International Journal of Approximate Reasoning 80 (2017) 428–442

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

Contraction in propositional logic

Thomas Caridroit *, Sébastien Konieczny, Pierre Marquis

CRIL, CNRS – Université d'Artois, France

ARTICLE INFO

Article history:
Received 14 November 2015
Received in revised form 21 June 2016
Accepted 22 June 2016
Available online 7 July 2016

Keywords:
Belief change
Revision
Contraction

ABSTRACT

The AGM model for the revision and contraction of belief sets provides rationality postulates for each of the two families of change operators. In the context of finite propositional logic, Katsuno and Mendelzon pointed out postulates for the revision of belief bases which correspond to the AGM postulates for the revision of beliefs sets. In this paper, we present postulates for the contraction of propositional belief bases which correspond to the AGM postulates for the contraction of belief sets. We highlight the existing connections with the revision of belief bases in the sense of Katsuno and Mendelzon thanks to Levi and Harper identities. We also present a representation theorem for contraction operators for propositional belief bases.

1. Introduction

Belief change has been studied for many years in philosophy, databases, and artificial intelligence. The AGM model, named after its three initiators Carlos Alchourrón, Peter Gärdenfors and David Makinson, is the main formal framework for modeling belief change [1]. Its key concepts and constructs have been the subject of significant developments [2–5]. Alchourrón, Gärdenfors and Makinson pointed out some postulates and representation theorems thereby establishing the basis for a framework suited to the belief change issue when beliefs are expressed using the language of any Tarskian logic, and given as belief sets (i.e., sets of beliefs closed under the consequence relation). Tarskian logics consider abstract consequence relations, that satisfy inclusion, monotony and idempotence (and the AGM framework adds also to them the supraclassicality, compactness and deduction conditions).

Katsuno and Mendelzon [6] presented a set of postulates for revision operators of belief bases in the framework of finite propositional logic. Generally speaking, a belief base is simply a non-deductively closed set of formulas. In [6] and in the present paper, a belief base can also be viewed as a single formula (the conjunction of its elements). Especially, belief base contraction could also be referred to as formula-based contraction. This departs from many works where the term "belief base contraction" is used for denoting syntax-dependent belief change [3].

Katsuno and Mendelzon [6] gave a representation theorem for revision operators in terms of faithful assignments.[1] This representation theorem is important because it is at the origin of the main approaches to iterated belief revision [4].

Revision and contraction operators of belief sets are closely related, as reflected by Levi and Harper identities. These identities can be used to define contraction operators from revision operators and vice versa. So works on contraction in

* Corresponding author.
E-mail addresses: caridroit@cril.fr (T. Caridroit), konieczny@cril.fr (S. Konieczny), marquis@cril.fr (P. Marquis).

[1] Such assignments correspond to a specific case of Grove's systems of spheres [7].

http://dx.doi.org/10.1016/j.ijar.2016.06.010
0888-613X/

the context of finite propositional logic might be expected. However, as far as we know, this issue has not been formally investigated up to now.

The aim of this paper is to define operators for propositional contraction of belief bases matching Katsuno and Mendelzon's revision operators and to check that these operators offer some expected properties. In the following, we give a set of postulates for contraction operators of belief bases in the framework of finite propositional logic, and establish a corresponding representation theorem.

The obtained results are not very surprising, since most of them can be obtained as corollaries of existing theorems (especially previous representation theorems and Levi and Harper identities). Nonetheless we consider that proving these results directly (and not as a byproduct of existing results) is important for a number of contexts where the whole AGM machinery cannot be applied directly.

Let us focus on two of them, which are quite significant. First, many works from the recent past years have been dedicated to the design and the study of belief revision and contraction operators in some weaker logical settings than the classical one, for instance Horn logic [8–11]. The obtained results form an important step towards the definition of change operators suited to Description Logics [12], or for other fragments of classical logic [13]. Indeed, in these weaker settings, there is no guarantee that the results from standard AGM theory still hold. In particular, some characterizations of belief change constructions that are equivalent (obtained by representation theorems) in the classical case are not equivalent any more. Furthermore, Harper and Levi identities do not necessarily hold (see [14] for the Horn case).

The second context we want to take as example is iterated change. Iterated revision has been extensively studied so far. This led to the development of revision operators on epistemic states, that are an epistemic representation that is more expressive than belief sets [4,15–17]. These works about iterated revision are based on Katsuno and Mendelzon's approach to the revision of propositional belief bases [6]. However, as far as we know, no study of what could be the corresponding iterated contraction operators has been conducted so far. This is not a straightforward task given that the Harper and Levi identities are not easily expressible on epistemic states. Accordingly, establishing direct formulations and proofs of results on propositional contraction appears as an important step for designing such iterated contraction operators (again one can not take advantage of the usual identities and representation results in this case).

The rest of the paper is organized as follows. In Section 2, some formal preliminaries are presented. In Section 3, the AGM and KM settings for belief contraction and revision are recalled. In Section 4, the standard connection between belief sets and belief bases is recalled as well. In Section 5, we define postulates that a contraction operator for propositional belief bases should satisfy. In Section 6 the correspondence between contraction of belief sets and contraction of belief bases is investigated; we check, using Levi and Harper identities, that there is a connection between propositional revision operators satisfying Katsuno and Mendelzon postulates and propositional contraction operators satisfying our postulates. Section 7 gives a representation theorem for the contraction of belief bases. Section 8 illustrates the representation theorem by defining a Dalal contraction operator. We conclude and discuss some perspectives for future work in Section 9.

2. Preliminaries

We consider a finite propositional language L built up from a (finite) set of propositional symbols P and the usual connectives. $\bot$ (resp. $\top$) is the Boolean constant `false` (resp. `true`). Formulas are interpreted in the standard way. $\vdash$ denotes logical deduction, and $\equiv$ denotes logical equivalence. $\mathcal{P}(L)$ denotes the power set of L, i.e., the set of all subsets of L.

A belief base is a finite set of propositional formulas $\{\varphi_1, \ldots, \varphi_n\}$. We suppose in this paper that a belief base is conjunctively interpreted, i.e., $\{\varphi_1, \ldots, \varphi_n\}$ is equivalent to $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_n$ (this is a usual harmless assumption when one supposes irrelevance of syntax (cf. postulate (C5)).

A belief set is a deductively closed set of formulas. This set is infinite, thus not convenient for an effective representation. Given any element K of $\mathcal{P}(L)$, $Cn(K)$ denotes the deductive closure of K. When K is a singleton $\{\varphi\}$, or more generally, when K is equivalent to a formula $\varphi \in L$, we also write $Cn(\varphi) = \{\psi \in L \mid \varphi \vdash \psi\}$ to denote the set of classical consequences of φ.

Fortunately, one can associate with any belief base φ a belief set K that is the set of all its consequences $K = Cn(\varphi)$. Reciprocally, we can always represent a belief set by a (finite) belief base consisting of a single formula (it is enough to select one representative of the logically strongest formulas in the belief set to get such a belief base).

An interpretation I is a mapping associating every symbol from P with a truth value. If φ is a formula from L, then $Mod(\varphi)$ denotes the set of its models. Conversely if M is a set of interpretations over P, then α_M denotes the formula (unique, up to logical equivalence) the models of which are those of M.

Given a preorder (i.e., a reflexive and transitive relation) $\leq_\varphi$ over the set of interpretations, $<_\varphi$ is its strict part defined by $I <_\varphi J$ if and only if $I \leq_\varphi J$ and $J \not\leq_\varphi I$ and $\simeq_\varphi$ is the associated equivalence relation defined by $I \simeq_\varphi J$ if and only if $I \leq_\varphi J$ and $J \leq_\varphi I$. $\min(X, \leq_\varphi)$ denotes the set of minimal elements of X for $\leq_\varphi$, i.e., $\min(X, \leq_\varphi) = \{x \in X \mid \nexists y \in X$ such that $y <_\varphi x\}$.

3. AGM and KM belief revision and contraction

Alchourrón, Gärdenfors and Makinson [1,2] pointed out the following postulates for the revision and the contraction of belief sets. These postulates are formulated in a very general framework, indeed they are valid for any Tarskian logic [18,19] that satisfies supraclassicality, deduction and compactness:

Definition 1. A consequence operation on a language L is a mapping $C : \mathcal{P}(L) \longrightarrow \mathcal{P}(L)$ satisfying the following conditions (K, K' are sets of formulas from L and α, β are formulas from L):

- $K \subseteq C(K)$ (inclusion)
- If $K \subseteq K'$ then $C(K) \subseteq C(K')$ (monotony)
- $C(K) = C(C(K))$ (iteration)
- If $K \vdash \alpha$, then $\alpha \in C(K)$ (supraclassicality)
- If $\beta \in C(K \cup \{\alpha\})$, then $(\alpha \rightarrow \beta) \in C(K)$ (deduction)
- If $\alpha \in C(K)$, then $\alpha \in C(K')$ for some finite subset K' of K (compactness)

Clearly enough, the classical deductive closure operation Cn is a consequence operation.

Given a belief set K and formula α, $K * \alpha$ denotes the revision of K by α. $+$ is the expansion operator [1,2], the result it gives is just the set of consequences of the union of K and $\{\alpha\}$ (i.e., $K + \alpha = C(K \cup \{\alpha\})$). Let β be any formula from L. An AGM revision [1,2] operator $*$ satisfies the following postulates:

(K ∗ 1) $K * \alpha$ is a belief set
(K ∗ 2) $\alpha \in K * \alpha$
(K ∗ 3) $K * \alpha \subseteq K + \alpha$
(K ∗ 4) If $\neg\alpha \notin K$, then $K + \alpha \subseteq K * \alpha$
(K ∗ 5) $K * \alpha = K_{\perp}$ iff $\vdash \neg\alpha$
(K ∗ 6) If $\alpha_1 \equiv \alpha_2$, then $K * \alpha_1 = K * \alpha_2$
(K ∗ 7) $K * (\alpha \wedge \beta) \subseteq (K * \alpha) + \beta$
(K ∗ 8) If $\neg\beta \notin K * \alpha$, then $(K * \alpha) + \beta \subseteq K * (\alpha \wedge \beta)$

The intuitive meaning of those postulates is as follows: $(K * 1)$ ensures that the result of the revision is a belief set. $(K * 2)$ states that the new piece of information belongs to the revised belief set. $(K * 3)$ implies that the revision by the new information cannot add beliefs that are not consequences of the new information and the belief set. $(K * 3)$ and $(K * 4)$ together means that, when the new information is consistent with the old belief set, the revision of the belief set must be an expansion. $(K * 5)$ expresses the fact that the only way to get to an inconsistent base by revision is to revise by contradictory information. $(K * 6)$ reflects the principle of independence of syntax. These six postulates are the base postulates for revision operators. $(K * 7)$ and $(K * 8)$ are called additional postulates. They express the expected behavior of revision operators in terms of minimality of change. They ensure that the revision by a conjunction of two pieces of information is the same as the revision by the first piece of information followed by the expansion by the second one when it does not lead to an inconsistency.

