y + n - z$.

By definition, $d\mathcal{NB}$ satisfies **(D4)**.

Let us show that $d\mathcal{NB}$ satisfies **(D5)**.
Let ϑ be a valuation, $n = m+1$ and $M_1 = M(\vartheta \to v)$ and $M_2 = M(\vartheta \to u)$ with $\text{height}_\text{M}(v) < \text{height}_\text{M}(u)$ and $V_v \neq \vartheta \neq V_u$. We clearly have here $\text{height}(M) = \text{height}(M_1) = \text{height}(M_2)$. Let $k = max(i|\mu(M) \leftrightarroweq_i \mu(M_1)$ and $i \in [\![0;n]\!])$ and $k' = max(i|\mu(M) \leftrightarroweq_i \mu(M_2)$ and $i \in [\![0;n]\!])$. From this, we have $k < k'$. Thus we have $\frac{1}{n}(n-k) > \frac{1}{n}(n-k')$ We can conclude that $d\mathcal{NB}(M, M_1) > d\mathcal{NB}(M, M_2)$.

Let us show that $d\mathcal{NB}$ does not satisfy **(D6)**.
Let $n = m+1$, $M_1 = M(\vartheta_1 \to v)$ and $M_2 = M(\vartheta_2 \to v)$ with $\vartheta_1 \neq V_v \neq \vartheta_2$. We clearly have here $\text{height}(M) = \text{height}(M_1) = \text{height}(M_2)$. Let $k = max(i|\mu(M) \leftrightarroweq_i \mu(M_1)$ and $i \in [\![0;n]\!])$ and $k' = max(i|\mu(M) \leftrightarroweq_i \mu(M_2)$ and $i \in [\![0;n]\!])$. From this, we have $k = k'$. Following the same reasonning that the previous point, we can conclude that $d\mathcal{NB}(M, M_1) = d\mathcal{NB}(M, M_2)$.

The fact that $d\mathcal{NB}$ does not satisfy **(D6)** allows us to conclude, using Proposition 3, that it does not satisfy **(D7)**.

□

The next distance is based on an approximation of the notion of bisimulation in which the valuations of the worlds may differ. Thus, two models quite close to each other are considered as ε-bisimilar. In this case, we first use a *propositional distance* d between valuations from $2^{|\mathbb{P}|} \times 2^{|\mathbb{P}|}$ to $\mathbb{N}$, supposed to satisfy the usual distance properties (indistinguishability, symmetry, subadditivity and nonnegativity) [23].

Definition 13 (**$d\varepsilon$-Bisimilarity).** *Let d be a propositional distance. Let $\varepsilon \in \mathbb{N}$. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models. M and M' are $d\varepsilon$-*bisimilar*, noted $M \leftrightarroweq^{d,\varepsilon} M'$, if and only if there is a $d\varepsilon$-bisimulation $Z \subseteq W \times W'$.*

Definition 14 (**$d\varepsilon$-Bisimulation).** *Let d be a propositional distance. Let $\varepsilon \in \mathbb{N}$. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models. Let $Z \subseteq W \times W'$. Z is a $d\varepsilon$-*bisimulation *if and only if $(w, w') \in Z$ and for all $(v, v') \in Z$:*

1. *$d(V_v, V_{v'}) \leq \varepsilon$ and*
2. *if $\exists u \in W$ such that $(v, u) \in R_a$, then $\exists u' \in W'$ such that $(v', u') \in R'_a$ and $(u, u') \in Z$, and*
3. *if $\exists u' \in W'$ such that $(v', u') \in R'_a$, then $\exists u \in W$ such that $(v, u) \in R_a$ and $(u, u') \in Z$.*

Similarly to the n-bisimulation (family of) distance(s), we can use the notion of $d\varepsilon$-bisimulation to establish a family of distances $\mathcal{D}_m^{d\mathcal{EB}_d}$. We seek here for the smallest possible ε so that both models are $d\varepsilon$-bisimilar.

Definition 15 (**$d\varepsilon$-Bisimulation-based Distance).** *Let d be a propositional distance. Let $\varepsilon \in \mathbb{N}$. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ two Kripke models containing at most m worlds. We denote by $d\mathcal{EB}_d(M, M')$ the distance between M and M', defined as follows:*

$$d\mathcal{EB}_d(M, M') = \min\{\varepsilon \mid \mu(M) \leftrightarroweq^{d,\varepsilon} \mu(M')\}.$$

It is clear that the distance $d\mathcal{EB}_d$ does not satisfy **(D5)**. Indeed, here, we seek for an epsilon regardless of the depth of the discordance between the models. But, as we somehow check the valuations of the worlds causing the discordance, if a non-drastic propositional distance d is used, $d\mathcal{EB}_d$ satisfies **(D7)** and thereby **(D6)**.

Proposition 9.

1. *Given any propositional distance d, $d\mathcal{EB}_d$ satisfies **(D1)-(D4)**.*
2. *$d\mathcal{EB}_d$ does not satisfy **(D5)** in general.*
3. *For any non-drastic distance d, $d\mathcal{EB}_d$ satisfies **(D6)** and **(D7)**.*

Proof. Let $M = \langle W, R, V, w\rangle$, $M' = \langle W', R', V', w'\rangle$ and $M'' = \langle W'', R'', V'', w''\rangle$ be three Kripke models.

Let us show that $d\mathcal{EB}_d$ satisfies **(D1)**.
From the definition of $d\mathcal{EB}_d$, we have, $d\mathcal{EB}_d(M, M') = 0$ if and only if $\min(\varepsilon \mid M \leftrightarroweq^{d,\varepsilon} M') = 0$. Hence $M \leftrightarroweq^{d,0} M'$. The definition 14 allows us to conclude that $M \leftrightarroweq M'$.

Let us show that $d\mathcal{EB}_d$ satisfies **(D2)**.
The fact that the ε-bisimulation is symmetric allows us to conclude directly.

Let us show that $d\mathcal{EB}_d$ satisfies **(D3)**.
Assume that $d : V \times V \to \mathbb{N}$, and that there are some $x, y, z \in \mathbb{N}$ such that $M \leftrightarroweq^{d,x} M''$, $M \not\leftrightarroweq^{d,x-1} M''$, $M \leftrightarroweq^{d,y} M'$, $M \not\leftrightarroweq^{d,y-1} M'$, $M' \leftrightarroweq^{d,z} M''$, $M' \not\leftrightarroweq^{d,z-1} M''$.

We want to show that $x \leq y + z$. To do so, we use an induction reasoning. *Induction Base:* $x \leq y + z$ for $x = y = z = 0$. *Induction Hypothesis:* there are some i, j, k such that $x \leq y + z$ for all $0 \leq i \leq x$, $0 \leq j \leq y$ and $0 \leq k \leq z$. *Induction Steps:* We have several cases to consider:

1. Let $x = i$, $y = j$ and $z = k+1$. In this case, we clearly have $x \leq y + z$.
2. Let $x = i$, $y = j+1$ and $z = k+1$. In this case, we clearly have $x \leq y + z$.
3. Let $x = i$, $y = j+1$ and $z = k$. In this case, we clearly have $x \leq y + z$.
4. Let $x = i+1$, $y = j+1$ and $z = k$. In this case, we clearly have $x \leq y + z$.
5. Let $x = i+1$, $y = j+1$ and $z = k+1$. In this case, we clearly have $x \leq y + z$.
6. Let $x = i+1$, $y = j$ and $z = k+1$. In this case, we clearly have $x \leq y + z$.
7. Let $x = i+1$, $y = j$ and $z = k$.
   (a) Assume that $i < j$ (and thus, $i + 1 \leq j$). Hence, we have $i + 1 \leq j + k$.
   (b) Assume that $i \geq j$. Hence we have
   $$M \leftrightarroweq^{d,i+1} M'' \text{ and } M \not\leftrightarroweq^{d,i} M'' \quad (\star)$$
   We prove a Lemma.
   Lemma 10. *Let M, M' and M'' be three Kripke models and d a propositional distance. If $M \leftrightarroweq^{d,\varepsilon_1} M'$ and $M' \leftrightarroweq^{d,\varepsilon_2} M''$, then $M \leftrightarroweq^{d,\varepsilon_1+\varepsilon_2} M''$.*
   We also have $M \leftrightarroweq^{d,j} M'$ and $M' \leftrightarroweq^{d,k} M''$. From Lemma 10, we have
   $$M \leftrightarroweq^{d,j+k} M'' \quad (\star\star)$$
   . From $(\star)$ and $(\star\star)$, we can deduce that $i + 1 \leq j + k$.

   Hence, $x \leq y + z$.

By definition, $d\mathcal{EB}_d$ satisfies **(D4)**.

