A Pigeon-Hole Based Encoding of Cardinality Constraints[1]

Saïd Jabbour **Lakhdar Saïs** Yakoub Salhi

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CRIL - CNRS, Université Lille Nord de France, Lens, France

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Motivation

- CP/SAT based data mining (Frequency constraint)
- Cardinality constraints appears in many other application domains
- Cross-fertilization between 0/1 linear programming and SAT

Goal : Find the most efficient and compact CNF encoding

- (Efficient) Maintain generalized arc consistency via unit propagation
- (Compact) Encoding of smallest size? (w.r.t. number of variables and clauses)

CNF encodings of the cardinality constraint

$$\sum_{i=1}^{n} x_i \geq \lambda, \qquad x_i \in \{0, 1\}$$

Several polynomial CNF encodings have been proposed:

- **Joost P. Warners [1996]** [Wrong formulation in page 12]
- BDD encoding [Bailleux at al. 2003]
- ...
- Cardinality networks [Asín et al. 2011]

Joost P. Warners [1996] - page 12

5 Horn cardinality clauses

In this section we will consider a special class of linear inequalities, the *Horn cardinality clauses*, which have the form

$$\sum_{i=1}^{m} x_i \geq b. \tag{22}$$

This is the only form of inequalities that we are aware of, for which there exists a polynomial CNF expansion (Hooker [7]). The CNF equivalent of (22) is

$$\neg z_{ik} \vee p_{x_i}, \qquad i = 1, \ldots, m, k = 1, \ldots, b, \tag{23}$$

$$\bigvee_{i=1}^{m} z_{ik}, \qquad k = 1, \ldots, b, \tag{24}$$

$$\neg z_{ik} \vee \neg z_{jk}, \qquad i, j = 1, \ldots, n, i \neq j, k = 1, \ldots, b. \tag{25}$$

Here (23) says that x_i is true if some z_{ik} is true, and (24) combined with (25) say that for each k exactly one z_{ik} must be true.

[7] J.N. Hooker. Unpublished note.

Polynomial Encoding of $\sum_{i=1}^{n} x_i \geq \lambda$ to CNF

$$\bigwedge_{k=1}^{\lambda} (\neg p_{ki} \vee x_i), \quad i = 1, \ldots, n \tag{1}$$

$$\bigvee_{i=1}^{n} p_{ki}, \quad k = 1, \ldots, \lambda \tag{2}$$

$$\bigwedge_{1 \leq k < k' \leq \lambda} (\neg p_{ki} \vee \neg p_{k'i}), \quad i = 1, \ldots, n \tag{3}$$

(2) and (3) encode the **Pigeon Hole** problem PHP_n^{λ}

- p_{ki} expresses that pigeon k is in hole i
- x_i is true if the hole i contains one of the pigeons k for $k = 1, \ldots, \lambda$

Complexity

- number of additional variables: $\lambda \times n$
- number of clauses: $O(n \times \lambda^2)$.

Symmetry Breaking on $\mathcal{P}_n^\lambda$

Let $\mathcal{S}ym(\mathcal{P}_n^\lambda)$ be the set of symmetries of $\mathcal{P}_n^\lambda$:

$$\bigcup_{1 \le i < j \le \lambda} \sigma(i,j) = (p_{i1}, p_{j1}), (p_{i2}, p_{j2}), \ldots, (p_{in}, p_{jn}) \tag{4}$$

$$\begin{pmatrix} p_{11} & \cdots & & [p_{1\lambda} & \cdots & p_{1n}] \\ & \cdots & [p_{2(\lambda-1)} & \cdots & p_{2(n-1)}] & \\ \vdots & \iddots & & \iddots & & \vdots \\ [p_{\lambda 1} & \cdots & p_{\lambda(n-\lambda+1)}] & & \cdots & p_{\lambda n} \end{pmatrix} \tag{5}$$

Symmetry Breaking on $\mathcal{P}_n^\lambda$ (cont.)