Similarly [1,2] provide postulates for characterizing rational contraction operators. Given a belief set K and a formula α, $K \div \alpha$ denotes the contraction of K by α. Let β be any formula from L. An AGM contraction operator $\div$ [1,2] satisfies the following postulates:

(K÷1) $K \div \alpha$ is a belief set
(K÷2) $K \div \alpha \subseteq K$
(K÷3) If $\alpha \notin K$, then $K \div \alpha = K$
(K÷4) If $\nvdash \alpha$, then $\alpha \notin K \div \alpha$
(K÷5) If $\alpha \in K$, then $K \subseteq (K \div \alpha) + \alpha$
(K÷6) If $\alpha \equiv \beta$, then $K \div \alpha = K \div \beta$
(K÷7) $(K \div \alpha) \cap (K \div \beta) \subseteq K \div (\alpha \wedge \beta)$
(K÷8) If $\alpha \notin K \div (\alpha \wedge \beta)$, then $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$

$(K \div 1)$ ensures that the result of the contraction is a belief set. $(K \div 2)$ guarantees that during the contraction no new information is added to the belief set $(K \div 3)$ states that if the information α is not in K, then there is nothing to do to remove α from K. $(K \div 4)$ ensures the success of contraction, that is to say, if α is not a tautology, then the contraction succeeds. $(K \div 5)$ ensures that the contraction of K by α followed by the expansion by α gives the belief set K as result (the converse inclusion being a consequence of $(K \div 1)$–$(K \div 4)$). $(K \div 6)$ states that the result of the contraction does not depend on the syntax of the information. These six postulates are the base postulates for contraction operators. $(K \div 7)$ and $(K \div 8)$ are called supplementary postulates. $(K \div 7)$ states that if an information is in both the contraction by α and the contraction by β, then it must be in the contraction by the conjunction $\alpha \wedge \beta$. $(K \div 8)$ expresses the minimality of change for the conjunction.

Let us now recall that AGM belief revision and belief contraction are closely related. Actually every belief revision operator induces a belief contraction one, and vice-versa. This is given by the well-known Levi and Harper identities (K is a belief set and α is a formula from L):

(Levi identity) $K \star \alpha = (K \div \neg\alpha) + \alpha$

(Harper identity) $K \div \alpha = K \cap (K \star \neg\alpha)$

So basically a belief revision operator can be defined from a contraction operator (of the negation of the new piece of information) followed by an expansion. And a contraction operator can be defined from a revision operator: the consequences of K that do not depend of the fact that the new piece of information is true are the ones that belong to the current belief set and are in the set obtained by revising the current belief set by the negation of the new piece of information.

The following theorems show that operators obtained using Levi identity (resp. Harper identity) are revision (resp. contraction) operators.

Theorem 1 *([2]). If the contraction operator $\div$ satisfies $(K \div 1)$–$(K \div 4)$ and $(K \div 6)$, then the revision operator defined using Levi identity satisfies $(K * 1)$–$(K * 6)$. Furthermore if $(K \div 7)$ is satisfied by $\div$, then $(K * 7)$ is satisfied by $*$, and if $(K \div 8)$ is satisfied by $\div$, then $(K * 8)$ is satisfied by $*$.*

Theorem 2 *([2]). If the revision operator $*$ satisfies $(K * 1)$–$(K * 6)$, then the contraction operator defined using Harper identity satisfies $(K \div 1)$–$(K \div 6)$. Furthermore, if $(K * 7)$ is satisfied by $*$, then $(K \div 7)$ is satisfied by $\div$, and if $(K * 8)$ is satisfied by $*$, then $(K \div 8)$ is satisfied by $\div$.*

Let us now recall Katsuno and Mendelzon's propositional counterpart of belief revision postulates and their representation theorem in terms of faithful assignment [6]. Let φ and α be two propositional formulas from L, where φ represents the current belief base of the agent and α is the new piece of information. A KM propositional revision operator $\circ$ is a mapping associating a formula $\varphi \circ \alpha$ with φ and α, and satisfying the following postulates [6]:

(R1) $\varphi \circ \alpha \vdash \alpha$
(R2) If $\varphi \wedge \alpha$ is consistent, then $\varphi \circ \alpha \equiv \varphi \wedge \alpha$
(R3) If α is consistent, then $\varphi \circ \alpha$ is consistent
(R4) If $\varphi_1 \equiv \varphi_2$ and $\alpha_1 \equiv \alpha_2$, then $\varphi_1 \circ \alpha_1 \equiv \varphi_2 \circ \alpha_2$
(R5) $(\varphi \circ \alpha) \wedge \beta \vdash \varphi \circ (\alpha \wedge \beta)$
(R6) If $(\varphi \circ \alpha) \wedge \beta$ is consistent, then $\varphi \circ (\alpha \wedge \beta) \vdash (\varphi \circ \alpha) \wedge \beta$

From here and for the rest of the paper, we limit the discussion to the case of finite propositional logic. The following theorem shows that operators satisfying revision postulates $(R1)$–$(R6)$ correspond to the revision operators satisfying the AGM postulates $(K * 1)$–$(K * 8)$.

Theorem 3 *([6]). Let $*$ be a revision operator on belief sets and $\circ$ be a revision operator on corresponding belief bases (i.e., for all φ and all α, $Cn(\varphi) * \alpha = Cn(\varphi \circ \alpha)$). $*$ satisfies $(K * 1)$–$(K * 8)$ if and only if $\circ$ satisfies (R1)–(R6).*

A representation theorem is a way to associate with a set of postulates a constructive approach to build the corresponding family of operators. Katsuno and Mendelzon presented such a theorem in terms of faithful assignments, which associates with each belief base a pre-order that ranks the interpretations from the most plausible ones to the least plausible ones.

Definition 2. A faithful assignment is a mapping that associates with any belief base φ a pre-order $\leq_\varphi$ on the set of all interpretations such that:

1. If $I \models \varphi$ and $J \models \varphi$, then $I \simeq_\varphi J$
2. If $I \models \varphi$ and $J \not\models \varphi$, then $I <_\varphi J$
3. If $\varphi_1 \equiv \varphi_2$, then $\leq_{\varphi_1} = \leq_{\varphi_2}$

Theorem 4 *([6]). A revision operator $\circ$ satisfies the postulates (R1)–(R6) if and only if there exists a faithful assignment that associates with each belief base φ a total pre-order $\leq_\varphi$ such that $Mod(\varphi \circ \alpha) = min(Mod(\alpha), \leq_\varphi)$.*

So this theorem shows that the revision of φ by a new piece of information α amounts to select in the models of α the interpretations that are the most plausible given the current beliefs φ (thanks to the total pre-order $\leq_\varphi$).

4. From belief sets to belief bases

Our purpose is now to define contraction operators on belief bases in the framework of finite propositional logic. Let φ and α be two formulas. $\varphi - \alpha$ denotes the contraction of φ by α, which is the new formula obtained by removing the piece of beliefs α from the (consequences of the) belief base φ.

In order to relate AGM belief set contraction and our notion of propositional belief base contraction, we first have to make formal the link between belief sets and belief bases. This is the aim of the following (easy) proposition:

Proposition 5. *In the finite propositional setting, the mapping Cn from $L/\equiv$ to the set of all belief sets, associating a belief set $K = Cn(\varphi)$ with every belief base represented by a formula φ considered up to logical equivalence, is bijective.*

Thus, for a belief base φ, the notation $K_\varphi = Cn(\varphi)$ and for a belief set K, the notation $\varphi_K \equiv Cn^{-1}(K)$ are safe.

Corollary 6. *We have $\varphi \equiv \varphi_{K_\varphi}$. Similarly we have $K = K_{\varphi_K}$.*

On this ground, in the finite propositional logic setting, a correspondence between AGM contraction operators for belief sets and contraction operators for belief bases can be established:

Definition 3. Given a contraction operator for belief sets $\div$, the operator $-_{(\div)}$ for belief bases is defined by: $\varphi -_{(\div)} \alpha = \varphi_{K_\varphi \div \alpha}$. Conversely, given a contraction operator on belief bases $-$, the operator $\div_{(-)}$ on belief sets is defined by: $K \div_{(-)} \alpha = K_{\varphi_K - \alpha}$.

Let $\div$ be a contraction operator on belief sets and $-$ be a contraction operator on belief bases. The operators $\div$ and $-$ are said to correspond to each other if $\div = \div_{(-)}$ and $- = -_{(\div)}$. The following proposition shows that if we use a contraction operator on belief sets $\div$ to define, via Definition 3, a contraction operator on belief bases $-_{(\div)}$, then the corresponding contraction operator on belief sets defined via Definition 3 is the initial contraction operator $\div$ (and vice-versa):

Proposition 7. *We have $-_{(\div_{(-)})} = -$. Similarly we have $\div_{(-_{(\div)})} = \div$.*

5. Postulates for propositional contraction

We now define a set of postulates for the contraction of propositional belief bases. In the following, φ, α and β are formulas from L:

(C1) $\varphi \vdash \varphi - \alpha$
(C2) If $\varphi \nvdash \alpha$, then $\varphi - \alpha \vdash \varphi$
(C3) If $\varphi - \alpha \vdash \alpha$, then $\vdash \alpha$
(C4) $(\varphi - \alpha) \wedge \alpha \vdash \varphi$
(C5) If $\varphi_1 \equiv \varphi_2$ and $\alpha_1 \equiv \alpha_2$, then $\varphi_1 - \alpha_1 \equiv \varphi_2 - \alpha_2$
(C6) $\varphi - (\alpha \wedge \beta) \vdash (\varphi - \alpha) \vee (\varphi - \beta)$
(C7) If $\varphi - (\alpha \wedge \beta) \nvdash \alpha$, then $\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$

The intuitive meaning of these postulates is as follows: (C1) ensures that after contraction, no new information is added to the belief base. (C2) indicates that if α is not deducible from φ, then no change is made during the contraction. (C3) ensures that the only possibility for the contraction of φ by α to fail is when α is a tautology. (C4) states that the conjunction of the contraction of φ by α and α gives a propositional formula which is equivalent to φ (the converse implication is a consequence of (C1) when $\varphi \vdash \alpha$). (C5) reflects the principle of independence of syntax. (C6) and (C7) express the minimality of change for the conjunction. (C6) states that the contraction by a conjunction always implies the disjunction of the contractions by the conjuncts. (C7) states that if α has not been removed during the contraction by $\alpha \wedge \beta$, then the contraction by α must imply the contraction by the conjunction.

The following proposition shows that the contraction operators satisfying postulates (C1)–(C7) correspond to the contraction operators satisfying the AGM postulates $(K \div 1)$–$(K \div 8)$.

Proposition 8. *Let $\div$ be a contraction operator on belief sets and $-$ $(= -_{(\div)})$ its corresponding operator on belief bases. Then $\div$ satisfies $(K \div 1)$–$(K \div 8)$ if and only if $-$ satisfies (C1)–(C7).*

Furthermore, it turns out that the contraction of a formula φ by a conjunction $(\alpha \wedge \beta)$ of formulas can have only three different outcomes (up to logical equivalence). Such a trichotomy result is similar to the one obtained in the classical AGM framework [2].