Let us show that $d\mathcal{EB}_d$ does not satisfy **(D5)**.
Consider the following example, $\mathbb{P} = \{x, y\}$, $W = \{w, v\}$, $R = \{R_1, R_2\}$ with $R_1 = \{(w, v), (v, v)\}$ and $R_2 = \{(w, v), (v, w)\}$, $V_w(x) = 0$, $V_w(y) = 1$ and $V_v(x) = 0$, $V_v(y) = 0$.
Let ϑ be a valuation such that $\vartheta(x) = 1$ and $\vartheta(y) = 0$. Let

$M_1 = M(\vartheta \to w)$ and $M_2 = M(\vartheta \to v)$. We clearly have $V_w \neq \vartheta \neq V_v$ and $\text{height}_{\text{M}}(w) < \text{height}_{\text{M}}(v)$. Let d be the drastic distance, we can see that $d\mathcal{EB}_d(M, M_1) = d(V_w, \vartheta) = 1 = d(V_v, \vartheta) = d\mathcal{EB}_d(M, M_2)$. Let d be a non-drastic distance, we can easily adapt ϑ such that $d(V_w, \vartheta) \geq d(V_v, \vartheta)$ and conclude that $d\mathcal{EB}_d(M, M_1) \geq d\mathcal{EB}_d(M, M_2)$. Which contradicts **(D5)**.

Let us show that $d\mathcal{EB}_d$ satisfies **(D7)** for any non-drastic distance d.
Let d be a non-drastic propositional distance. Let ϑ_1 and ϑ_2 be two valuations. Let M_1 and M_2 be two Kripke models. Assume that $M_1 = M(\vartheta_1 \to v)$ and $M_2 = M(\vartheta_2 \to v)$ such that $d(\vartheta_1, V_v) < d(\vartheta_2, V_v)$. By definition of $d\mathcal{EB}_d$, we clearly have $d\mathcal{EB}_d(M, M_1) = d(\vartheta_1, V_v)$ and $d\mathcal{EB}_d(M, M_2) = d(\vartheta_2, V_v)$. This allows us to conclude that $d\mathcal{EB}_d(M, M_1) < d\mathcal{EB}_d(M, M_2)$.

The fact that, for any non-drastic distance, $d\mathcal{EB}_d$ satisfies **(D7)** allows us to conclude, using Proposition 3, that it satisfies **(D6)** as well.

□

The ideas of the two previous "weak" bisimulations can be taken together:

Definition 16 (**$d\varepsilon$-n-Bisimilarity**). *Let d be a propositional distance. Let $\varepsilon \in \mathbb{N}$. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models. M and M' are $d\varepsilon$-n-*bisimilar*, noted $M \leftrightarroweq^{d,\varepsilon}_n M'$, if and only if there is a $d\varepsilon$-n-bisimulation $Z \subseteq W \times W'$.*

Definition 17 (**$d\varepsilon$-n-Bisimilation**). *Let d be a propositional distance. Let $\varepsilon \in \mathbb{N}$. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models. Let $Z \subseteq W \times W'$. :*

- *Z is a $d\varepsilon$-0-*bisimulation.
- *Z is a $d\varepsilon$-1-*bisimulation *if and only if $(w, w') \in Z$ and $d(V_w, V'_{w'}) \leq \varepsilon$.*
- *Z is a $d\varepsilon$-$(n+1)$-*bisimulation *if and only if $(w, w') \in Z$ and for all $(v, v') \in Z$:*
  1. *$d(V_v, V'_{v'}) \leq \varepsilon$ and*
  2. *if $\exists u \in W$ such that $(v, u) \in R_a$, then $\exists u' \in W'$ such that $(v', u') \in R'_a$ and $(u, u') \in Z$, and*
  3. *if $\exists u' \in W'$ such that$(v', u') \in R'_a$, then $\exists u \in W$ such that $(v, u) \in R_a$ and $(u, u') \in Z$,*

  and Z is a $d\varepsilon$-n-bisimulation.

Clearly, we can also take advantage of the notion of $d\varepsilon$-n-bisimulation to establish another family of distances $\mathcal{D}_m^{d\mathcal{ENB}_d^\gamma}$. Here, for each depth p, we look for the smallest ε such that both models are $d\varepsilon$-p-bisimilar. We also apply a discounting factor $\gamma \in (0; 1]$ to each of these distances. Thus, the more a difference between two models is at a high depth, the less it is important for the distance between them.

Definition 18 (**$d\varepsilon$-n-Bisimulation-based Distance**). *Let d be a propositional distance. Let $\varepsilon \in \mathbb{N}$. Let $M = \langle W, R, V, w\rangle$ and $M' = \langle W', R', V', w'\rangle$ be two Kripke models containing at most m worlds. Let $\gamma \in (0; 1]$. We denote by $d\mathcal{ENB}_d^\gamma(M, M')$ the distance between M and M', defined as follows:*

$$d\mathcal{ENB}_d^\gamma(M, M') = \sum_{i=1}^{m} (\min(\varepsilon \mid \mu(M) \leftrightarroweq^{d,\varepsilon}_i \mu(M')) \times \gamma^{(i-1)})$$

It is easy to show that, for a small enough discounting factor, $d\mathcal{ENB}_d^\gamma$ satisfies **(D5)**. Like we did it with $d\mathcal{EB}_d$, we check the valuations of the worlds causing the discordance. Hence, again, if a non-drastic propositional distance d is used, $d\mathcal{ENB}_d^\gamma$ satisfies **(D7)** and **(D6)**.

Proposition 11.

1. *Given any propositional distance d, and any discounting factor γ, $d\mathcal{ENB}_d^\gamma$ satisfies **(D1)-(D4)**.*
2. *Given any propositional distance d, there is a $\lambda \in (0; 1]$ such that, for all $\gamma < \lambda$, $d\mathcal{ENB}_d^\gamma$ satisfies **(D5)**.*
3. *For any non-drastic distance d, $d\mathcal{ENB}_d^\gamma$ satisfies **(D6)** and **(D7)**.*

Proof. Let $M = \langle W, R, V, w\rangle$, $M' = \langle W', R', V', w'\rangle$ and $M'' = \langle W'', R'', V'', w''\rangle$ be three Kripke models. Let $n = \max(|W|, |W'|) + 1$.

Let us show that $d\mathcal{ENB}_d^\gamma$ satisfies **(D1)**.
From the definition of $d\mathcal{ENB}_d^\gamma$, we have, $d\mathcal{ENB}_d^\gamma(M, M') = 0$ if and only if $\forall i \in [\![1; n]\!], \min(\varepsilon \mid M \leftrightarroweq^{d,\varepsilon}_i M') = 0$. Hence, $M \leftrightarroweq^{d,0}_n M'$. Definition 17 allows us to conclude that $M \leftrightarroweq_n M'$. Thus, from Proposition 7, we have $M \leftrightarroweq M'$.

Let us show that $d\mathcal{ENB}_d^\gamma$ satisfies **(D2)**.
The fact that the ε-n-bisimulation is symmetric allows us to conclude directly.

Let us show that $d\mathcal{ENB}_d^\gamma$ satisfies **(D3)**.
Following the same reasoning as third point of the proof of the proposition 9, using the following Lemma

Lemma 12. *Let M, M' and M'' be three Kripke models. Let d be a propositional distance. For all i in $\mathbb{N}$, if $M \leftrightarroweq^{d,\varepsilon_1}_i M'$ and $M' \leftrightarroweq^{d,\varepsilon_2}_i M''$, then $M \leftrightarroweq^{d,\varepsilon_1+\varepsilon_2}_i M''$.*

we deduce that $d\mathcal{ENB}_d^\gamma$ satisfies **(D3)**.

By definition, $d\mathcal{ENB}_d^\gamma$ satisfies **(D4)**.

Let us show that, there is a $\lambda \in (0; 1]$ such that, for all $\gamma < \lambda$, $d\mathcal{ENB}_d^\gamma$ satisfies **(D5)**.
Let M_1 and M_2 be two Kripke models such that $M_1 = M(\vartheta \to v)$ and $M_2 = M(\vartheta \to u)$ with $p_1 = \text{height}_{\text{M}}(v) < \text{height}_{\text{M}}(u) = p_2$ and $V_v \neq \vartheta \neq V_u$. Assume that $d\mathcal{ENB}_d^\gamma(M, M_1) \leq d\mathcal{ENB}_d^\gamma(M, M_2)$ for all $\gamma \in (0; 1]$.