Let $\sigma(i,j) = (p_{i1}, p_{j1}), (p_{i2}, p_{j2}), \ldots, (p_{in}, p_{jn}), \quad 1 \leq i < j \leq \lambda$
$sbp_{\sigma(i,j)} =$

- $(p_{i1} \leq p_{j1}) \wedge$
- $(p_{i1} = p_{j1}) \rightarrow (p_{i2} \leq p_{j2}) \wedge$
- $\ldots$
- $(p_{i1} = p_{j1}) \ldots (p_{i(n-1)} = p_{j(n-1)}) \rightarrow (p_{in} \leq p_{jn})$

Property

$\mathcal{P}_n^\lambda$ is satisfiable iff $\mathcal{P}_n^\lambda \wedge sbp(\mathcal{S}ym(\mathcal{P}_n^\lambda))$ is satisfiable

$\rightarrow$ Instead of adding SBP to the formula, we apply resolution between clauses from $\mathcal{P}_n^\lambda$ and $sbp(\mathcal{S}ym(\mathcal{P}_n^\lambda))$

Symmetry Breaking on $\mathcal{P}_n^\lambda$ (cont.)

Eliminating the upper-left corner triangle

- $\mathcal{P}_n^\lambda \wedge sbp_{\sigma(1,2)} \models \neg p_{11}$:
  - $\sigma = (p_{11}, p_{21}) \subset \sigma(1,2)$
  - $sbp_\sigma = (p_{12} \leq p_{22}) = (1)(\neg p_{11} \vee p_{21})$
  - $c = (\neg p_{11} \vee \neg p_{21}) \in \mathcal{P}_n^\lambda$
  - $\eta[p_{21}, (1), c] = \neg p_{11}$.
- $\mathcal{P}_n^b \wedge sbp(\mathcal{S}ym(\mathcal{P}_n^b)) \models \neg p_{21} \wedge \neg p_{31} \wedge, \ldots, \neg p_{(\lambda-1)1}$
  - Use similar reasoning
- All the binary clauses from (3) involving $\neg p_{11}, \neg p_{21}, \ldots, \neg p_{(\lambda-1)1}$ are eliminated
- $\ldots$

Symmetry Breaking on $\mathcal{P}_n^\lambda$ (cont.)

Eliminating the lower-right corner triangle

- $\mathcal{P}_n^\lambda \wedge sbp(\mathcal{S}ym(\mathcal{P}_n^\lambda)) \models \neg p_{2n} \wedge \neg p_{3n}, \ldots, \neg p_{\lambda n}$
  - $\mathcal{P}_n^\lambda \wedge sbp(\mathcal{S}ym(\mathcal{P}_n^\lambda)) \models \neg p_{2n}$:
    - $\eta[p_{1n}, (p_{1\lambda} \vee, \ldots, \vee p_{1(n-1)} \vee p_{1n}), (\neg p_{1n} \vee \neg p_{2n})] = r_1 = (p_{1\lambda} \vee, \ldots, \vee p_{1(n-1)} \vee \neg p_{2n})$.
    - $\eta[p_{2n}, r_1, (p_{2(\lambda-1)} \vee, \ldots, \vee p_{2(n-1)} \vee p_{2n})] = r_2 = (\mathbf{p_{1\lambda}} \vee, \ldots, \vee \mathbf{p_{1(n-2)}} \vee \mathbf{p_{1(n-1)}} \vee p_{2(\lambda-1)} \vee, \ldots, \vee p_{2(n-1)})$.
    - To eliminate the first $n-1$ literals from r_2, we exploit $sbp_{\sigma(1,i)}$ with $2 \leq i \leq \lambda$.
    - Let $s_1 = (p_{2(\lambda-1)} \vee p_{1\lambda} \vee p_{2\lambda} \vee \ldots, \vee p_{1(n-2)} \vee p_{2(n-2)} \vee \neg p_{1(n-1)} \vee p_{2(n-1)}) \in sbp_{\sigma(1,2)}$. $\eta[p_{1(n-1)}, r_2, s_1] = r_3 = (\mathbf{p_{1\lambda}} \vee, \ldots, \vee \mathbf{p_{1(n-2)}} \vee p_{2(\lambda-1)} \vee, \ldots, \vee p_{2(n-1)})$.
    - Now, $p_{1(n-2)}$ can be eliminated from r_3. Let $s_2 = (p_{2(\lambda-1)} \vee p_{1\lambda} \vee p_{2\lambda} \vee \ldots, \vee p_{1(n-3)} \vee p_{2(n-3)} \vee \neg p_{1(n-2)} \vee p_{2(n-1)}) \in sbp_{\sigma(1,2)}$. We obtain $\eta[p_{1(n-2)}, r_3, s_2] = r_4 = (\mathbf{p_{1\lambda}} \vee, \ldots, \vee \mathbf{p_{1(n-3)}} \vee p_{2(\lambda-1)} \vee, \ldots, \vee p_{2(n-1)})$.
    - Similarly, we eliminate $\{p_{1\lambda}, \ldots, p_{1(n-3)}\}$ from r_4.
    - ...
  - ...