Proposition 9. *In the presence of (C1)–(C5), the conjunction of (C6) and (C7) is equivalent to (Tri):*

$$\textbf{(Tri)} \quad \varphi - (\alpha \wedge \beta) \equiv \begin{cases} \varphi - \alpha \text{ or} \\ \varphi - \beta \text{ or} \\ (\varphi - \alpha) \vee (\varphi - \beta) \end{cases}$$

Looking at the proof of this proposition, we can observe that if $\varphi - (\alpha \wedge \beta) \vdash \beta$, then $\varphi - (\alpha \wedge \beta) \equiv \varphi - \alpha$. This means that when β is more entrenched (i.e., more important/plausible) than α, then when we are asked to remove $\alpha \wedge \beta$, the contraction to be done is exactly the contraction by α alone.

6. A correspondence between contraction and revision

Now that we have defined postulates for contraction operators on belief bases, we can check that the contraction operators satisfying these postulates correspond to revision operators in the sense of Katsuno and Mendelzon [6].

We first show that Levi and Harper identities hold also in this propositional setting. We note $\circ_{(-)}$ the revision operator on belief bases defined from $-$ via Levi identity and $-_{(\circ)}$ the contraction operator on belief bases defined from $\circ$ via Harper identity.

Definition 4. Levi and Harper identities for belief bases can be expressed as follows. Let φ and α be two formulas from L:

$\varphi \circ_{(-)} \alpha = (\varphi - \neg\alpha) \wedge \alpha$ (**Levi identity**)

$\varphi -_{(\circ)} \alpha = \varphi \vee (\varphi \circ \neg\alpha)$ (**Harper identity**)

These identities are direct translations of usual ones. Remark simply that the expansion operator usually used in Levi identity is translated here (as expected) by a conjunction.

Operators obtained by means of these identities satisfy the expected properties:

Proposition 10. *If the contraction operator $-$ satisfies (C1)–(C5) then the revision operator $\circ_{(-)}$ defined using Levi identity satisfies (R1)–(R4). Furthermore if (C6) is satisfied by $-$, then (R5) is satisfied by $\circ_{(-)}$, and if (C7) is satisfied by $-$, then (R6) is satisfied $\circ_{(-)}$.*

Therefore, the KM revision operators for propositional belief bases can be defined using Levi identity from the contraction operators for propositional belief bases we have introduced. Reciprocally, contraction operators for propositional belief bases can be defined using Harper identity, from KM revision operators for belief bases.

Proposition 11. *If the revision operator $\circ$ satisfies (R1)–(R4) then the contraction operator $-_{(\circ)}$ defined using Harper identity satisfies (C1)–(C5). Furthermore, if (R5) is satisfied by $\circ$, then (C6) is satisfied by $-_{(\circ)}$ and if (R6) is satisfied by $\circ$, then (C7) is satisfied by $-_{(\circ)}$.*

The following proposition shows that if we use a revision operator $\circ$ to define, via Harper identity, a contraction operator $-_{(\circ)}$, then the revision operator defined via Levi identity, from $-_{(\circ)}$ is the initial revision operator $\circ$. The other way around, if we use a contraction operator $-$ to define, via Levi identity a revision operator $\circ_{(-)}$, then the contraction operator defined via Harper identity from $\circ_{(-)}$ is the initial contraction operator $-$.

Proposition 12.

- *if $\circ$ is a revision operator, then $\circ_{(-_{(\circ)})} = \circ$*
- *if $-$ is a contraction operator, then $-_{(\circ_{(-)})} = -$*

Our postulates for contraction of belief bases are thus in close correspondence with the revision postulates for belief bases defined by Katsuno and Mendelzon.

7. A representation theorem

Let us now check that we can state a representation theorem for contraction within the framework of finite propositional logic, which is a counterpart of the representation theorem of Katsuno and Mendelzon for revision.

Lemma 13. *Let $-$ be a contraction operator satisfying (C1)–(C7). Let I be an interpretation and φ be a formula.*

$$\varphi - \neg\alpha_{\{I\}} \equiv \varphi \vee \alpha_{\{I\}}$$

This lemma states that with a formula α, with only one model, the contraction of φ by the negation of α is equivalent to the disjunction of φ and α (the interesting case is when α does not imply φ).

The idea of the representation theorem is to express the set of models of the contraction of a base φ by a new piece of information α as the union of the models of φ and of the minimal counter-models of α with respect to $\leq_\varphi$.

Theorem 14. *A contraction operator $-$ satisfies the postulates **(C1)–(C7)** if and only if there exists a faithful assignment that associates with each belief base φ a total pre-order $\leq_\varphi$ on the set of all interpretations such that*

$$Mod(\varphi - \alpha) = Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi)$$

Note that a similar construction has been used in [20] for the contraction of Horn belief sets.

8. An example of propositional contraction operator

Let I and J be two interpretations and φ be a formula from L. The Dalal distance [21] d_H (that is a Hamming distance [22] on interpretations) between I and J is the number of propositional symbols on which the two interpretations differ:

$$d_H(I, J) = |\{x \in \mathcal{A} | I(x) \neq J(x)\}|$$

The distance between the formula φ and the interpretation I is defined as

$$d(\varphi, I) = \min_{J \in Mod(\varphi)} d(J, I)$$

We can now define a faithful assignment $\leq_\varphi$ such that $I \leq_\varphi J$ if and only if $d(\varphi, I) \leq_\varphi d(\varphi, J)$.

This distance, and the corresponding faithful assignment can be used to define a revision operator, known as Dalal revision operator $\circ_D$ [21], defined by $Mod(\varphi \circ_D \alpha) = min(Mod(\alpha), \leq_\varphi)$. From Theorem 4, $\circ_D$ satisfies (R1)–(R6).

But we can also define a Dalal contraction operator $-_D$ by $Mod(\varphi -_D \alpha) = Mod(\varphi) \cup Mod(min(Mod(\neg\alpha), \leq_\varphi))$. And thanks to Theorem 14, we known that Dalal contraction operator $-_D$ satisfies (C1)–(C7).

Let us consider the following example for illustrating both operators.

Example 1. Let P be a set consisting of the three propositional variables *bird*, *flies*, *has_a_beak*, ordered in this way. Looking at a bird, we may first assume that it can fly (hence the belief base is $\varphi = bird \wedge flies$). Getting closer to the bird, we realize that it looks as a penguin, and penguins cannot fly. So we must revise our beliefs with the new piece of information $\alpha = \neg flies$. The third column of the following table makes precise the Dalal distance between each model of α ($Mod(\alpha) = \{000, 001, 100, 101\}$) and φ ($Mod(\varphi) = \{110, 111\}$).

	110	111	φ
000	2	3	2
001	3	2	2
100	1	2	1
101	2	1	1

The set of models of the result of the revision consists of the models of α which are the closest to φ: $Mod(\varphi \circ_D \alpha) = \{100, 101\}$, hence $\varphi \circ_D \alpha \equiv bird \wedge \neg flies$.

Finally, it is not clear that the observed bird is a penguin, it could also be a hawk and hawks can fly. So we now want to contract our belief base with α. The third column of the following table gives the Dalal distance between each model of $\neg\alpha$ ($Mod(\neg\alpha) = \{011, 010, 110, 111\}$) and $\varphi \circ_D \alpha$ ($Mod(\varphi \circ_D \alpha) = \{100, 101\}$).

	100	101	$\varphi \circ_D \alpha$
011	3	2	2
010	2	3	2
110	1	2	1
111	2	1	1

The set of models of the contracted base is the union of models of $\varphi \circ_D \alpha$ with the set of models of $\neg\alpha$ which are the closest to $\varphi \circ_D \alpha$: $Mod((\varphi \circ_D \alpha) -_D \alpha) = \{100, 101, 111, 110\}$, hence $(\varphi \circ_D \alpha) -_D \alpha \equiv bird$.

We now illustrate the representation theorem (Theorem 14) on Fig. 1. The interpretations (depicted as dots) are located at different levels L_i, two interpretations at the same level are equally plausible (i.e., $I \simeq_\varphi J$) and an interpretation I appearing at a lower level than another J is strictly more plausible (i.e., $I <_\varphi J$). The interpretations appearing at the lowest level (L_0) are the models of the belief base $\psi = \varphi \circ_D \alpha$.

When ψ is contracted by α, the result consists of all models of ψ to which are added the most plausible models of $\neg\alpha$ according to the pre-order of plausibility $\leq_\psi$ associated with ψ by the faithful assignment. This represents the minimal change required for not implying α any longer. These interpretations are located at L_1 on Fig. 1. The minimal interpretations of $\neg\alpha$ (at L_1) are added to the interpretations of ψ (at L_0).

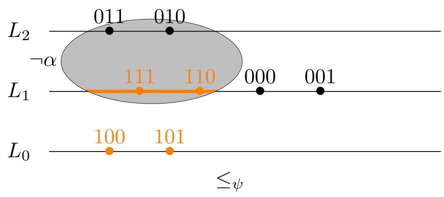

Fig. 1. Contraction of ψ by α.

9. Conclusion and perspectives

In this paper we have investigated belief contraction in the framework of finite propositional logic. The aim was, like in Katsuno and Mendelzon work for revision, to define postulates for contraction operators. We have checked that the operators of contraction characterized by our postulates correspond to the revision operators characterized by Katsuno and Mendelzon postulates. We also gave a representation theorem in terms of faithful assignment.

Defining iterated contraction operators is the main perspective of this work. Indeed, the translation by Katsuno and Mendelzon of the AGM postulates is the basis of the study of iterated revision operators following Darwiche and Pearl [4,15,16,23]. There has been very few work on iterated contraction: to the best of our knowledge, only one paper [24] addresses this problem, but in a different framework from the one of Darwiche and Pearl. Defining "Darwiche and Pearl"-like iterated contraction operators will be a first step in the investigation of the relationships between [24] and [4].

Appendix

Proposition 5. *In the finite propositional setting, the mapping Cn from $L/\equiv$ to the set of all belief sets, associating a belief set $K = Cn(\varphi)$ with every belief base represented by a formula φ considered up to logical equivalence, is bijective.*

Proof. Let $E = L/\equiv$ be the set of formulas considered up to logical equivalence and F be the set of belief sets. We want to show that Cn is a bijection from E to F.

We first show that Cn is a surjection from E to F. This amounts to show that for each belief set K of F, we can construct a formula φ such that φ is equivalent the conjunction of all formulas of K. We can suppose w.l.o.g. that every formula α of K is a CNF formula, so that K is equivalent to the (infinite) set of all clauses δ appearing in the representation of at least one α of K. In finite propositional logic, this infinite set of clauses is equivalent to the finite set of its logically strongest elements. And since this latter set is finite, the conjunction of its elements is a formula φ satisfying the expected condition: $K = Cn(\varphi)$.