By the definition of $d\mathcal{ENB}_d^\gamma$, we have $d\mathcal{ENB}_{d_V}^\gamma(M, M_1) = \sum_{i=1}^n (\min(\varepsilon \mid \mu(M) \leftrightarroweq^{d_V,\varepsilon}_i \mu(M_1)) \times \gamma^{(i-1)})$ and $d\mathcal{ENB}_{d_V}^\gamma(M, M_2) = \sum_{i=1}^n (\min(\varepsilon \mid \mu(M) \leftrightarroweq^{d_V,\varepsilon}_i \mu(M_2)) \times \gamma^{(i-1)})$. By construction of M_1 and M_2, we have $d\mathcal{ENB}_{d_V}^\gamma(M, M_1) = \sum_{i=p_1}^n (d_V(v, \vartheta) \times \gamma^{(i-1)})$ and $d\mathcal{ENB}_{d_V}^\gamma(M, M_2) = \sum_{i=p_2}^n (d_V(u, \vartheta) \times \gamma^{(i-1)})$. Thus, we have

$$\sum_{i=p_1}^{n} (d_V(v, \vartheta) \times \gamma^{(i-1)}) \leq \sum_{i=p_2}^{n} (d_V(u, \vartheta) \times \gamma^{(i-1)})$$

$$d_V(v, \vartheta) \times \sum_{i=p_1}^{n} (\gamma^{(i-1)}) \leq d_V(u, \vartheta) \times \sum_{i=p_2}^{n} (\gamma^{(i-1)})$$

$$d_V(v, \vartheta) \times \gamma^{(p_1-1)} \cdot (\frac{1 - \gamma^{(n-p_1+1)}}{1 - \gamma}) \leq$$

$$d_V(u, \vartheta) \times \gamma^{(p_2-1)} \cdot (\frac{1 - \gamma^{(n-p_2+1)}}{1 - \gamma})$$

$$d_V(v, \vartheta) \times (\gamma^{(p_1-1)} - \gamma^n) \leq d_V(u, \vartheta) \times (\gamma^{(p_2-1)} - \gamma^n) \quad (\star)$$

Now, assume that $\gamma = \frac{\alpha}{n\beta}$ where $\alpha = \min\{d_V(w,w') \neq 0\}$, $\beta = \max\{d_V(w,w')\}$ and $n = \max(\text{height}(M), \text{height}(M')) + 2$. Note that, as $\text{height}(M) = \text{height}(M_1) = \text{height}(M_2)$, we can use the same n. We have $n > p_2 > p_1 \leq 0$ and $0 < \alpha \leq \beta$. Furthermore, assume that $d_V(v,\vartheta) = \alpha$ and $d_V(u,\vartheta) = \beta$.

From $(\star)$ we have

$$\alpha \times ((\frac{\alpha}{n\beta})^{(p_1-1)} - (\frac{\alpha}{n\beta})^n) \leq \beta \times ((\frac{\alpha}{n\beta})^{(p_2-1)} - (\frac{\alpha}{n\beta})^n)$$

$$\frac{\alpha^{p_1}(n\beta)^{(n-p_1+1)} - \alpha^{(n+1)}}{(n\beta)^n} \leq \frac{\alpha^{(p_2-1)}(n\beta)^{(n-p_2+1)} - \alpha^n\beta}{(n\beta)^n}$$

$$\alpha^{p_1}(n\beta)^{(n-p_1+1)} - \alpha^{(n+1)} \leq \alpha^{(p_2-1)}(n\beta)^{(n-p_2+1)} - \alpha^n\beta$$

$$\alpha^{p_1}(n\beta)^{(n-p_1+1)} + \alpha^n\beta \leq \alpha^{(p_2-1)}(n\beta)^{(n-p_2+1)} + \alpha^{(n+1)}(\star\star)$$

- As $\beta \geq \alpha$, we have $\alpha^n\beta \geq \alpha^{(n+1)}$ (1) ;
- As $\beta \geq \alpha$, we have

$$\beta^{(p_2-p_1-1)} \geq \alpha^{(p_2-p_1-1)}$$

$$\beta^{(p_2-p_1)} \geq \alpha^{(p_2-p_1-1)}\beta$$

$$(n\beta)^{(p_2-p_1)} > \alpha^{(p_2-p_1-1)}\beta$$

$$\alpha^{p_1}(n\beta)^{(p_2-p_1)} > \alpha^{(p_2-1)}\beta$$

$$\alpha^{p_1}(n\beta)^{(n-p_1+1)} > \alpha^{(p_2-1)}\beta(n\beta)^{(n-p_2+1)}(2)$$

From (1) and (2) we deduce that $\alpha^{p_1}(n\beta)^{(n-p_1+1)} + \alpha^n\beta > \alpha^{(p_2-1)}(n\beta)^{(n-p_2+1)} + \alpha^{(n+1)}$ which contradicts $(\star\star)$. This contradiction, taken with the fact that we took the minimal value for $d_V(v,\vartheta)$ and the maximal one for $d_V(u,\vartheta)$, allows us to conclude that $d\mathcal{ENB}_d^\gamma$ satisfies **(D5)**.

Let us show that $d\mathcal{ENB}_d^\gamma$ satisfies **(D7)** for any non-drastic distance d.
Let d be a non-drastic propositional distance. Let ϑ_1 and ϑ_2 be two valuations. Let M_1 and M_2 be two Kripke models. Assume that $M_1 = M(\vartheta_1 \to v)$ and $M_2 = M(\vartheta_2 \to v)$ such that $d(\vartheta_1, V_v) < d(\vartheta_2, V_v)$. Let $\text{height}_M(v) = p$. Thus, we have $(n-p) \times d(\vartheta_1, V_v) < (n-p) \times d(\vartheta_2, V_v)$. Furthermore, we have $(n-p) \times d(\vartheta_1, V_v) \times \sum_{i=p+1}^{n}(\gamma^{(i-1)}) < (n-p) \times d(\vartheta_2, V_v) \times \sum_{i=p+1}^{n}(\gamma^{(i-1)})$. Therefore, the definition of $d\mathcal{ENB}_d^\gamma$, allows us to conclude that $d\mathcal{ENB}_d^\gamma(M, M_1) < d\mathcal{ENB}_d^\gamma(M, M_2)$.

The fact that, for any non-drastic distance, $d\mathcal{ENB}_d^\gamma$ satisfies **(D7)** allows us to conclude, using Proposition 3, that it satisfies **(D6)** as well.

□

5.2 Tree-based Distance

We define a family of distances between Kripke models based on the tree models that correspond to them.

The idea is to unveil the Kripke models into trees and to compare how much these trees can be matched, by looking at the best matching.

Definition 19 (Tree Model). *Let $M = \langle W, R, V, w_0 \rangle$ be a finite pointed Kripke model. The* tree model *corresponding to M is a tuple $\langle W', R', V', w_0 \rangle$ where:*

(i) $W' = \{w_0\} \cup \{\sigma = w_0 a_1 w_1 a_2 \cdots a_{n-1} w_{n-1} a_n w_n \mid (w_0, w_1) \in R_{a_1}, \ldots, (w_{n-1}, w_n) \in R_{a_n}\}$
(ii) $R' = \{R'_a \mid a \in \mathbb{A}\}$
(iii) $R'_a = \{(\sigma, \sigma a w) \mid \sigma, \sigma a w \in W'\}$
(iv) $V'_{w_0}(p) = V_{w_0}(p)$
(v) $V'_{\sigma a w}(p) = V_w(p)$

Definition 20 (Tree Function). *Let $M = \langle W, R, V, w_0 \rangle$ be a minimal finite pointed Kripke model. We denote by $\tau(M)$ the tree model corresponding to M.*

Proposition 13. *Let $M = \langle W, R, V, w_0 \rangle$ be a minimal finite pointed Kripke model. $\tau(M)$ is bisimilar to M.*

Proof. Let $M = \langle W, R, V, w_0 \rangle$ be a minimal finite pointed Kripke model and $M' = \langle W', R', V', w_0 \rangle$ the corresponding tree model. Let $Z \subseteq W \times W'$, such that $Z = \{(w_0, w_0)\} \cup \{(w, \sigma a w') \mid \exists \sigma' \in W'$ such that $\sigma = \sigma' a_i w\}$. By construction of M', it is clear that Z is a bisimulation. Thus, $M \leftrightarroweq M'$. □

Assume that we want to measure the distance between M and M'. First, we generate the corresponding tree models A and A'. We take advantage of the Hamming distance d_h to compare valuations. We measure the distance between the roots of two trees and make the sum with the distance between the sub-trees of A and A' as follows: for each sub-tree α of A, we recursively seek for the sub-tree α' of A' whose distance with α is the smallest. Once all pairs (α, α') are found, we make the sum of the distances and apply a discounting factor $\gamma \in (0;1]$. Note that an α can only match one α'. In the event that a sub-tree α does not have a corresponding sub-tree, we make it correspond to a fictitious sub-tree π which, is at a distance d_{max} of α, where d_{max} is the maximum Hamming distance between two valuations.

Let $\mathcal{D}_m^{d\mathcal{T}\pi^\gamma}$ be a family of distances defined on a finite set $\mathcal{M}$ of finite tree models containing at most m worlds.