$ph\mathcal{P}_n^\lambda$ encoding of the cardinality constraint

$$\neg p_{(\lambda-k+1)(i+k-1)} \vee x_{(i+k-1)}, \quad 1 \leq i \leq n-\lambda+1, \; 1 \leq k \leq \lambda \tag{6}$$

$$\bigvee_{i=1}^{n-\lambda+1} p_{(\lambda-k+1)(i+k-1)}, \quad 1 \leq k \leq \lambda \tag{7}$$

$$p_{(\lambda-k+1)k} \vee \cdots \vee p_{(\lambda-k+1)(i+k)} \vee \neg p_{(\lambda-k)(i+k+1)}, \quad 0 \leq i \leq n-\lambda-1, 1 \leq k \leq \lambda-1 \tag{8}$$

Soundness and Unit Propagation

Property

If ρ is a model of $ph\mathcal{P}_n^\lambda$ then ρ is a model of $\sum_{i=1}^n x_i \geq \lambda$.

Property

If ρ is a model of $\sum_{i=1}^n x_i \geq \lambda$, then there exists a model ρ' of $ph\mathcal{P}_n^\lambda$ such that for all $i \in \{1, \ldots, n\}$, $\rho(x_i) = \rho'(x_i)$.

Property (Unit propagation)

Let ρ be a model of $ph\mathcal{P}_n^\lambda$ assigning 0 to the elements of a set $S = \{x_{i_1}, \ldots, x_{i_{n-\lambda}}\}$ of $n - \lambda$ propositional variables included in $X = \{x_1, \ldots, x_n\}$. Unit propagation is sufficient to deduce that for all variable $x \in X \backslash S$, $\rho(x) = 1$.

Theoretical Comparison with other encodings

Encoding	#Clauses	# Variables	Decided
Sequential unary counter [Sinz05]	$\mathcal{O}(\lambda \times n)$	$\mathcal{O}(\lambda \times n)$	UP
Parallel binary counter [Sinz05]	$7n - 3log(n) - 6$	$2n - 2$	Search
Totalizer [Bailleux03]	$\mathcal{O}(n^2)$	$\mathcal{O}(n \times log_2(n))$	UP
Buttner & Rintanen [Buttner05]	$\mathcal{O}(\lambda^2 \times n)$	$\mathcal{O}(n \times log_2(n))$	UP
Sorting Network [EenS06]	$\mathcal{O}(n \times log_2^2(n))$	$\mathcal{O}(n \times log_2^2(n))$	UP
Cardinality Network [AsinNOR11]	$\mathcal{O}(n \times log_2^2(\lambda))$	$\mathcal{O}(n \times log_2^2(\lambda))$	UP
Warners [Warners96]	$8n$	$2n$	Search
$ph\mathcal{P}_n^\lambda$	$\mathcal{O}(\lambda \times (n - \lambda))$	$\mathcal{O}(\lambda \times (n - \lambda))$	UP

Table : Comparison of CNF encodings of $\sum_{i=1}^{n} x_i \leq \lambda$

Conclusion & future works

- Pigeon Hole Based formulation of the cardinality constraint
- Competitive with most of the previous encoding
- Erratum to Joost P. Warners formulation [1996 paper]
- A nice methodology: reduction of the encoding modulo symmetry, redundant constraints, resolution...

Future works

- CNF encoding of $\sum_{i=1}^{n} \alpha_i x_i \geq \lambda$ for both $x_i \in \{0, 1\}$ and $x_i \in \{1 \ldots, n_i\}$ (**done - Best encoding**)
- Use of the same methodology to encode global constraints to CNF (e.g. allDifferent constraint)