It remains to show that Cn is an injection from E to F. Consider any φ and ψ in E such that $\varphi \not\equiv \psi$. Since $\varphi \not\equiv \psi$, there exists a model I of φ which is not a model of ψ, or a model of ψ which is not a model if φ. Suppose that I is a model of $\varphi \wedge \neg\psi$ (the remaining case is symmetric to this one). On the one hand, since $I \models \varphi$ it cannot be the case that $\varphi \models \neg\alpha_I$, hence $\neg\alpha_I \notin Cn(\varphi)$. On the other hand, since $I \models \neg\psi$, we have $\psi \models \neg\alpha_I$, hence $\neg\alpha_I \in Cn(\psi)$. Accordingly, $Cn(\varphi) \neq Cn(\psi)$, and this concludes the proof. $\square$

Corollary 6. *We have $\varphi \equiv \varphi_{K_\varphi}$. Similarly we have $K = K_{\varphi_K}$.*

Proof. This result follows trivially from the fact that Cn is a bijection, as stated by Proposition 5. Indeed, from Proposition 5, $\varphi_{K_\varphi} = Cn^{-1}(K_\varphi) = Cn^{-1}(Cn(\varphi)) = \varphi$. And $K_{\varphi_K} = Cn(\varphi_K) = Cn(Cn^{-1}(K)) = K$. $\square$

Proposition 7. *We have $-_{(\div_{(-)})} = -$. Similarly we have $\div_{(-_{(\div)})} = \div$.*

Proof. Let φ and α be two formulas from L. We want to show that $\varphi -_{(\div_{(-)})} \alpha \equiv \varphi - \alpha$. $\varphi -_{(\div_{(-)})} \alpha = \varphi_{K_\varphi \div_{(-)} \alpha} = \varphi_{K_{\varphi_{K_\varphi} - \alpha}}$ from Definition 3. $\varphi_{K_{\varphi_{K_\varphi} - \alpha}} \equiv \varphi_{K_{\varphi - \alpha}}$ using Corollary 6. $\varphi_{K_{\varphi-\alpha}} = Cn^{-1}(K_{\varphi-\alpha}) = Cn^{-1}(Cn(\varphi - \alpha)) \equiv \varphi - \alpha$.

So, we have $\varphi -_{(\div_{(-)})} \alpha \equiv \varphi - \alpha$.

We now show that $K \div_{(-_{(\div)})} \alpha = K \div \alpha$.

$K \div_{(-_{(\div)})} \alpha = K_{\varphi_K -_{(\div)} \alpha} = K_{\varphi_{K_{\varphi_K} \div \alpha}}$ from Definition 3. $K_{\varphi_{K_{\varphi_K} \div \alpha}} = K_{\varphi_{K \div \alpha}}$ using Corollary 6. $K_{\varphi_{K\div\alpha}} = Cn(\varphi_{K\div\alpha}) = Cn(Cn^{-1}(K \div \alpha)) = K \div \alpha$.

So, we have $K \div_{(-_{(\div)})} \alpha = K \div \alpha$. $\square$

Proposition 8. *Let $\div$ be a contraction operator on belief sets and $-$ $(= -_{(\div)})$ its corresponding operator on belief bases. Then $\div$ satisfies $(K \div 1)$–$(K \div 8)$ if and only if $-$ satisfies (C1)–(C7).*

Proof. Let K be a belief set, φ be a formula from L such that $K = Cn(\varphi)$, and α, β, φ_1, φ_2, α_1, and α_2 be formulas from L.

- We first show that $\div$ satisfies $(K\div 2)$ if and only if $-$ satisfies (C2):
  $K\div\alpha\subseteq K \Leftrightarrow Cn(\varphi)\div\alpha\subseteq Cn(\varphi)\Leftrightarrow Cn(\varphi-\alpha)\subseteq Cn(\varphi)\Leftrightarrow \varphi\vdash\varphi-\alpha$
- We show that if $\div$ satisfies $(K\div 3)$ then $-$ satisfies (C2) and if $-$ satisfies (C1) and (C2) then $\div$ satisfies $(K\div 3)$:
  - We first show that $(K\div 3)$ implies (C2).
    If $\alpha\notin K$ then $K\div\alpha=K$. So by $(K\div 3)$, if $\alpha\notin K$ then $K\subseteq K\div\alpha$, meaning for the corresponding operator $-$ that if $\varphi\nvdash\alpha$ then $\varphi-\alpha\vdash\varphi$.
  - We now show that (C1) and (C2) imply $(K\div 3)$.
    (If $\varphi\nvdash\alpha$ then $\varphi-\alpha\vdash\varphi$) and $\varphi\vdash\varphi-\alpha\Rightarrow$ (If $\alpha\notin K$ then $K\subseteq K\div\alpha$) and $K\div\alpha\subseteq K\Rightarrow$ If $\alpha\notin K$ then ($K\subseteq K\div\alpha$ and $K\div\alpha\subseteq K$) $\Rightarrow$ If $\alpha\notin K$ then $K=K\div\alpha$
- We show that $\div$ satisfies $(K\div 4)$ if and only if $-$ satisfies (C3):
  If $\nvdash\alpha$ then $\alpha\notin K\div\alpha\Leftrightarrow$ if $\alpha\in K\div\alpha$ then $\vdash\alpha$. $\alpha\in K\div\alpha\Leftrightarrow\alpha\in Cn(\varphi)\div\alpha\Leftrightarrow\varphi-\alpha\vdash\alpha$. So if $\alpha\in K\div\alpha$ then $\vdash\alpha\Leftrightarrow$ if $\varphi-\alpha\vdash\alpha$ then $\vdash\alpha$
- We show that if $-$ satisfies (C4) then $\div$ satisfies $(K\div 5)$, and that if $\div$ satisfies $(K\div 5)$ and $(K\div 3)$ then $-$ satisfies (C4):
  If $\alpha\in K$ then $K\subseteq(K\div\alpha)+\alpha$
  $\alpha\in K\Leftrightarrow\varphi\vdash\alpha$
  $K\subseteq(K\div\alpha)+\alpha\Leftrightarrow Cn(\varphi)\subseteq(Cn(\varphi)\div\alpha)+\alpha\Leftrightarrow Cn(\varphi)\subseteq Cn((\varphi-\alpha)\wedge\alpha)\Leftrightarrow(\varphi-\alpha)\wedge\alpha\vdash\varphi$. So if $\alpha\in K$ then $K\subseteq(K\div\alpha)+\alpha\Leftrightarrow$ if $\varphi\vdash\alpha$ then $(\varphi-\alpha)\wedge\alpha\vdash\varphi$.
  If $\alpha\notin K$ then by $(K\div 3)$ $K\subseteq(K\div\alpha)$, so $K\subseteq(K\div\alpha)+\alpha$, and we show similarly as above that if $\varphi\nvdash\alpha$ then $(\varphi-\alpha)\wedge\alpha\vdash\varphi$.
- We show that $\div$ satisfies $(K\div 6)$ if and only if $-$ satisfies (C5):
  Let φ_1 and φ_2 be two formulas such that $Cn(\varphi_1)=Cn(\varphi_2)=K$. If $\alpha_1\equiv\alpha_2$ then $K\div\alpha_1=K\div\alpha_2\Leftrightarrow$ If $\alpha_1\equiv\alpha_2$ and $Cn(\varphi_1)=Cn(\varphi_2)$ then $Cn(\varphi_1)\div\alpha_1=Cn(\varphi_2)\div\alpha_2\Leftrightarrow$ If $\alpha_1\equiv\alpha_2$ and $\varphi_1\equiv\varphi_2$ then $\varphi_1-\alpha_1\equiv\varphi_2-\alpha_2$
- We show that $\div$ satisfies $(K\div 7)$ if and only if $-$ satisfies (C6):
  $(K\div\alpha)\cap(K\div\beta)\subseteq K\div(\alpha\wedge\beta)$
  $(K\div\alpha)\cap(K\div\beta)=Cn(\varphi-\alpha)\cap Cn(\varphi-\beta)=Cn((\varphi-\alpha)\vee(\varphi-\beta))\equiv(\varphi-\alpha)\vee(\varphi-\beta)$
  $K\div(\alpha\wedge\beta)\equiv\varphi-(\alpha\wedge\beta)$
  So $(K\div\alpha)\cap(K\div\beta)\subseteq K\div(\alpha\wedge\beta)\Leftrightarrow\varphi-(\alpha\wedge\beta)\vdash(\varphi-\alpha)\vee(\varphi-\beta)$
- We finally show that $\div$ satisfies $(K\div 8)$ if and only if $-$ satisfies (C7):
  If $\alpha\notin K\div(\alpha\wedge\beta)$ then $K\div(\alpha\wedge\beta)\subseteq K\div\alpha$
  $\alpha\notin K\div(\alpha\wedge\beta)\Leftrightarrow\varphi-(\alpha\wedge\beta)\nvdash\alpha$
  $K\div(\alpha\wedge\beta)\subseteq K\div\alpha\Leftrightarrow Cn(\varphi)\div(\alpha\wedge\beta)\subseteq Cn(\varphi)\div\alpha\Leftrightarrow\varphi-\alpha\vdash\varphi-(\alpha\wedge\beta)$
  So if $\alpha\notin K\div(\alpha\wedge\beta)$ then $K\div(\alpha\wedge\beta)\subseteq K\div\alpha\Leftrightarrow$ If $\varphi-(\alpha\wedge\beta)\nvdash\alpha$ then $\varphi-\alpha\vdash\varphi-(\alpha\wedge\beta)$ □

Proposition 9. *In the presence of (C1)–(C5), the conjunction of (C6) and (C7) is equivalent to (Tri):*

$$\textbf{\textit{(Tri)}}\quad \varphi-(\alpha\wedge\beta)\equiv\begin{cases}\varphi-\alpha \text{ or}\\ \varphi-\beta \text{ or}\\ (\varphi-\alpha)\vee(\varphi-\beta)\end{cases}$$

Proof. Let φ, α, and β be formulas from L. ($\Rightarrow$)
Suppose that (C6) and (C7) are satisfied:
(C6): $\varphi-(\alpha\wedge\beta)\vdash(\varphi-\alpha)\vee(\varphi-\beta)$
(C7): If $\varphi-(\alpha\wedge\beta)\nvdash\alpha$ then $\varphi-\alpha\vdash\varphi-(\alpha\wedge\beta)$
If $\alpha\wedge\beta$ is valid, then α is valid and β is valid. We have therefore $\alpha\wedge\beta\equiv\alpha\equiv\beta$. From (C5), we must have: $\varphi-(\alpha\wedge\beta)\equiv\varphi-\alpha\equiv\varphi-\beta\equiv(\varphi-\alpha)\vee(\varphi-\beta)$.
If $\alpha\wedge\beta$ is not valid (i.e., α is not valid or β is not valid) then by (C3), $\varphi-(\alpha\wedge\beta)\nvdash\alpha\wedge\beta$.
So there are three cases:

1. $\varphi-(\alpha\wedge\beta)\nvdash\alpha$ and $\varphi-(\alpha\wedge\beta)\vdash\beta$
2. $\varphi-(\alpha\wedge\beta)\vdash\alpha$ and $\varphi-(\alpha\wedge\beta)\nvdash\beta$
3. $\varphi-(\alpha\wedge\beta)\nvdash\alpha$ and $\varphi-(\alpha\wedge\beta)\nvdash\beta$

We consider the three following cases:

1. (Hyp1) Suppose that

$$\varphi-(\alpha\wedge\beta)\nvdash\alpha \qquad (\lhd)$$
$$\varphi-(\alpha\wedge\beta)\vdash\beta \qquad (\rhd)$$

From (C6), $\varphi-(\alpha\wedge\beta)\vdash(\varphi-\alpha)\vee(\varphi-\beta)$. From ($\rhd$), $\varphi-(\alpha\wedge\beta)\vdash[(\varphi-\alpha)\vee(\varphi-\beta)]\wedge\beta$. Thus,