Definition 21 (Tree-based Distance). *Let $M = \langle W, R, V, w \rangle$ and $M' = \langle W', R', V', w' \rangle$ be two tree models containing at most m worlds. Let $\gamma \in (0;1]$.*

$$d\mathcal{T}\pi^\gamma(M, M') = h(w, w') \cdot \gamma^{\text{height}_M(w)} + \sum_{a \in \mathbb{A}} d\mathcal{T}\pi_a^\gamma(\tau(\mu(M)), \tau(\mu(M')))$$

where

$$h(w, w') = \begin{cases} d_{max} & \text{if } w = \pi \text{ or } w' = \pi \\ d_h(V_w, V'_{w'}) & \text{otherwise} \end{cases}$$

$$d\mathcal{T}\pi_a^\gamma(\tau(\mu(M)), \tau(\mu(M'))) = \min_{b \in B_a^{w,w'}} (\sum_{(v,v') \in b} d\mathcal{T}\pi^\gamma \begin{pmatrix} \langle W, R, V, v \rangle, \\ \langle W', R', V', v' \rangle \end{pmatrix})$$

$$B_a^{w,w'} = \{b | b \in \mathcal{P}((R_a(w) \cup \{\pi\}) \times (R'_a(w')) \cup \{\pi\}) \text{ and } R_a(w) \subseteq dom(b) \text{ and } R'_a(w') \subseteq img(b)\}$$

For a small enough discounting factor, $d\mathcal{T}\pi^\gamma$ satisfies **(D5)**. And, as we compare the valuations of the worlds causing the difference between the models, $d\mathcal{T}\pi^\gamma$ satisfies **(D7)** and so **(D6)**.

Proposition 14.

1. *Given any discounting factor γ, $d\mathcal{T}\pi^\gamma$ satisfies **(D1)-(D4)**.*
2. *$\exists \lambda \in (0;1]$ such that $\forall \gamma < \lambda$, $d\mathcal{T}\pi^\gamma$ satisfies **(D5)**.*
3. *Given any discounting factor γ, $d\mathcal{T}\pi^\gamma$ satisfies **(D6)** and **(D7)**.*

Proof. Let $M = \langle W, R, V, w \rangle$, $M' = \langle W', R', V', w' \rangle$ and $M'' = \langle W'', R'', V'', w'' \rangle$ be three tree models.

Let us show that $d\mathcal{T}\pi^\gamma$ satisfies **(D1)**. Suppose that $d\mathcal{T}\pi^\gamma(M, M') = 0$. From the definition of $d\mathcal{T}\pi^\gamma$,

$d\mathcal{T}\pi^\gamma(M,M')=0$ if and only if

$$\begin{cases} h(w,w')\cdot\gamma^{\text{height}_{\text{M}}(w)}=0 \Leftrightarrow V_w=V'_{w'} \text{ because } \gamma^{\text{height}_{\text{M}}(w)}\neq 0\\ d\mathcal{T}\pi^\gamma_a(\tau(M),\tau(M'))=0, \forall i\in\mathbb{N}\Leftrightarrow\forall a\in\mathbb{A},\\ \quad\begin{cases}\text{if } \exists v\in R_a(w), \text{ then } \exists v'\in R'_a(w') \text{ such that}\\ \quad h(v,v')\cdot\gamma^{\text{height}_{\text{M}}(v)}=0\\ \text{if } \exists v'\in R'_a(w'), \text{then}\exists v\in R_a(w) \text{ such that}\\ \quad h(v,v')\cdot\gamma^{\text{height}_{\text{M}}(v)}=0\end{cases}\end{cases}$$

So we have $d\mathcal{T}\pi^\gamma(M,M')=0$ if and only if $M\leftrightarrows M'$. Thus $d\mathcal{T}\pi^\gamma$ satisfies **(D1)**.

Let us show that $d\mathcal{T}\pi^\gamma$ satisfies **(D2)**.
Note that when the distance $d\mathcal{T}\pi^\gamma(M,M')$ is calculated, we can use the relation b of each level of the tree to build a new relation $\beta\subseteq W\times W'$ for the whole model. Note also that $d\mathcal{T}\pi^\gamma(M,M')=\sum_{(v,v')\in\beta} h(v,v')\cdot\gamma\text{height}_{\text{M}}(v)$, where β is this new relation. Let $d\mathcal{T}\pi^\gamma(M,M')=\sum_{(v,v')\in\beta_1} h(v,v')\cdot\gamma^{\text{height}_{\text{M}}(v)}$ and $d\mathcal{T}\pi^\gamma(M',M)=\sum_{(v',v)\in\beta_2} h(v',v)\cdot\gamma\text{height}_{\text{M'}}(v')$.
With an induction on the height of the tree, we show that from X to Y the relation b is the same as from Y to X.
Here, $\beta_1=\beta_2$, moreover, h (the Hamming distance) satisfies **(D2)**, which means that $d\mathcal{T}\pi^\gamma(M,M')=\sum_{(v,v')\in\beta_1} h(v,v')\cdot\gamma^{\text{height}_{\text{M}}(v)}=\sum_{(v',v)\in\beta_2} h(v',v)\cdot\gamma^{\text{height}_{\text{M'}}(v')}=d\mathcal{T}\pi^\gamma(M',M)$.
Thus $d\mathcal{T}\pi^\gamma$ satisfies **(D2)**.

Let us show that $d\mathcal{T}\pi^\gamma$ satisfies **(D3)**.
Let $d\mathcal{T}\pi^\gamma(M,M')=\sum_{(v,v')\in\beta_1} h(v,v')\cdot\gamma^{\text{height}_{\text{M'}}(v)}$ and $d\mathcal{T}\pi^\gamma(M',M'')=\sum_{(v',v'')\in\beta_2} h(v',v'')\cdot\gamma^{\text{height}_{\text{M'}}(v')}$ such that $\beta_1=\{(v,v')|v\in W\cup\{\varepsilon\}$ and $v'\in W'\cup\{\varepsilon\}\}$ and $\beta_2=\{(v',v'')|v'\in W'\cup\{\varepsilon\}$ and $v''\in W''\cup\{\varepsilon\}\}$. Let $\beta_3=\{(v,v'')|\exists v'\in W'\cup\varepsilon$ such that $(v,v')\in\beta_1$ and $(v',v'')\in\beta_2\}$.
Note that $\sum_{(v,v'')\in\beta_3} h(v,v'')\cdot\gamma^{\text{height}_{\text{M}}(v)}\geq d\mathcal{T}\pi^\gamma(M,M'')$, because β_3 is not necessarily the best relation possible to minimize the sum. Furthermore, we have $h(v,v'')\leq h(v,v')+h(v',v'')$, for each $v\in W\cup\{\varepsilon\}, v'\in W'\cup\{\varepsilon\}, v''\in W''\cup\{\varepsilon\}$, because h is subadditive. Thus, $\sum_{(v,v'')\in\beta_3} h(v,v'')\cdot\gamma^{\text{height}_{\text{M}}(v)}\leq \sum_{(v,v')\in\beta_1} h(v,v')\cdot\gamma^{\text{height}_{\text{M}}(v)}+\sum_{(v',v'')\in\beta_2} h(v',v'')\cdot\gamma^{\text{height}_{\text{M'}}(v')}$.
So, $d\mathcal{T}\pi^\gamma(M,M'')\leq d\mathcal{T}\pi^\gamma(M,M')+d\mathcal{T}\pi^\gamma(M',M''))$.

Let us show that $d\mathcal{T}\pi^\gamma$ satisfies **(D4)**. h being non-negative, it is easy to see that the same is true for $\sum_{(v,v')\in\beta} h(v,v')$. Thus $d\mathcal{T}\pi^\gamma$ is non-negative.

Let us show that, there is a $\lambda\in(0;1]$ such that, for all $\gamma<\lambda$, $d\mathcal{T}\pi^\gamma$ satisfies **(D5)**.
Let M_1 and M_2 be two Kripke models such that there corresponding tree models contains at most m worlds. By fixing $\lambda=\frac{1}{(m+1)\cdot|\mathbb{P}|}$, we ensure that, for all $\gamma<\lambda$, a discordance at a height p can not be offset by any errors at a height at least $p+1$. Indeed, with this discounting factor, even if two models M and M' differ on one propositional variable of the valuation of the pointed world and the models M and M'' differ on all propositional variables of the valuation of all the worlds except the pointed one, this γ ensure that $d\mathcal{T}\pi^\gamma(M,M')>d\mathcal{T}\pi^\gamma(M,M'')$. Thus $d\mathcal{T}\pi^\gamma$ satisfies **(D5)**.

Let us show that $d\mathcal{T}\pi^\gamma$ satisfies **(D7)**.
Let ϑ_1 and ϑ_2 be two valuations. Let M_1 and M_2 be two Kripke models. Assume that $M_1=M(\vartheta_1\to v)$ and $M_2=M(\vartheta_2\to v)$ such that $d(\vartheta_1,V_v)<d(\vartheta_2,V_v)$. Let $\text{height}_{\text{M}}(v)=p$.

Thus, we have $(n-p)\times d(\vartheta_1,V_v)<(n-p)\times d(\vartheta_2,V_v)$. Furthermore, we have $(n-p)\times d(\vartheta_1,V_v)\times\sum_{i=p+1}^{n}(\gamma^{(i-1)}<(n-p)\times d(\vartheta_2,V_v)\times\sum_{i=p+1}^{n}(\gamma^{(i-1)}$. Therefore, the definition of $d\mathcal{T}\pi^\gamma$, allows us to conclude that $d\mathcal{T}\pi^\gamma(M,M_1)<d\mathcal{T}\pi^\gamma(M,M_2)$.