$\varphi - (\alpha \wedge \beta) \vdash [(\varphi - \alpha) \wedge \beta] \vee [(\varphi - \beta) \wedge \beta] \qquad (\diamond)$

From (C4) and (C1), $(\varphi - \beta) \wedge \beta \vdash \beta$ and $\varphi \vdash \varphi - \alpha$. Thus, $(\varphi - \beta) \wedge \beta \vdash \varphi - \alpha$. And so, $(\varphi - \beta) \wedge \beta \vdash (\varphi - \alpha) \wedge \beta$. Which implies that $[(\varphi - \alpha) \wedge \beta] \vee [(\varphi - \beta) \wedge \beta] \equiv (\varphi - \alpha) \wedge \beta$.
From $(\diamond)$ we derive that $\varphi - (\alpha \wedge \beta) \vdash (\varphi - \alpha) \wedge \beta$. Thus

$\varphi - (\alpha \wedge \beta) \vdash (\varphi - \alpha) \qquad (\triangle)$

From (C7) and $(\triangleleft)$, we derive that

$\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta) \qquad (\triangledown)$

Finally, from $(\triangle)$ and $(\triangledown)$ we deduce that $\varphi - (\alpha \wedge \beta) \equiv (\varphi - \alpha)$.

$$\text{(C6) and (C7)} \Rightarrow \varphi - (\alpha \wedge \beta) \equiv (\varphi - \alpha) \tag{1}$$

2. (Hyp2) Suppose that $\varphi - (\alpha \wedge \beta) \nvdash \beta$ and $\varphi - (\alpha \wedge \beta) \vdash \alpha$.
This case being symmetric with the previous one, we have analogously:

$$\text{(C6) and (C7)} \Rightarrow \varphi - (\alpha \wedge \beta) \equiv (\varphi - \beta) \tag{2}$$

3. (Hyp3) Suppose that $\varphi - (\alpha \wedge \beta) \nvdash \alpha$ and $\varphi - (\alpha \wedge \beta) \nvdash \beta$.
(C7) and $\varphi - (\alpha \wedge \beta) \nvdash \alpha \Rightarrow \varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$. (C7) and $\varphi - (\alpha \wedge \beta) \nvdash \beta \Rightarrow \varphi - \beta \vdash \varphi - (\alpha \wedge \beta)$. $\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$ and $\varphi - \beta \vdash \varphi - (\alpha \wedge \beta) \Rightarrow (\varphi - \alpha) \vee (\varphi - \beta) \vdash \varphi - (\alpha \wedge \beta)$. (C6) gives the converse implication.

$$\text{(C6) and (C7)} \Rightarrow \varphi - (\alpha \wedge \beta) \equiv (\varphi - \alpha) \vee (\varphi - \beta) \tag{3}$$

(1) + (2) + (3) implies that, in presence of (C1)–(C5),

$$\text{(C6) and (C7)} \Rightarrow \varphi - (\alpha \wedge \beta) \equiv \begin{cases} \varphi - \alpha \text{ or} \\ \varphi - \beta \text{ or} \\ (\varphi - \alpha) \vee (\varphi - \beta) \end{cases} \tag{Δ_1}$$

$(\Leftarrow)$

$$\varphi - (\alpha \wedge \beta) \equiv \begin{cases} \varphi - \alpha \text{ or} \\ \varphi - \beta \text{ or} \\ (\varphi - \alpha) \vee (\varphi - \beta) \end{cases}$$

Three cases must be considered:

1. $\varphi - (\alpha \wedge \beta) \equiv \varphi - \alpha$
$\varphi - \alpha \vdash (\varphi - \alpha) \vee (\varphi - \beta) \Rightarrow \varphi - (\alpha \wedge \beta) \vdash (\varphi - \alpha) \vee (\varphi - \beta)$ (C6)
As $\varphi - (\alpha \wedge \beta) \equiv \varphi - \alpha$, we have $\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$ so (C7) is trivially satisfied.

$$\varphi - (\alpha \wedge \beta) \equiv \varphi - \alpha \Rightarrow \text{(C6) and (C7)} \tag{4}$$

2. $\varphi - (\alpha \wedge \beta) \equiv \varphi - \beta$
This case being symmetric with the previous one, we obtain analogously:

$$\varphi - (\alpha \wedge \beta) \equiv \varphi - \beta \Rightarrow \text{(C6) and (C7)} \tag{5}$$

3. $\varphi - (\alpha \wedge \beta) \equiv (\varphi - \alpha) \vee (\varphi - \beta) \Rightarrow \varphi - (\alpha \wedge \beta) \vdash (\varphi - \alpha) \vee (\varphi - \beta)$ (C6)
and $(\varphi - \alpha) \vee (\varphi - \beta) \vdash \varphi - (\alpha \wedge \beta)$ which implies that $\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$, so (C7) is trivially satisfied.

$$\varphi - (\alpha \wedge \beta) \equiv (\varphi - \alpha) \vee (\varphi - \beta) \tag{6}$$

(4) + (5) + (6) implies that, in presence of (C1)–(C5),

$$\varphi - (\alpha \wedge \beta) \equiv \begin{cases} \varphi - \alpha \text{ or} \\ \varphi - \beta \text{ or} \\ (\varphi - \alpha) \vee (\varphi - \beta) \end{cases} \Rightarrow \text{(C6) and (C7)} \qquad \square \tag{Δ_2}$$

Proposition 10. *If the contraction operator $-$ satisfies (C1)–(C5) then the revision operator $\circ_{(-)}$ defined using Levi identity satisfies (R1)–(R4). Furthermore if (C6) is satisfied by $-$, then (R5) is satisfied by $\circ_{(-)}$, and if (C7) is satisfied by $-$, then (R6) is satisfied $\circ_{(-)}$.*

Proof. Let φ, α, β, and γ be formulas from L. Assume that (C1)–(C5) are satisfied.

- (R1): we trivially have $(\varphi - \neg\alpha) \wedge \alpha \vdash \alpha$. Which gives us $\varphi \circ \alpha \vdash \alpha$.
- (R2): assume that $\varphi \wedge \alpha$ is consistent, so we have $\varphi \nvdash \neg\alpha$. According to (C1) and (C2), we have $\varphi - \neg\alpha \equiv \varphi$. Hence $(\varphi - \neg\alpha) \wedge \alpha \equiv \varphi \wedge \alpha$. Which gives us $\varphi \circ \alpha \equiv \varphi \wedge \alpha$.
- (R3): assume that α is consistent, so we have $\nvdash \neg\alpha$. According to (C3), we have $\varphi - \neg\alpha \nvdash \neg\alpha$. Hence $(\varphi - \neg\alpha) \wedge \alpha \nvdash \bot$. Which gives us $\varphi \circ \alpha \nvdash \bot$.
- (R4): suppose now $\varphi_1 \equiv \varphi_2$ and $\alpha_1 \equiv \alpha_2$. According to (C5), we have $\varphi_1 - \neg\alpha_1 \equiv \varphi_2 - \neg\alpha_2$. So $(\varphi_1 - \neg\alpha_1) \wedge \alpha_1 \equiv (\varphi_2 - \neg\alpha_2) \wedge \alpha_2$. Which gives us $\varphi_1 \circ \alpha_1 \equiv \varphi_2 \circ \alpha_2$.

Suppose that (C6) is satisfied and let γ be such that $\varphi \circ (\alpha \wedge \beta) \vdash \gamma$. We want to show that $(\varphi \circ \alpha) \wedge \beta \vdash \gamma$.

As $\neg\alpha \equiv \neg(\alpha \wedge \beta) \wedge (\alpha \to \beta)$, according to (C5), it is sufficient to show, using Levi identity, that $[\varphi - (\neg(\alpha \wedge \beta) \wedge (\alpha \to \beta))] \wedge \alpha \wedge \beta \vdash \gamma$. According to (C6), it is enough to show that $[\varphi - \neg(\alpha \wedge \beta)] \wedge \alpha \wedge \beta \vdash \gamma$ and $[\varphi - (\alpha \to \beta)] \wedge \alpha \wedge \beta \vdash \gamma$.

- As $(\varphi - \neg(\alpha \wedge \beta)) \wedge (\alpha \wedge \beta) \equiv \varphi \circ (\alpha \wedge \beta)$ according to Levi identity, by hypothesis, $\varphi \circ (\alpha \wedge \beta) \vdash \gamma$. So we also have $(\varphi \circ \alpha) \wedge \beta \vdash \gamma$, and the conclusion follows.
- A consequence of (R2) is $\varphi \wedge \alpha \vdash \varphi \circ \alpha$ (R2.2)
  As $\varphi \circ (\alpha \wedge \beta) \vdash \gamma$, according to (R2.2) $\varphi \wedge (\alpha \wedge \beta) \vdash \gamma$. Hence $\varphi \vdash (\alpha \wedge \beta) \to \gamma$.
  There are two cases:
  - If $\varphi \nvdash (\alpha \to \beta)$, according to (C2) we have $\varphi - (\alpha \to \beta) \vdash \varphi$. Hence $\varphi - (\alpha \to \beta) \vdash (\alpha \wedge \beta) \to \gamma$. So $[\varphi - (\alpha \to \beta)] \wedge \alpha \wedge \beta \vdash \gamma$.
  - If $\varphi \vdash (\alpha \to \beta)$, according to (C4) we have $(\varphi - (\alpha \to \beta)) \wedge (\alpha \to \beta) \vdash \varphi$. Hence $(\varphi - (\alpha \to \beta)) \wedge (\alpha \to \beta) \vdash (\alpha \wedge \beta) \to \gamma$. So $\varphi - (\alpha \to \beta) \vdash (\alpha \to \beta) \to ((\alpha \wedge \beta) \to \gamma)$. Now we have $(\alpha \to \beta) \to ((\alpha \wedge \beta) \to \gamma) \equiv \neg(\neg\alpha \vee \beta) \vee (\neg(\alpha \wedge \beta) \vee \gamma) \equiv (\alpha \wedge \neg\beta) \vee \neg\alpha \vee \neg\beta \vee \gamma \equiv \neg\alpha \vee \neg\beta \vee \gamma$. Which gives us $\varphi - (\alpha \to \beta) \vdash (\alpha \wedge \beta) \to \gamma$. So we have $[\varphi - (\alpha \to \beta)] \wedge (\alpha \wedge \beta) \vdash \gamma$

Assume that (C7) is satisfied. We want to show that (R6) is satisfied. As $\neg\alpha \equiv (\neg\alpha \vee \neg\beta) \wedge \neg\alpha$, according to (C5), we have $\varphi - \neg\alpha \equiv \varphi - [(\neg\alpha \vee \neg\beta) \wedge \neg\alpha]$. Now suppose that $\varphi \circ \alpha \nvdash \neg\beta$. As $\varphi \circ \alpha \equiv (\varphi - \neg\alpha) \wedge \alpha$, we have $\varphi - \neg\alpha \nvdash \neg\alpha \vee \neg\beta$. (C5) gives us $\varphi - [(\neg\alpha \vee \neg\beta) \wedge \neg\alpha] \nvdash \neg\alpha \vee \neg\beta$. According to (C7), $\varphi - (\neg\alpha \vee \neg\beta) \vdash \varphi - [(\neg\alpha \vee \neg\beta) \wedge \neg\alpha]$. Hence $(\varphi - \neg(\alpha \wedge \beta)) \wedge \alpha \wedge \beta \vdash (\varphi - \neg\alpha) \wedge \alpha \wedge \beta$. So $\varphi \circ (\alpha \wedge \beta) \vdash (\varphi \circ \alpha) \wedge \beta$. □

Proposition 11. *If the revision operator $\circ$ satisfies (R1)–(R4) then the contraction operator $-_{(\circ)}$ defined using Harper identity satisfies (C1)–(C5). Furthermore, if (R5) is satisfied by $\circ$, then (C6) is satisfied by $-_{(\circ)}$ and if (R6) is satisfied by $\circ$, then (C7) is satisfied by $-_{(\circ)}$.*

Proof. Let φ, α, β, and γ be formulas from L. Assume that (R1)–(R4) are satisfied.

- (C1) is satisfied: Clearly $\varphi \vdash \varphi \vee (\varphi \circ \neg\alpha)$. Which gives us $\varphi \vdash \varphi - \alpha$ using Harper identity.
- (C2): Assume that $\varphi \nvdash \alpha$. According to (R2), $\varphi \circ \neg\alpha \vdash \varphi \wedge \neg\alpha$. Hence $\varphi \circ \neg\alpha \vdash \varphi$. So $\varphi \vee (\varphi \circ \neg\alpha) \vdash \varphi$. Which gives us $\varphi - \alpha \vdash \varphi$.
- (C3): Assume that $\nvdash \alpha$. We have that $\neg\alpha$ is consistent. So according to (R3), $\varphi \circ \neg\alpha$ is consistent. Moreover, according to (R1), $\varphi \circ \neg\alpha \vdash \neg\alpha$. So $\varphi \circ \neg\alpha \nvdash \alpha$. Hence $\varphi \vee (\varphi \circ \neg\alpha) \nvdash \alpha$. Harper identity gives us $\varphi - \alpha \nvdash \alpha$.
- (C4): From (R1) we have that $\varphi \circ \neg\alpha \vdash \neg\alpha$. So this gives $(\varphi \circ \neg\alpha) \vee \varphi \vdash \neg\alpha \vee \varphi$, that gives $[(\varphi \circ \neg\alpha) \vee \varphi] \wedge \alpha \vdash \varphi$. From Harper identity we have that $(\varphi \circ \neg\alpha) \vee \varphi = \varphi - \alpha$, so $[\varphi - \alpha] \wedge \alpha \vdash \varphi$.
- (C5): Assume $\varphi_1 \equiv \varphi_2$ and $\alpha_1 \equiv \alpha_2$. According to (R4), we have $\varphi_1 \circ \neg\alpha_1 \equiv \varphi_2 \circ \neg\alpha_2$. So $\varphi_1 \vee (\varphi_1 \circ \neg\alpha_1) \equiv \varphi_2 \vee (\varphi_2 \circ \neg\alpha_2)$. Which gives us $\varphi_1 - \alpha_1 \equiv \varphi_2 - \alpha_2$.
- (C6): Assume that (R5) is satisfied and let γ be such that $(\varphi - \alpha) \vee (\varphi - \beta) \vdash \gamma$. We want to show that $\varphi - (\alpha \wedge \beta) \vdash \gamma$.
  - Suppose that $\varphi \vdash \alpha \wedge \beta$. As $\alpha \equiv \neg((\neg\alpha \vee \neg\beta) \wedge \neg\alpha)$, from (R4), we have $\varphi - \alpha \equiv \varphi - \neg((\neg\alpha \vee \neg\beta) \wedge \neg\alpha)$. Hence $\varphi - \neg((\neg\alpha \vee \neg\beta) \wedge \neg\alpha) \vdash \gamma$. According to (R5), we have $[\varphi \circ (\neg\alpha \vee \neg\beta)] \wedge \neg\alpha \vdash \varphi \circ [(\neg\alpha \vee \neg\beta) \wedge \neg\alpha]$. Levi identity gives that $\varphi \circ [(\neg\alpha \vee \neg\beta) \wedge \neg\alpha] \vdash \varphi - \neg((\neg\alpha \vee \neg\beta) \wedge \neg\alpha)$. Hence $\varphi \circ [(\neg\alpha \vee \neg\beta) \wedge \neg\alpha] \vdash \gamma$. Which gives us, from (R5), $[\varphi \circ (\neg\alpha \vee \neg\beta)] \wedge \neg\alpha \vdash \gamma$. Equivalently $[\varphi \circ (\neg\alpha \vee \neg\beta)] \vdash \alpha \vee \gamma$. By symmetry on α and β, we have $[\varphi \circ (\neg\alpha \vee \neg\beta)] \vdash \beta \vee \gamma$. Moreover, from (R1), we have $\varphi \circ (\neg\alpha \vee \neg\beta) \vdash \neg\alpha \vee \neg\beta$. We can deduce that $\varphi \circ (\neg\alpha \vee \neg\beta) \vdash \gamma$. Levi identity gives that $(\varphi - (\alpha \wedge \beta)) \wedge \neg(\alpha \wedge \beta) \vdash \gamma$ (*). Having assumed that $\varphi \vdash \alpha \wedge \beta$, (C4) gives us $(\varphi - (\alpha \wedge \beta)) \wedge (\alpha \wedge \beta) \vdash \varphi$. According to (C1) $\varphi \vdash \varphi - \alpha$. As we assumed $\varphi - \alpha \vdash \gamma$ and $\varphi - \beta \vdash \gamma$, we have $\varphi \vdash \gamma$. By transitivity of $\vdash$, $(\varphi - (\alpha \wedge \beta)) \wedge (\alpha \wedge \beta) \vdash \gamma$ (**). According to (*) and (**), we have $\varphi - (\alpha \wedge \beta) \vdash \gamma$.
  - Suppose now that $\varphi \nvdash \alpha \wedge \beta$, from (C2) $\varphi - (\alpha \wedge \beta) \vdash \varphi$. Besides, according to (C1), $\varphi \vdash \varphi - \alpha$. We have $\varphi \vdash (\varphi - \alpha) \vee (\varphi - \beta)$. Hence $\varphi - (\alpha \wedge \beta) \vdash (\varphi - \alpha) \vee (\varphi - \beta)$. Which gives us $\varphi - (\alpha \wedge \beta) \vdash \gamma$.
- (C7): Assume that (R6) is satisfied and $\varphi - (\alpha \wedge \beta) \nvdash \alpha$. We want to show that $\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$.
  - Suppose that $\varphi \nvdash \alpha$, from (C2) we have $\varphi - \alpha \vdash \varphi$. Moreover, according to (C1), $\varphi \vdash \varphi - (\alpha \wedge \beta)$. So we obtain $\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$.
  - Suppose now that $\varphi \vdash \alpha$. Towards a contradiction assume that $[\varphi - (\alpha \wedge \beta)] \wedge \neg(\alpha \wedge \beta) \vdash \alpha$. Equivalently $\varphi - (\alpha \wedge \beta) \vdash (\alpha \wedge \beta) \vee \alpha$. This is equivalent to $\varphi - (\alpha \wedge \beta) \vdash \alpha$, which contradicts the assumption $\varphi - (\alpha \wedge \beta) \nvdash \alpha$. Hence we have $[\varphi - (\alpha \wedge \beta)] \wedge \neg(\alpha \wedge \beta) \nvdash \alpha$. Levi identity gives us $\varphi \circ \neg(\alpha \wedge \beta) \nvdash \alpha$. According to (R6), as $[\varphi \circ \neg(\alpha \wedge \beta)] \wedge \neg\alpha$ is consistent, we have $\varphi \circ [\neg(\alpha \wedge \beta) \wedge \neg\alpha] \vdash [\varphi \circ \neg(\alpha \wedge \beta)] \wedge \neg\alpha$. However, from (R4), we have $\varphi \circ [\neg(\alpha \wedge \beta) \wedge \neg\alpha] \equiv$

$\varphi \circ \neg\alpha$. Furthermore, Levi identity gives us $\varphi \circ \neg\alpha \equiv (\varphi - \alpha) \wedge \neg\alpha$. Hence $(\varphi - \alpha) \wedge \neg\alpha \vdash [\varphi \circ \neg(\alpha \wedge \beta)] \wedge \neg\alpha$. We have $(\varphi - \alpha) \wedge \neg\alpha \vdash \varphi \circ \neg(\alpha \wedge \beta)$. Levi identity gives us $(\varphi - \alpha) \wedge \neg\alpha \vdash [\varphi - (\alpha \wedge \beta)] \wedge \neg(\alpha \wedge \beta)$. We have $(\varphi - \alpha) \wedge \neg\alpha \vdash \varphi - (\alpha \wedge \beta)$ (1). From (C1), $\varphi \vdash \varphi - (\alpha \wedge \beta)$. As $\varphi \vdash \alpha$, according to (C4), we have $(\varphi - \alpha) \wedge \alpha \vdash \varphi$. By transitivity of $\vdash$, we get $(\varphi - \alpha) \wedge \alpha \vdash \varphi - (\alpha \wedge \beta)$ (2). (1) and (2) give us $\varphi - \alpha \vdash \varphi - (\alpha \wedge \beta)$. □

Proposition 12.

- *if $\circ$ is a revision operator, then $\circ_{(-_{(\circ)})} = \circ$*
- *if $-$ is a contraction operator, then $-_{(\circ_{(-)})} = -$*

Proof. We first show that $\circ_{(-_{(\circ)})} = \circ$. We just have to show that for any formulas φ and α, $\varphi \circ_{(-_{(\circ)})} \alpha \equiv \varphi \circ \alpha$. $\varphi \circ_{(-_{(\circ)})} \alpha \equiv (\varphi -_{(\circ)} \neg\alpha) \wedge \alpha \equiv [\varphi \vee (\varphi \circ \alpha)] \wedge \alpha \equiv (\varphi \wedge \alpha) \vee [(\varphi \circ \alpha) \wedge \alpha]$. From (R1), we have $\varphi \circ_{(-_{(\circ)})} \alpha \equiv (\varphi \wedge \alpha) \vee (\varphi \circ \alpha)$.