The fact that $d\mathcal{T}\pi^\gamma$ satisfies **(D7)** allows us to conclude, using Proposition 3, that it satisfies **(D6)** as well.

□

5.3 Worlds Sets Distance

Finally, we define a family of distances $\mathcal{D}_m^{d\mathcal{WS}_d^\gamma}$. Each distance of this family is based on distances d between sets of worlds[5], which is itself based on a propositional distance (also noted d) between the valuations associated with the worlds. Here we calculate, for each height p, the distance between the two sets of valuations at a height p. We also apply a discounting factor $\gamma\in(0;1]$ to each of the intermediate distances.

Definition 22 (Worlds Sets Distance). *Let $M=\langle W,R,V,w\rangle$ and $M'=\langle W',R',V',w'\rangle$ be two Kripke models containing at most m worlds and d be a distance between world sets. Let $\gamma\in(0;1]$. We denote by $d\mathcal{WS}_d^\gamma(M,M')$ the distance between M and M', defined as follows:*
$d\mathcal{WS}_d^\gamma(M,M')=\mathcal{F}(\sigma_0(\mu(M),\mu(M'))),\ldots,\sigma_n(\mu(M),\mu(M')))$
where:

$$\begin{aligned}\sigma_0(M,M') &= d(\{w\},\{w'\});\\ \sigma_1(M,M') &= avg\{d(R_a(w),R'_a(w'))\mid a\in\mathbb{A}\};\\ &\vdots\\ \sigma_n(M,M') &= avg\{d(R_{a_{i_1}}\circ\cdots\circ R_{a_{i_n}}(w),R'_{a_{i_1}}\circ\cdots\circ R'_{a_{i_n}}(w'))\\ &\qquad \mid a_{i_k}\neq a_{i_{k+1}}\in\mathbb{A}\};\\ \mathcal{F}(\sigma_0,\ldots,\sigma_n) &= \sum_{i=0}^{m}(\sigma_i\cdot\gamma^i).\end{aligned}$$

For example, we can take advantage of the Hausdorff distance [13] that we adapt to KD45_n models.

Definition 23 (Hausdorff Distance). *Let W and W' be two sets of worlds. We define the Hausdorff distance between W and W' by:*

$$\mathcal{H}(W,W')=\max\begin{pmatrix}\max(\min(d_h(w,w')|w'\in W')|w\in W)\\ \max(\min(d_h(w,w')|w\in W)|w'\in W')\end{pmatrix}$$

We denote by $d\mathcal{WS}_\mathcal{H}^\gamma$ the distance defined by Definition 22 using the Hausdorff distance between worlds sets.

For any propositional distance d and a small enough discounting factor, $d\mathcal{WS}_d^\gamma$ satisfies **(D5)**. As we check the valuations of the worlds using a non-drastic distance d, $d\mathcal{WS}_d^\gamma$ also satisfies **(D7)** and so **(D6)**. Contrastingly, if a drastic propositional distance D is used, $d\mathcal{WS}_\mathcal{D}^\gamma$ does not satisfy **(D7)** nor **(D6)**. Indeed, in this case, we do not look at the discordance between the valuations of the worlds.

Proposition 15.

1. *Given any propositional distance d and any discounting factor γ, $d\mathcal{WS}_d^\gamma$ satisfies **(D1)**-**(D4)**.*

[5] d is supposed to satisfy the usual distance properties (indistinguishability, symmetry, subadditivity and nonnegativity).

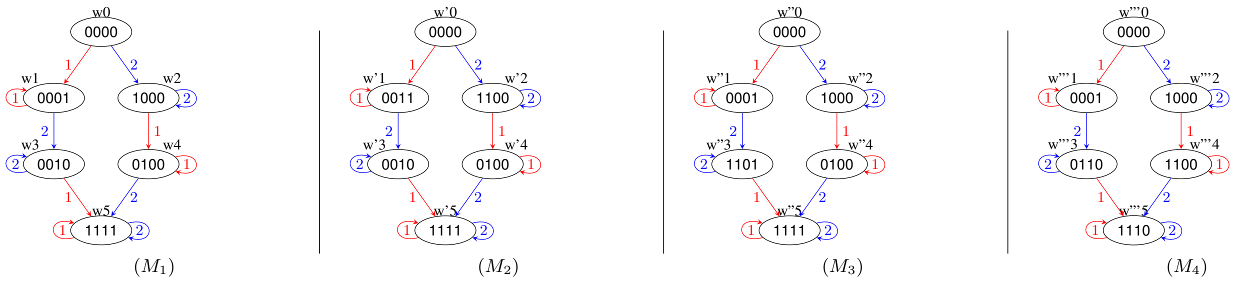

Figure 1: Four Kripke models

$\gamma = 1/2$	M_1, M_2	M_1, M_3	M_1, M_4	M_2, M_3	M_2, M_4	M_3, M_4
$d\mathcal{NB}$	6	5	5	6	6	5
$d\mathcal{EB}_h$	1,000	4,000	1,000	4,000	1,000	3,000
$d\mathcal{ENB}_h^\gamma$	0,938	1,750	0,438	2,250	0,938	1,312
$d\mathcal{T}\pi^\gamma$	2,500	1,000	0,500	3,500	3,000	1,000
$d\mathcal{WS}_\mathcal{D}^{\gamma\,7}$	0,500	0,125	0,469	0,625	0,969	0,469
$d\mathcal{WS}_\mathcal{H}^\gamma$	0,500	0,500	0,469	1,000	0,969	0,719

Table 1: Distances between models of Figure 1

	(D5)	**(D6)**	**(D7)**
$d\mathcal{NB}$	$\surd$	$\times$	$\times$
$d\mathcal{EB}_d$	$\times$	$\surd^\#$	$\surd^\#$
$d\mathcal{ENB}_d^\gamma$	$\surd^\gamma$	$\surd^\#$	$\surd^\#$
$d\mathcal{T}\pi^\gamma$	$\surd^\gamma$	$\surd$	$\surd$
$d\mathcal{WS}_d^\gamma$	$\surd^\gamma$	$\surd^\#$	$\surd^\#$

Table 2: Distances and some properties they satisfy. $\surd$ means satisfied by the distance, $\surd^\gamma$ means satisfied for a small enough discounting factor, $\surd^\#$ means satisfied if a non-drastic distance is used, and $\times$ means not satisfied.

2. *Given any propositional distance d, $\exists\lambda \in (0;1]$ such that $\forall\gamma < \lambda$, $d\mathcal{WS}_d^\gamma$ satisfies* ***(D5)****.*
3. *And, for any non-drastic distance d, $d\mathcal{WS}_d^\gamma$ satisfies* ***(D6)*** *and* ***(D7)****.*

Proof. Let $M = \langle W, R, V, w\rangle$, $M' = \langle W', R', V', w'\rangle$ and $M'' = \langle W'', R'', V'', w''\rangle$ be three Kripke models.

Let us show that $d\mathcal{WS}_d^\gamma$ satisfies **(D1)**.
By definition, we have $M \leftrightarrows M'$ if and only if $\exists Z \subseteq W \times W'$ such that $(w, w') \in Z$ and $\forall(v, v') \in Z$:

1. $V_v = V'_{v'}$
2. if $\exists u \in W$ such that $(v, u) \in R_a$, then $\exists u' \in W'$ such that $(v', u') \in R'_a$ and $(u, u') \in Z$.
3. if $\exists u' \in W'$ such that $(v', u') \in R'_a$, then $\exists u \in W$ such that $(v, u) \in R_a$ and $(u, u') \in Z$.

As d satisfies **(D1)**, this is equivalent to, $d(\{w\}, \{w'\}) = 0$ and $\forall a \in \mathbb{A}, d(R_a(w), R'_a(w')) = 0$.
Likewise, $\forall a_k \neq a_{k+1}, d(R_{a_0} \circ \cdots \circ R_{a_n}(w), R'_{a_0} \circ \cdots \circ R'_{a_n}(w')) = 0$.
Which gives us, $\mathcal{F}(\sigma_0(M, M'), \ldots, \sigma_n(M, M')) = 0 = d\mathcal{WS}_d^\gamma(M, M')$.

Let us show that $d\mathcal{WS}_d^\gamma$ satisfies **(D2)**.
The fact that d satisfies **(D2)** tells us that $\forall\Omega, \Omega'$ sets of worlds, $d(\Omega, \Omega') = d(\Omega', \Omega)$. Thus, $\forall\Omega, \Omega'$ sets of worlds, $\sum d(\Omega, \Omega') = \sum d(\Omega', \Omega)$. Which leads us to $\mathcal{F}(\sigma_0(M, M'), \sigma_1(M, M'), \ldots, \sigma_n(M, M')) = \mathcal{F}(\sigma_0(M', M), \sigma_1(M', M), \ldots, \sigma_n(M', M))$, so $d\mathcal{WS}_d^\gamma(M, M') = d\mathcal{WS}_d^\gamma(M', M)$.

Let us show that $d\mathcal{WS}_d^\gamma$ satisfies **(D3)**.
The fact that d satisfies **(D3)** tells us that $\sum_{a\in\mathbb{A}} d(R_a(w), R''_a(w'')) \leq \sum_{a\in\mathbb{A}} [d(R_a(w), R'_a(w')) + d(R'_a(w'), R''_a(w''))]$. So $\sigma_0(M, M'') \leq \sigma_0(M, M') + \sigma_0(M', M'')$.
Likewise, $\forall i, \sigma_i(M, M'') \leq \sigma_i(M, M') + \sigma_i(M', M'')$.
We clearly have $d\mathcal{WS}_d^\gamma(M, M'') \leq d\mathcal{WS}_d^\gamma(M, M') + d\mathcal{WS}_d^\gamma(M', M'')$.

The fact that d satisfies **(D4)**, allows us to conclude that $d\mathcal{WS}_d^\gamma$ satisfies **(D4)**.

Let us show that, there is a $\lambda \in (0;1]$ such that, for all $\gamma < \lambda$, $d\mathcal{WS}_d^\gamma$ satisfies **(D5)**.
Let M_1 and M_2 be two Kripke models such that there corresponding tree models contains at most m worlds. Let $d_{min} = min(d(\Omega, \Omega') > 0 | \Omega \text{ and } \Omega' \text{ two sets of worlds})$ and $d_{max} = max(d(\Omega, \Omega') | \Omega \text{ and } \Omega' \text{ two sets of worlds})$. By fixing $\lambda = \frac{d_{min}}{(m+1)\cdot d_{max}}$, we ensure that, for all $\gamma < \lambda$, a discordance at a height p can not be offset by any errors at a height at least $p + 1$. Indeed, with this discounting factor, even if two models M and M' differ on one propositional variable of the valuation of the pointed world and the models M and M'' differ on all propositional variables of the valuation of all the worlds except the pointed one, this γ ensure that $d\mathcal{WS}_d^\gamma(M, M') > d\mathcal{WS}_d^\gamma(M, M'')$. Thus $d\mathcal{WS}_d^\gamma$ satisfies **(D5)**.

Let us show that $d\mathcal{WS}_d^\gamma$ satisfies **(D7)**.
Let ϑ_1 and ϑ_2 be two valuations, M_1 and M_2 be two Kripke models and d a non-drastic distance between sets of worlds. We can assume that d is based on a non-drastic propositional distance d_V. Assume that $M_1 = M(\vartheta_1 \to v)$ and $M_2 = M(\vartheta_2 \to v)$ such that $d_V(V_v, \vartheta_1) < d_V(V_v, \vartheta_2)$. Let $\text{height}_M(v) = p$.

Thus, we have $\sigma_i(M, M_1) = 0 = \sigma_i(M, M_2)$, for all $i \neq p$. Furthermore, we have $\sigma_p(M, M_1) = avg\{d(\Omega, \Omega_1) | \Omega = R_{a_{i_1}} \circ \cdots \circ R_{a_{i_n}}(w), \Omega_1 = R_{a_{i_1}} \circ \cdots \circ R_{a_{i_n}}(w) \text{ and } a_{i_k} \neq a_{i_{k+1}} \in \mathbb{A}\}$ and $\sigma_p(M, M_2) = avg\{d(\Omega, \Omega_2) | \Omega = R_{a_{i_1}} \circ \cdots \circ R_{a_{i_n}}(w), \Omega_2 = R_{a_{i_1}} \circ \cdots \circ R_{a_{i_n}}(w) \text{ and } a_{i_k} \neq a_{i_{k+1}} \in \mathbb{A}\}$. If $v \notin \Omega$, then $d(\Omega, \Omega_1) = 0 = d(\Omega, \Omega_2)$. If $v \in \Omega$, then $d(\Omega, \Omega_1) = d_V(V_v, \vartheta_1) < d_V(V_v, \vartheta_2) = d(\Omega, \Omega_2)$. This gives us $\sigma_p(M, M_1) < \sigma_p(M, M_2)$ and allows us to conclude that $d\mathcal{WS}_d^\gamma(M, M_1) < d\mathcal{WS}_d^\gamma(M, M_2)$.

The fact that $d\mathcal{WS}_d^\gamma$ satisfies **(D7)** allows us to conclude, using Proposition 3, that it satisfies **(D6)** as well.

$\square$

5.4 Comparing Distances

Our distances capture different intuitions about how close two Kripke models are. A key question when dealing with distances d is to determine how fine-grained they are. Stating it formally calls for the following notion of refinement:

Definition 24 (**Refinement).** *Let d_1 and d_2 be two distances. d_1 is* at least as fine *as d_2 (denoted $d_1 \geq_f d_2$) if and only if $\forall a, b, c$, if $d_1(a,b) < d_1(a,c)$ then $d_2(a,b) < d_2(a,c)$ and if $d_1(a,b) = d_1(a,c)$ then $d_2(a,b) = d_2(a,c)$.*

Basically, a distance refines another one if it allows to obtain a finer distinction between models. So, if a distance can be refined by another (sensible) one, this can be seen as a flaw of the first, that does not do the full discrimination work.

We can show that there is no such refinement relation between the distances we have introduced:

Proposition 16. *$d\mathcal{NB}$, $d\mathcal{EB}_h$, $d\mathcal{ENB}_h^\gamma$, $d\mathcal{T}\pi^\gamma$, $d\mathcal{WS}_\mathcal{D}^\gamma$ and $d\mathcal{WS}_\mathcal{H}^\gamma$ are pairwise incomparable with respect to $\geq_f$.*

This result shows that we have obtained six truly different types of distances.

Let us now illustrate the differences between these distances on a small example.

Example 1. *Consider the four models in Figure 1. The differences between M_1 and M_2 are in w_1' and w_2' (height $= 1$). In the first model agent 1 believes 0001 and agent 2 believes 1000, and in the second model agent 1 believes 0011 and agent 2 believes 1100. The difference between M_1 and M_3 is in w_3'' (height $= 2$). The differences between M_1 and M_4 are in w_3''', w_4''' and w_5''' (height $= 2$). Table 1 reports the distances between those models. One can check that $d\mathcal{NB}$, $d\mathcal{EB}_d$, $d\mathcal{ENB}_d^\gamma$, $d\mathcal{T}\pi^\gamma$, $d\mathcal{WS}_\mathcal{D}^\gamma$ and $d\mathcal{WS}_\mathcal{H}^\gamma$ do not order these four models in the same way. Note also that the discounting factor $\gamma = 1/2$ is not small enough to ensure that **(D5)** is satisfied by each distance.*

Another way to compare the distances under consideration is to focus on the satisfaction of expected properties **(D5)-(D7)**. Table 2 summarizes the obtained results.

6 USE OF DISTANCES FOR BELIEF REVISION

We show now how the families of distances considered in the previous sections can be exploited to revise Kripke models, or more generally finite sets of Kripke models.

Revising a Kripke model by a formula could lead to several (but a finite number of) models, being able to take account for such sets is indeed essential in order to possibly iterate the revision.

Let α be a formula such that $\deg(\alpha) = p$ and $\mathcal{M}'$ be a finite set of (finite, pointed KD45$_n$) Keipke models containing at most m' worlds.

In this case, each distance d of a family $\mathcal{D}_m^d$ is defined on a finite set $\mathcal{M}$ of finite Kripke models containing at most m worlds such that $m = max(m', |\mathbb{A}|^p \cdot |\mathbb{P}|^{p+1})$. Doing so, we ensure that we can compare all models of $\mathcal{M}'$ with the models of α.