- If $\varphi \wedge \alpha$ is consistent, from (R2) we have $\varphi \circ \alpha \equiv \varphi \wedge \alpha$. Hence $\varphi \circ_{(-_{(\circ)})} \alpha \equiv (\varphi \wedge \alpha) \vee (\varphi \wedge \alpha) \equiv \varphi \wedge \alpha \equiv \varphi \circ \alpha$.
- If $\varphi \wedge \alpha$ is not consistent, we have $\varphi \circ_{(-_{(\circ)})} \alpha \equiv \varphi \circ \alpha$

So we have $\varphi \circ_{(-_{(\circ)})} \alpha \equiv \varphi \circ \alpha$, hence $\circ_{(-_{(\circ)})} = \circ$. We now show that $-_{(\circ_{(-)})} = -$. To do this, we show that for any φ and α, $\varphi -_{(\circ_{(-)})} \alpha \equiv \varphi - \alpha$. $\varphi -_{(\circ_{(-)})} \alpha \equiv \varphi \vee (\varphi \circ_{(-)} \neg\alpha) \equiv \varphi \vee [(\varphi - \alpha) \wedge \neg\alpha] \equiv (\varphi \vee \neg\alpha) \wedge [\varphi \vee (\varphi - \alpha)]$. First, from (C1), we have $\varphi \vee (\varphi - \alpha) \equiv \varphi - \alpha$. Furthermore:

- If $\varphi \vdash \alpha$ then by (C4) we have $\varphi - \alpha \vdash \neg\alpha \vee \varphi$. In this case, we have $\varphi -_{(\circ_{(-)})} \alpha \equiv \varphi - \alpha$
- If $\varphi \nvdash \alpha$ then by (C1) and (C2), $\varphi - \alpha \equiv \varphi$. So $(\varphi \vee \neg\alpha) \wedge (\varphi - \alpha) \equiv \varphi \equiv \varphi - \alpha$. Thus $\varphi -_{(\circ_{(-)})} \alpha \equiv \varphi - \alpha$. □

Lemma 13. *Let $-$ be a contraction operator satisfying (C1)–(C7). Let I be an interpretation and φ be a formula.*

$$\varphi - \neg\alpha_{\{I\}} \equiv \varphi \vee \alpha_{\{I\}}$$

Proof. First, if $\alpha_{\{I\}} \vdash \varphi$, then $\varphi \nvdash \neg\alpha_{\{I\}}$. So by (C1) and (C2) we have that $\varphi - \neg\alpha_{\{I\}} \equiv \varphi$. From $\alpha_{\{I\}} \vdash \varphi$ we also know that $\varphi \vee \alpha_{\{I\}} \equiv \varphi$. The conclusion follows.

The interesting case is when $\alpha_{\{I\}} \nvdash \varphi$. Then we have $\varphi \vdash \neg\alpha_{\{I\}}$. From (C4), $(\varphi - \neg\alpha_{\{I\}}) \wedge \neg\alpha_{\{I\}} \vdash \varphi$, hence $\varphi - \neg\alpha_{\{I\}} \vdash \varphi \vee \alpha_{\{I\}}$. From (C1), $\varphi \vdash \varphi - \neg\alpha_{\{I\}}$. Furthermore, the postulate (C3) gives us $\varphi - \neg\alpha_{\{I\}} \nvdash \neg\alpha_{\{I\}}$. Thus we get $\alpha_{\{I\}} \vdash \varphi - \neg\alpha_{\{I\}}$. Therefore $\varphi - \neg\alpha_{\{I\}} \equiv \varphi \vee \alpha_{\{I\}}$. □

Theorem 14. *A contraction operator $-$ satisfies the postulates **(C1)–(C7)** if and only if there exists a faithful assignment that associates with each belief base φ a total pre-order $\leq_\varphi$ on the set of all interpretations such that*

$$Mod(\varphi - \alpha) = Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi)$$

Proof. Let $-$ be a contraction operator which satisfies the postulates (C1) to (C7). For each formula φ, we define a total pre-order $\leq_\varphi$ using the operator $-$: $\forall I, J$ two interpretations, we define the relation $\leq_\varphi$ by $I \leq_\varphi J$ if and only if $I \in Mod(\varphi - \neg\alpha_{\{I,J\}})$.

We first show that $\leq_\varphi$ is a total pre-order.

- **Total:** let I and J be two interpretations. As $\alpha_{\{I,J\}}$ has at least one model, $\neg\alpha_{\{I,J\}}$ has at least one counter-model. We deduce that $\nvdash \neg\alpha_{\{I,J\}}$, which allows us to conclude from (C3) that $\varphi - \neg\alpha_{\{I,J\}} \nvdash \neg\alpha_{\{I,J\}}$. So we know that there is $L \in Mod(\varphi - \neg\alpha_{\{I,J\}})$ such that $L \in Mod(\alpha_{\{I,J\}}) = \{I, J\}$. Therefore, either $I \in Mod(\varphi - \neg\alpha_{\{I,J\}})$ and thus $I \leq_\varphi J$, or $J \in Mod(\varphi - \neg\alpha_{\{I,J\}})$ and thus $J \leq_\varphi I$. Hence $\leq_\varphi$ is total.
- **Reflexive:** Every binary relation which is total necessarily is reflexive.
- **Transitive:** Suppose that $I \leq_\varphi J$ and $J \leq_\varphi L$. Let us consider the case when I, J and L are pairwise distinct, and none of them is a model of φ. Indeed, in the remaining case when at least two of them are equal, transitivity is trivially satisfied. If one of them is a model of φ, then the result also trivially holds by (C1). If $J \models \varphi$, then by (C1) $I \models \varphi$. And if $L \models \varphi$, then by the assumptions and (C1) we deduce that J and I are also models of φ.
  And if $I \models \varphi$ then by construction $I \leq_\varphi M$ for every interpretation M, so especially for $M = L$.
  So now let us consider the general case. Towards a contradiction, suppose $I \not\leq_\varphi L$. As $\leq_\varphi$ is total, we have $L <_\varphi I$, therefore $L \models \varphi - \neg\alpha_{\{I,L\}}$ and $I \not\models \varphi - \neg\alpha_{\{I,L\}}$. By (Tri) we have that $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{I,L\}}$ or $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{J\}}$ or $\varphi - \neg\alpha_{\{I,J,L\}} \equiv (\varphi - \neg\alpha_{\{I,L\}}) \vee (\varphi - \neg\alpha_{\{J\}})$.
  • Case (1) $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{I,L\}}$. From (C6) and Lemma 13 we have that $\varphi - \neg\alpha_{\{I,L\}} \vdash \varphi - \neg\alpha_{\{I\}} \vee \varphi - \neg\alpha_{\{L\}} \equiv \varphi \vee \alpha_{\{I\}} \vee \alpha_{\{L\}}$. Since $J \not\models \varphi \vee \alpha_{\{I\}} \vee \alpha_{\{L\}}$, we have $J \not\models \varphi - \neg\alpha_{\{I,L\}}$, so $J \not\models \varphi - \neg\alpha_{\{I,J,L\}}$. Since $L \models \varphi - \neg\alpha_{\{I,J,L\}}$ and $L \not\models \neg\alpha_{\{J,L\}}$, we deduce that $\varphi - \neg\alpha_{\{I,J,L\}} \nvdash \neg\alpha_{\{J,L\}}$. So by (C7) we have that $\varphi - \neg\alpha_{\{J,L\}} \vdash \varphi - \neg\alpha_{\{I,J,L\}}$. As $J \not\models \varphi - \neg\alpha_{\{I,J,L\}}$, we have $J \not\models \varphi - \neg\alpha_{\{J,L\}}$, which means by definition that $J \not\leq_\varphi L$. Contradiction.

• Case (2) $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{J\}} \equiv \varphi \vee \alpha_{\{J\}}$. This means in particular that $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$ and $J \models \varphi - \neg\alpha_{\{I,J,L\}}$. So we know that $\varphi - \neg\alpha_{\{I,J,L\}} \nvdash \neg\alpha_{\{I,J\}}$. So by (C7) we have that $\varphi - \neg\alpha_{\{I,J\}} \vdash \varphi - \neg\alpha_{\{I,J,L\}}$. As $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$, we have $I \not\models \varphi - \neg\alpha_{\{I,J\}}$, which means by definition that $I \nleq_\varphi J$. Contradiction.
• Case (3) $\varphi - \neg\alpha_{\{I,J,L\}} \equiv (\varphi - \neg\alpha_{\{I,L\}}) \vee (\varphi - \neg\alpha_{\{J\}}) \equiv (\varphi - \neg\alpha_{\{I,L\}}) \vee (\varphi \vee \alpha_{\{J\}})$. This equivalence implies that $J \models \varphi - \neg\alpha_{\{I,J,L\}}$, $L \models \varphi - \neg\alpha_{\{I,J,L\}}$, and $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$. Since $J \models \varphi - \neg\alpha_{\{I,J,L\}}$ and $J \not\models \neg\alpha_{\{I,J\}}$, we deduce that $\varphi - \neg\alpha_{\{I,J,L\}} \nvdash \neg\alpha_{\{I,J\}}$. So by (C7) we have that $\varphi - \neg\alpha_{\{I,J\}} \vdash \varphi - \neg\alpha_{\{I,J,L\}}$. As $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$, we have $I \not\models \varphi - \neg\alpha_{\{I,J\}}$, which means by definition that $I \nleq_\varphi J$. Contradiction.

We have shown that $\leq_\varphi$ is a total, reflexive and transitive relation. It is therefore a total pre-order. Then we show that the mapping $\varphi \mapsto \leq_\varphi$ is a faithful assignment.

- The third condition (if $\varphi_1 \equiv \varphi_2$, then $\leq_{\varphi_1} = \leq_{\varphi_2}$) comes from (C5). Indeed, if $\varphi_1 \equiv \varphi_2$ then $\varphi_1 - \neg\alpha_{\{I_1,I_2\}} \equiv \varphi_2 - \neg\alpha_{\{I_1,I_2\}}$, hence $I_1 \leq_{\varphi_1} I_2$ iff $I_1 \leq_{\varphi_2} I_2$, so $\leq_{\varphi_1} = \leq_{\varphi_2}$.
- The first condition comes from (C1): $\varphi \vdash \varphi - \neg\alpha$, so if $I_1 \in Mod(\varphi)$ then $I_1 \in Mod(\varphi - \neg\alpha_{\{I_1,I_2\}})$ and if $I_2 \in Mod(\varphi)$ then $I_2 \in Mod(\varphi - \neg\alpha_{\{I_1,I_2\}})$. So by definition, we have $I_1 \leq_\varphi I_2$ and $I_2 \leq_\varphi I_1$, hence $I_1 \simeq_\varphi I_2$.
- Let us now show that the second condition (if $I_1 \models \varphi$ and $I_2 \not\models \varphi$ then $I_1 <_\varphi I_2$) is satisfied. From the definition of $\leq_\varphi$ and (C1), we can deduce from $I_1 \models \varphi$ that $I_1 \leq_\varphi I_2$. It remains to show that $I_2 \nleq_\varphi I_1$. We consider two cases:
  - If $\varphi \vdash \neg\alpha_{\{I_1,I_2\}}$, then we have $\varphi \vdash \neg\alpha_{\{I_1\}} \wedge \neg\alpha_{\{I_2\}}$. So, in particular, $\varphi \vdash \neg\alpha_{\{I_1\}}$, which contradicts the fact that $I_1 \models \varphi$, showing that this case is impossible.
  - If $\varphi \nvdash \neg\alpha_{\{I_1,I_2\}}$, then, from (C2), $\varphi - \neg\alpha_{\{I_1,I_2\}} \vdash \varphi$. We therefore deduce that $I_2 \not\models \varphi - \neg\alpha_{\{I_1,I_2\}}$, hence $I_2 \nleq_\varphi I_1$.

  The second condition for the assignment to be faithful is thus satisfied.