We denote by $Mod(\alpha)$ the set of Kripke models M that satisfy α. The revision of $\mathcal{M}$ by α is a set of Kripke models, noted $\mathcal{M} \circ \alpha$. We expect from the revision operator $\circ$ that it satisfies a set of rationality conditions, reminiscent of those proposed by Katsuno and Mendelzon in the case of classical propositional logic [17]:

(R1) $\mathcal{M} \circ \alpha \subseteq Mod(\alpha)$
(R2) if $\mathcal{M} \cap Mod(\alpha) \neq \emptyset$, then $\mathcal{M} \circ \alpha = \mathcal{M} \cap Mod(\alpha)$
(R3) if $Mod(\alpha) \neq \emptyset$, then $\mathcal{M} \circ \alpha \neq \emptyset$
(R4) if $Mod(\alpha) = Mod(\beta)$, then $\mathcal{M} \circ \alpha = \mathcal{M} \circ \beta$
(R5) $(\mathcal{M} \circ \alpha) \cap Mod(\beta) \subseteq \mathcal{M} \circ (\alpha \wedge \beta)$
(R6) if $(\mathcal{M} \circ \alpha) \cap Mod(\beta) \neq \emptyset$,
then $\mathcal{M} \circ (\alpha \wedge \beta) \subseteq (\mathcal{M} \circ \alpha) \cap Mod(\beta)$

In the case of classical propositional logic, Katsuno and Mendelzon gave a representation theorem for characterizing all revision operators satisfying the expected conditions. This theorem is based on the concept of faithful assignment. It is interesting to adapt this concept to our framework to obtain conditions which are sufficient to ensure the rationality of a revision operator:

Definition 25 (**Faithful assignment).** *A* faithful assignment *is a mapping that associates with any finite set $\mathcal{M}$ of Kripke models a pre-order $\leq_\mathcal{M}$ on the set of Kripke models, such as:*

- *if $M_1 \in \mathcal{M}$ and $M_2 \in \mathcal{M}$, then $M_1 \simeq_\mathcal{M} M_2$;*
- *if $M_1 \in \mathcal{M}$ and $M_2 \notin \mathcal{M}$, then $M_1 <_\mathcal{M} M_2$;*
- *if $\mathcal{M}_1 = \mathcal{M}_2$, then $\leq_{\mathcal{M}_1} = \leq_{\mathcal{M}_2}$*

We have the following result:

Proposition 17. *Let $\circ$ be a revision operator that associates with any finite set $\mathcal{M}$ of Kripke models and any formula α of $\mathcal{L}$ a set of Kripke models. If there exists a faithful assignment that associates with each finite set of Kripke models $\mathcal{M}$ a nœtherian[6] total pre-order $\leq_\mathcal{M}$ such that $\mathcal{M} \circ \alpha = min(Mod(\alpha), \leq_\mathcal{M})$, then $\circ$ satisfies (R1)-(R6).*

Proof. The proof is identical to the proof of the representation theorem in [17] except that interpretations are replaced by finite sets of KD45$_n$ pointed finite Kripke models and propositional formulas are replaced by formulas of $\mathcal{L}$.

Assume that there is a faithful assignment that maps $\mathcal{M}$ to a nœtherian total pre-order $\leq_\mathcal{M}$. We define a revision operation $\circ$ by $\mathcal{M} \circ \alpha = Min(Mod(\alpha), \leq_\mathcal{M})$. We show that $\circ$ satisfies postulates (R1)-(R6). It is obvious that postulate (R1) follows from the definition of the revision operation $\circ$. It is also obvious that postulates (R3) and (R4) follow from the definition of the faithful assignment.

We show postulate (R2). It suffices to show that if $\mathcal{M} \cap Mod(\alpha)$ is not empty then $\mathcal{M} \cap Mod(\alpha) = Min(Mod(\alpha), \leq_\mathcal{M})$. $\mathcal{M} \cap Mod(\alpha) \subseteq Min(Mod(\alpha), \leq_\varphi)$ follows from the conditions of the faithful assignment. To prove the other containment, we assume that $M \in Min(Mod(\alpha), \leq_\varphi)$ and $M \notin \mathcal{M} \cap Mod(\alpha)$. Since $\mathcal{M} \cap Mod(\alpha)$ is not empty, there is a KD45$_n$ Kripke model M' such that $M' \in \mathcal{M} \cap Mod(\alpha)$. Then $M \not\leq_\mathcal{M} M'$ follows from the conditions of the faithful assignment. Morover, $M' \leq_\mathcal{M} M$ follows from the conditions of the faithful assignment. Hence, M is not minimal in $Mod(\alpha)$ with respect to $\leq_\mathcal{M}$. This is a contradiction.

We show postulates (R5) and (R6). It is obvious that if $(\mathcal{M} \circ \alpha) \cap Mod(\beta)$ is empty then (R6) holds. Hence, it suffices to show that if $Min(Mod(\alpha), \leq_\mathcal{M}) \cap Mod(\beta)$ is not empty then $Min(Mod(\alpha), \leq_\mathcal{M}) \cap Mod(\beta) = Min(Mod(\alpha \wedge \beta), \leq_\mathcal{M})$ holds.

Assume that $M \in Min(Mod(\alpha), \leq_\mathcal{M}) \cap Mod(\beta)$ and $M \notin Min(Mod(\alpha \wedge \beta), \leq_\mathcal{M})$. Then, since $M \in Mod(\alpha \wedge \beta)$, there is a KD45$_n$ finite Kripke model M' such that $M' \in Mod(\alpha \wedge \beta)$ and $M' <_\mathcal{M} M$. This contradicts $M \in Min(Mod(\alpha), \leq_\mathcal{M})$. Therefore, we obtain $Min(Mod(\alpha), \leq_\mathcal{M}) \cap Mod(\beta) \subseteq Min(Mod(\alpha \wedge \beta), \leq_\mathcal{M})$.

To prove the other containment, we asume that $M \in Min(Mod(\alpha \wedge \beta), \leq_\mathcal{M})$ and $M \notin Min(Mod(\alpha), \leq_\mathcal{M}) \cap Mod(\beta)$. Since $M \in Mod(\beta)$, $M \notin Min(Mod(\alpha), \leq_\mathcal{M})$ holds. Since we assume that $Min(Mod(\alpha), \leq_\mathcal{M})$ is not empty, suppose that M' is a KD45$_n$ finite pointed Kripke model of $Min(Mod(\alpha), \leq_\mathcal{M}) \cap Mod(\beta)$. Then $M' \in Mod(\alpha \wedge \beta)$ holds. Since we assume that $M \in Min(Mod(\alpha \wedge \beta), \leq_\mathcal{M})$ and $\leq_\mathcal{M}$ is

[6] A pre-order on a set E is nœtherian if there is no sequence of element of E that is infinite and strictly decreasing for the pre-order.

total, $M \leq_{\mathcal{M}} M'$ holds. Thus $M \in Min(Mod(\alpha), \leq_{\mathcal{M}}$ follows from $M' \in Min(Mod(\alpha), \leq_{\mathcal{M}})$. This is a contradiction. □

Given a distance d between Kripke models and a finite set of Kripke models $\mathcal{M}$, we note for any Kripke models M, $d(M, \mathcal{M}) = \underset{M' \in \mathcal{M}}{min}(d(M, M'))$ and $\text{height}(\mathcal{M}) = \underset{M \in \mathcal{M}}{max}(\text{height}(M))$. As $\mathcal{M}$ is finite, we can ensure that $d(M, \mathcal{M})$ and $\text{height}(\mathcal{M})$ are defined. On this basis, one can easily associate with d and $\mathcal{M}$ a total pre-order $\leq_{\mathcal{M}}$ by stating that $M_1 \leq_{\mathcal{M}} M_2$ if and only if $d(M_1, \mathcal{M}) \leq d(M_2, \mathcal{M})$.

To ensure that $\leq_{\mathcal{M}}$ is nœtherian, we consider two additional conditions on d :

Definition 26 (Bounded distance). *A distance d is said to be bounded if and only if for all finite set of Kripke models $\mathcal{M}$, for all formula α of $\mathcal{L}$ such that $\deg(\alpha) = k$, for all Kripke models M_2 such that*

- *M_2 satisfies α ;*
- *$\text{height}(M_2) > \max(k+1, \text{height}(\mathcal{M}))$,*

there is a Kripke model M_1 such that

- *M_1 satisfies α ;*
- *$\text{height}(M_1) \leq \max(k+1, \text{height}(\mathcal{M}))$;*
- *$d(M_1, \mathcal{M}) \leq d(M_2, \mathcal{M})$.*

Definition 27 (Minimal model condition). *A distance d between Kripke models satisfies the minimal model condition if and only if for all models M_1 and M_2, $d(M_1, M_2) = d(\mu(M_1), \mu(M_2))$.*

When d is bounded, for any model $M_2 \in Mod(\alpha)$, we know that there is a model $M_1 \in Mod(\alpha)$ such that $height(M_1) \leq max(deg(\alpha)+1, height(\mathcal{M}))$ and $d(M_1, \mathcal{M}) \leq d(M_2, \mathcal{M})$. Yet, there is a finite number of models M_1 of $Mod(\alpha)$ (up to bisimulation) verifying $height(M_1) \leq max(deg(\alpha)+1, height(\mathcal{M}))$, this ensure that $\leq_{\mathcal{M}}$ is nœtherian.

Definition 28 (Revision Operator). *Let d be a bounded distance verifying the minimal model condition. Let $\mathcal{M}$ be a set of KD45$_n$ finite pointed Kripke models and let α be a formula. We assign to $\mathcal{M}$ a nœtherian total pre-order $\leq^d_{\mathcal{M}}$ on KD45$_n$ finite Kripke models defined as follows: $M_1 \leq^d_{\mathcal{M}} M_2$ if and only if $d(M_1, \mathcal{M}) \leq d(M_2, \mathcal{M})$.*
The revision operation $\circ_d$ associated with this pre-order $\leq^d_{\mathcal{M}}$ is defined semantically:

$$\mathcal{M} \circ_d \alpha = min(Mod(\alpha), \leq^d_{\mathcal{M}}).$$

So M is closer to $\mathcal{M}$ than M' when its distance with the models of $\mathcal{M}$ is lower than the distance of M' with the models of $\mathcal{M}$.

Proposition 18. *Let d be any bounded distance verifying the minimal model condition. The operator $\circ_d$ satisfies (R1)-(R6).*

Proof. From proposition 17, it suffices to show that the assignment defined in Definition 28 is a faithful assignment. Clearly $\leq^d_{\mathcal{M}}$ is a total pre-order because $\leq$ is a total pre-order. We are going to show that it is faithful.

- If $M \in \mathcal{M}$ and $M' \in \mathcal{M}$ then $d(M, \mathcal{M}) = d(M', \mathcal{M}) = 0$ by definition of $\leq^d_{\mathcal{M}}$. So we can not have $M <^d_{\mathcal{M}} M'$.
- If $M \in \mathcal{M}$ and $M' \notin \mathcal{M}$ then $d(M, \mathcal{M}) = 0$ and $d(M', \mathcal{M}) = k$ with $k > 0$. So $d(M, \mathcal{M}) < d(M', \mathcal{M})$, i.e. $M \leq^d_{\mathcal{M}} M'$.
- Finally, if $\mathcal{M} \equiv \mathcal{M}'$ then clearly $\leq^d_{\mathcal{M}} = \leq^d_{\mathcal{M}'}$. □

The last point is to check whether some of the distances introduced satisfy the relevant conditions. Fortunately, this is the case:

Proposition 19. *$d\mathcal{NB}$, $d\mathcal{EB}_d$, $d\mathcal{ENB}^\gamma_h$, $d\mathcal{T}\pi^\gamma$ and $d\mathcal{WS}^\gamma_{\mathcal{H}}$ are bounded and satisfy the minimal model condition.*

Proof. Let us give the proof for $d\mathcal{ENB}^\gamma_h$ (the proofs for the other distances are similar).

Let φ be a formula of $\mathcal{L}$ such that $\deg(\varphi) = k$. Let $M = \langle W, R, V, w_0\rangle$ and $M_2 = \langle W_2, R_2, V_2, w^2_0\rangle$ be two Kripke models such that $\text{height}(M) = k'$, M_2 satisfies φ and $\text{height}(M_2) > \max(k+1, k')$.

- If $M \models \varphi$, if suffices to take $M_1 = M$.
- Else, let $\kappa = \max(k+1, k')$ and M_1 be the restriction of M to κ (noted $(M \restriction \kappa)$) define as usual [7]. So $M_1 = \langle W_1, R_1, V_1, w^1_0\rangle$ such that $W_1 = \{w \in W_2 | \text{height}_(w) \leq \kappa\}$, $R_1 = R_2 \cap (W_1 \times W_1)$, $V_1 = \{V_w \in V_2 | w \in W_1\}$ and $w^1_0 = w^2_0$. Thus, M_1 is a copy of M_2 up to a modal depth of κ. Clearly, M_1 satisfies φ and $\text{height}(M_1) \leq \kappa$.
  - We show that $d\mathcal{ENB}^\gamma_h$ is bounded. To do so, we show that $d\mathcal{ENB}^\gamma_h(M, M_1) \leq d\mathcal{ENB}^\gamma_h(M, M_2)$. We have two cases to consider:
    1. If all the differences between M and M_2 are at a depth higher than κ, then, clearly, $M \models \varphi$ so we have a contradiction.
    2. If some of the differences between M and M_2 are at a depth lower than κ, then, by definition of $d\mathcal{ENB}^\gamma_h$, the fact that $\text{height}(M_1) < \text{height}(M_2)$ implies that $d\mathcal{ENB}^\gamma_h(M, M_1) < d\mathcal{ENB}^\gamma_h(M, M_2)$.

From the definition of these distances, it is obvious that they satisfy the minimal model condition. □

Consequently, we can define AGM revision operators on the sets of KD45$_n$ models based on the five distances $d\mathcal{NB}$, $d\mathcal{EB}_d$, $d\mathcal{ENB}^\gamma_h$, $d\mathcal{T}\pi^\gamma$ and $d\mathcal{WS}^\gamma_{\mathcal{H}}$. The distances $d\mathcal{ENB}^\gamma_h$, $d\mathcal{T}\pi^\gamma$ et $d\mathcal{WS}^\gamma_{\mathcal{H}}$ appear as the most interesting ones (among those considered here), because they also satisfy all the expected properties **(D1)-(D7)**.

We now illustrate the five revision operators corresponding to these distances.

Example 2. *Let us consider the Kripke model M_0 in Figure 2. In this situation, agent 1 believes $\neg x \wedge y$, agent 2 believes $x \wedge \neg y$ and agent 3 believes $x \wedge y$. Agent 1 believes that agents 2 and 3 believe $x \wedge y$, agent 2 believes that agent 1 believes $x \wedge y$ and that agent 3 believes $\neg x \wedge y$, agent 3 believes that agents 1 and 2 believe $\neg x \wedge y$.*

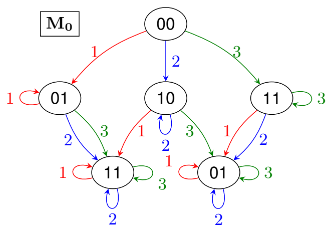

Figure 2: Finite Kripke Model Before Revision

Let us look at the outcome of the revision of this model by the formula $\alpha = x \wedge \mathrm{B}_1(\neg y) \wedge \mathrm{B}_2(\neg x) \wedge \mathrm{B}_3(\neg x \wedge \neg y)$ for our five revision operators $\circ_{d\mathcal{NB}}$, $\circ_{d\mathcal{EB}_d}$, $\circ_{d\mathcal{ENB}^\gamma_d}$, $\circ_{d\mathcal{T}\pi^\gamma}$ and $\circ_{d\mathcal{WS}^\gamma_{\mathcal{H}}}$. We are revising the model by changing the real world and beliefs about the real world of the three agents.

Figure 3 shows two Kripke models M_1 and M_2. Although both models M_1 and M_2 are selected as models resulting of the revision [7] *by operators $\circ_{d\mathcal{NB}}$, $\circ_{d\mathcal{EB}_d}$ and $\circ_{d\mathcal{ENB}_d^\gamma}$, M_2 is the only model resulting from the revision of M_0 by α for the revision operators $\circ_{d\mathcal{WS}_\mathcal{H}^\gamma}$ and $\circ_{d\mathcal{T}\pi^\gamma}$.*

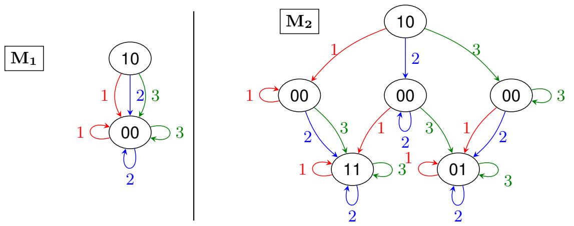

Figure 3: Finite Kripke Models After Revision

Let us consider $\circ_{d\mathcal{NB}}$. Since the valuation of the pointed world must change to satisfy α, all the models of α are equidistant from M_0. Let us now consider $\circ_{d\mathcal{EB}_d}$ and $\circ_{d\mathcal{ENB}_d^\gamma}$. Since the valuation of the world accessible for agent 3 must completely change (from xy to $\overline{xy}$) to satisfy α, this time again, all models of α are equidistant from M_0. Finally, for $\circ_{d\mathcal{T}\pi^\gamma}$ and $\circ_{d\mathcal{WS}_\mathcal{H}^\gamma}$, as the associated distances consider the valuation of each world at each height of the model, the closest model is one which coincides with M_0 but for the valuations. Thus, $d\mathcal{T}\pi^\gamma$ and $d\mathcal{WS}_\mathcal{H}^\gamma$ appear as the most appropriate distances for defining distance-based revision operators.

In [2] the modeled revision is a subjective revision, which means that the new information is received by one of the agents of the system (thus, after the revision, subjective models of this agent will be modified). Here, the revision which is defined is that of the observer of the multi-agent system, which describes the real world and the beliefs of the agents.

7 Conclusion

In this paper we have investigated distances between KD45$_n$ Kripke models. The aim was to characterize revision operators based on these distances. We have identified properties that expected distances should satisfu, introduced distances verifying those properties, and showed that these distances are incomparable with respect to refinement. Then, we have showed that the representation theorem in terms of faithful assignment defined by Katsuno and Mendelzon [16] can be adapted to define the revision of a KD45$_n$ Kripke model by a formula. Finally, we have showed that all the distances we defined can be used to define distance-based revision operators.

Clearly enough, the distances defined here make sense for other classes of Kripke models than KD45$_n$ ones. However it is not clear that the set of expected properties should remain the same. Identifying reasonable conditions to be satisfied by distances when revising for example preferences, programs, etc. is a perspective for further research.

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