Finally, it remains to show that

$$Mod(\varphi - \alpha) = Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi).$$

We consider two cases:

- $\varphi \nvdash \alpha$. From (C1) and (C2), we have $Mod(\varphi - \alpha) = Mod(\varphi)$. Let $J \in min(Mod(\neg\alpha), \leq_\varphi)$. For every $L \in Mod(\neg\alpha)$, we have $J \leq_\varphi L$. Now, since $\varphi \nvdash \alpha$, there exists $I \in Mod(\varphi)$ such that $I \in Mod(\neg\alpha)$. Hence, $J \leq_\varphi I$ holds, which implies that $I \nless_\varphi J$. Then, by the second condition on faithful assignment, we must have $J \in Mod(\varphi)$. This shows that $min(Mod(\neg\alpha), \leq_\varphi) \subseteq Mod(\varphi)$, and the conclusion follows.
- $\varphi \vdash \alpha$. By (C4), we have $\varphi - \alpha \vdash \varphi \vee \neg\alpha$. Let us thus successively consider the two cases (a) and (b) below.

  (a) Suppose first that $\vdash \alpha$. Then we have $\varphi - \alpha \vdash \varphi$ since $\neg\alpha$ is inconsistent. Furthermore, by (C1), we have $\varphi \vdash \varphi - \alpha$. Therefore, $\varphi - \alpha \equiv \varphi$. Accordingly, $Mod(\varphi - \alpha) = Mod(\varphi) = Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi)$ holds since $Mod(\neg\alpha) = \emptyset = min(Mod(\neg\alpha), \leq_\varphi)$ when α is valid.

  (b) Suppose then that $\nvdash \alpha$. We want to show first that $Mod(\varphi - \alpha) \subseteq Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi)$. Let I be any interpretation such that $I \models \varphi - \alpha$: we want to show that $I \in Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi)$. If $I \models \varphi$, then this membership statement holds directly. So suppose that $I \not\models \varphi$. In this case we want to show that $I \in min(Mod(\neg\alpha), \leq_\varphi)$. Since we have shown that $\varphi - \alpha \vdash \varphi \vee \neg\alpha$, from $I \models \varphi - \alpha$ and $I \not\models \varphi$, we get that $I \models \neg\alpha$.
  Towards a contradiction, suppose that $I \notin min(Mod(\neg\alpha), \leq_\varphi)$. This means that there is $J \models \neg\alpha$ such that $J <_\varphi I$. From the definition of $\leq_\varphi$ stated above, this means that $I \not\models \varphi - \neg\alpha_{\{I,J\}}$ (*) and that $J \models \varphi - \neg\alpha_{\{I,J\}}$.
  Now, let us consider the formula $\gamma = \neg\alpha \wedge \neg\alpha_{\{I,J\}}$. Clearly, we have $\neg\alpha \equiv \gamma \vee \alpha_{\{I,J\}}$. So by (C5) we get that $\varphi - \alpha \equiv \varphi - \neg(\gamma \vee \alpha_{\{I,J\}}) \equiv \varphi - (\neg\gamma \wedge \neg\alpha_{\{I,J\}})$. Now by (C6) we have that $\varphi - (\neg\gamma \wedge \neg\alpha_{\{I,J\}}) \vdash (\varphi - \neg\gamma) \vee (\varphi - \neg\alpha_{\{I,J\}})$. We have supposed that $I \models \varphi - \alpha$, hence $I \models \varphi - (\neg\gamma \wedge \neg\alpha_{\{I,J\}})$. From the previous implication, we obtain that $I \models (\varphi - \neg\gamma) \vee (\varphi - \neg\alpha_{\{I,J\}})$. But by (*) we have that $I \not\models \varphi - \neg\alpha_{\{I,J\}}$, so we must have that $I \models (\varphi - \neg\gamma)$. However, since $\varphi \vdash \alpha$, we also have that $\varphi \vdash \neg\gamma$. Then by (C4) we obtain that $(\varphi - \neg\gamma) \wedge \neg\gamma \vdash \varphi$. Since $I \models \neg\gamma$, if $I \models \varphi - \neg\gamma$, then we must have $I \models \varphi$, a contradiction.
  Therefore, we have $Mod(\varphi - \alpha) \subseteq Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi)$.
  Let us now show that $Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi) \subseteq Mod(\varphi - \alpha)$.
  - If $I \in Mod(\varphi)$, then since from (C1), we have $\varphi \vdash \varphi - \alpha$, we conclude that $I \in Mod(\varphi - \alpha)$.
  - Suppose now that $I \notin Mod(\varphi)$ and $I \in min(Mod(\neg\alpha), \leq_\varphi)$ and suppose that $I \notin Mod(\varphi - \alpha)$. In this case, $min(Mod(\neg\alpha), \leq_\varphi)$ is not empty, which means that $\nvdash \alpha$. So, from (C3), $\varphi - \alpha \nvdash \alpha$. We can deduce that $\exists J \in Mod(\varphi - \alpha)$ such that $J \in Mod(\neg\alpha)$.
    Let us consider the two possible cases: $J \in Mod(\varphi)$ and $J \notin Mod(\varphi)$. If $J \in Mod(\varphi)$, then by the second condition of the faithful assignment we have that $J <_\varphi I$. But as $J \in Mod(\neg\alpha)$, this means that

    $$I \notin min(Mod(\neg\alpha), \leq_\varphi).$$

    Contradiction. If $J \notin Mod(\varphi)$, then we have that $J \in Mod(\varphi - \alpha)$ and $I \notin Mod(\varphi - \alpha)$. So $\varphi - \alpha \nvdash \neg\alpha_{\{I,J\}}$, hence by (C7) we have that $\varphi - \neg\alpha_{\{I,J\}} \vdash \varphi - \alpha$. As $I \notin Mod(\varphi - \alpha)$, we have $I \notin Mod(\varphi - \neg\alpha_{\{I,J\}})$. Then by definition (and (C3)) this means that $J <_\varphi I$. But we also know that $J \in Mod(\neg\alpha)$, so this implies that $I \notin min(Mod(\neg\alpha), \leq_\varphi)$. Contradiction.

($\Leftarrow$) Suppose we have a faithful assignment that associates φ with a total pre-order $\leq_\varphi$. A contraction operator $-$ is defined by

$$Mod(\varphi - \alpha) = Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi) \quad (\delta)$$

We now show that $-$ satisfies the postulates (C1)–(C7).

(C1) is satisfied. Indeed, from $Mod(\varphi-\alpha) = Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi)$, we obtain easily $Mod(\varphi) \subseteq Mod(\varphi-\alpha)$ hence $\varphi \vdash \varphi - \alpha$.

(C2) is satisfied. If $\varphi \nvdash \alpha$, then $\varphi \wedge \neg\alpha$ has a model. Since the assignment is faithful, $min(Mod(\neg\alpha), \leq_\varphi) = Mod(\varphi \wedge \neg\alpha)$. So $Mod(\varphi - \alpha) = Mod(\varphi)$. This shows that (C2) is satisfied.

(C3) is satisfied. Suppose that $\nvdash \alpha$, we have $Mod(\neg\alpha) \neq \emptyset$. From (δ), there is an interpretation I belonging to $Mod(\varphi-\alpha)$ such that I also belongs to $Mod(\neg\alpha)$. This clearly shows that $Mod(\varphi - \alpha) \nsubseteq Mod(\alpha)$. This shows that (C3) is satisfied.

(C4) is satisfied. We have $Mod((\varphi - \alpha) \wedge \alpha) = Mod(\varphi - \alpha) \cap Mod(\alpha)$. So (δ) gives us $Mod((\varphi - \alpha) \wedge \alpha) = (Mod(\varphi) \cap Mod(\alpha)) \cup (Min(Mod(\neg\alpha), \leq_\varphi) \cap Mod(\alpha))$. We have $min(Mod(\neg\alpha), \leq_\varphi) \cap Mod(\alpha) = \emptyset$. Hence $Mod((\varphi-\alpha) \wedge \alpha) = Mod(\varphi) \cap Mod(\alpha)$. So $Mod((\varphi - \alpha) \wedge \alpha) \subseteq Mod(\varphi)$. This shows that (C4) is satisfied.

(C5) is satisfied. Suppose that $\varphi_1 \equiv \varphi_2$ and $\alpha_1 \equiv \alpha_2$. We have $Mod(\varphi_1) = Mod(\varphi_2)$ and $Mod(\alpha_1) = Mod(\alpha_2)$. By definition of the faithful assignment, we have $\leq_{\varphi_1} = \leq_{\varphi_2}$. Hence $min(Mod(\neg\alpha_1), \leq_{\varphi_1}) = min(Mod(\neg\alpha_2), \leq_{\varphi_2})$. So $Mod(\varphi_1) \cup min(Mod(\neg\alpha_1), \leq_{\varphi_1}) = Mod(\varphi_2) \cup min(Mod(\neg\alpha_2), \leq_{\varphi_2})$. Therefore $Mod(\varphi_1 - \alpha_1) = Mod(\varphi_2 - \alpha_2)$. This shows that (C5) is satisfied.

(C6) is satisfied. $Mod(\varphi - (\alpha \wedge \beta)) = Mod(\varphi) \cup min(Mod(\neg(\alpha \wedge \beta)), \leq_\varphi) = Mod(\varphi) \cup min(Mod(\neg\alpha) \cup Mod(\neg\beta), \leq_\varphi)$. However, we have $min(Mod(\neg\alpha) \cup Mod(\neg\beta), \leq_\varphi) \subseteq min(Mod(\neg\alpha), \leq_\varphi) \cup min(Mod(\neg\beta), \leq_\varphi)$.

So we have that $Mod(\varphi - (\alpha \wedge \beta)) \subseteq Mod(\varphi) \cup min(Mod(\neg\alpha), \leq_\varphi) \cup min(Mod(\neg\beta), \leq_\varphi) \subseteq Mod(\varphi - \alpha) \cup Mod(\varphi - \beta)$. This shows that (C6) is satisfied.

(C7) is satisfied. Let us suppose that $\varphi - (\alpha \wedge \beta) \nvdash \alpha$. So there is $I \in Mod(\varphi) \cup min(Mod(\neg(\alpha \wedge \beta)), \leq_\varphi)$ such that $I \in Mod(\neg\alpha)$. There are two cases:

- $I \in Mod(\varphi \wedge \neg\alpha)$. In this case, since the assignment is faithful, we have $Mod(\varphi - \alpha) = Mod(\varphi)$. And the conclusion follows from the fact that $-$ satisfies (C1).
- $I \in Mod(\neg\varphi \wedge \neg\alpha) \cap min(Mod(\neg(\alpha \wedge \beta), \leq_\varphi)$. To prove (C7), it is enough to show that $min(Mod(\neg\alpha), \leq_\varphi) \subseteq min(Mod(\neg(\alpha \wedge \beta)), \leq_\varphi)$. (A).
  Since $\neg\alpha \vdash \neg(\alpha \wedge \beta)$, we necessarily have $I \in min(Mod(\neg\alpha), \leq_\varphi)$ since $I \models \neg\alpha$. Since $\leq_\varphi$ is a total pre-order, every $J \in min(Mod(\neg\alpha), \leq_\varphi)$ satisfies $J \simeq_\varphi I$. Assume that there exists $L \in Mod(\neg(\alpha \wedge \beta))$ such that $L <_\varphi J$. This contradicts the fact that $I \in min(Mod(\neg(\alpha \wedge \beta)), \leq_\varphi)$. Thus (A) is satisfied and therefore (C7) is satisfied as well. $\square$

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