Fundamenta Informaticae XXX-(2004) 1–25
IOS Press

Quasi-possibilistic logic and its measures of information and conflict

Didier Dubois

Sébastien Konieczny

Henri Prade

Institut de Recherche en Informatique de Toulouse
31062 Toulouse - France
{dubois,konieczny,prade}@irit.fr

Abstract. Possibilistic logic and quasi-classical logic are two logics that were developed in artificial intelligence for coping with inconsistency in different ways, yet preserving the main features of classical logic. This paper presents a new logic, called quasi-possibilistic logic, that encompasses possibilistic logic and quasi-classical logic, and preserves the merits of both logics. Indeed, it can handle plain conflicts taking place at the same level of certainty (as in quasi-classical logic), and take advantage of the stratification of the knowledge base into certainty layers for introducing gradedness in conflict analysis (as in possibilistic logic). When querying knowledge bases, it may be of interest to evaluate the extent to which the relevant available information is precise and consistent. The paper review measures of (im)precision and inconsistency/conflict existing in possibilistic logic and quasi-classical logic, and proposes generalized measures in the unified framework.

Keywords: possibilistic logic, paraconsistent logic, measures of information, inconsistency, uncertainty.

1. Introduction

Information is often pervaded with uncertainty or inconsistency. This state of affairs has led to the development of important research trends in artificial intelligence in the last thirty years in order to design inference tools capable of coping with uncertainty and/or inconsistency. In the presence of inconsistency, two general approaches can be conceived, namely either to restore consistency by "getting rid" of a part of the information in one way or another (see e.g. [24, 25, 11, 19, 21]), or to "live" with it by still being able to draw inferences of interest (see e.g. [31, 6, 2, 3, 14, 15]).

In this paper, we are interested in the second type of approach. There is indeed a need for handling contradictory information in a safe way in inference processes. Inconsistency situations may be due to information coming from different sources, or to the cohabitation of recent information with older one, as suggested by the following toy example. Let us suppose we have some pieces of information about Peter, maybe coming from different sources. *Peter works in Grenoble* ; *Peter lives in Marseilles* ; *Peter is in his forties.* Moreover we have the general knowledge that *if somebody works in some place, (s)he cannot live in another place if the two places are distant.* Besides, *Grenoble and Marseilles are distant places.* In such a case, inconsistency-free information such as *Peter is in his forties* should not be lost in the inference process. Moreover, it would be useful to state that the available information enables us to conclude that both *Peter works in Grenoble* and *Peter works in Marseilles,* as well as *Peter lives in Grenoble* and *Peter lives in Marseilles.* Note that having contradictory information is not the same as having no information (e. g., the available information does not enable us to conclude if *Peter has blue eyes* is true or not). Lastly, it may be desirable in such an example to stratify the information in layers corresponding to different level of certainty. For instance, the piece of information *Peter lives in Marseilles* may be rather old and as such would be not regarded as completely certain, then this should lead to a preference for the conclusions *Peter works in Grenoble* and *Peter lives in Grenoble.* In this paper, we develop a logical framework rich enough for handling these different issues in a rigorous and efficient way.

Possibilistic logic and quasi-classical logic are two logics that have been developed in artificial intelligence for coping with inconsistency in different ways. Possibilistic logic (ΠL) extends classical logic by considering classical formulas associated with certainty levels. These levels are at the core of the inference mechanism. They allow us to compute global inconsistency levels for such knowledge bases. Paraconsistent logics aim to handle inconsistent pieces of information by isolating them, avoiding the trivialization of the inference for the whole base. Quasi-classical logic (QCL) is one of those logics. It has the nice feature of possessing a semantics close to the one of classical logic. These two logics share a valuable feature, since both of them remain as close as possible to classical logic, which is clearly advantageous, both from modeling and computational complexity points of view.

This paper presents a new logic, called quasi-possibilistic logic, which encompasses possibilistic logic and quasi-classical logic as particular cases, and preserves the merits of each logic. Indeed, we can handle local conflicts taking place at a given level of certainty (as in quasi-classical logic), and take advantage of the stratification of the knowledge base into certainty layers for introducing gradedness in conflict analysis (as in possibilistic logic).

Section 2 provides a refresher on ΠL and QCL. Section 3 introduces quasi-possibilistic logic ($Q\Pi L$) which provides a joint framework for dealing with uncertain and paraconsistent information. Information measures pertaining to uncertainty or to inconsistency are also briefly discussed in ΠL, QCL and $Q\Pi L$.

2. Background

We consider a propositional language $\mathcal{L}_{\mathcal{PS}}$ based on a finite set of propositional symbols PS and the connectives $\{\neg, \vee, \wedge, \rightarrow\}$. We will denote the formulas in $\mathcal{L}_{\mathcal{PS}}$ by lower Greek letters $\varphi, \psi, \ldots$ We will denote the atoms (propositional symbols) of the language by $a, b, c, \ldots$ For each atom $a \in \mathcal{L}_{\mathcal{PS}}$, a is a literal and $\neg a$ is a literal. We will denote literals by $l, l_1, \ldots$ We will denote $\vdash_{CL}$ the inference relation of classical logic.

Let $(L, <)$ be a totally ordered set, we will denote the elements of L by $n, m, n_1, n_2, \ldots$. A possibilistic formula is a pair (φ, n) where φ is a propositional formula and n is the "weight" of the formula (i.e. an element of L). In the following L will be often taken as the $[0, 1]$ interval for the sake of simplicity, however a bounded integer interval would be enough for the results to hold.

A knowledge base K will be a finite set of formulas. Depending on the framework those formulas will be classical formulas (in Section 2.2) or possibilistic formulas, but it will never be ambiguous.

Let K be a knowledge base, we note $[K]$ the set of models of K and we note $Atoms(K)$ the set of propositional symbols of PS appearing in K. Let A be a set, $|A|$ will denote the cardinality of this set.

2.1. Possibilistic logic

We now recall the main features of possibilistic logic before introducing measures of (im)precision for possibilistic knowledge bases and discussing some paraconsistent extensions of possibilistic logic.

Possibilistic logic is a weighted logic of incomplete knowledge. It partitions a classical knowledge base into subsets of formulas according to their levels of certainty. Since some formulas are more certain than others, it is possible to isolate a consistent subset of sufficiently certain formulas from an inconsistent knowledge base, and inference becomes non-trivial in the presence of inconsistency.

A possibilistic logic [8, 7] formula is a classical logic formula φ weighted in terms of a lower bound $n \in (0, 1]$ of a necessity measure, i.e., the possibilistic logic formula (φ, n) is understood as $N(\varphi) \geq n$, where N is a necessity measure.

Basically, possibilistic logic inference aims at deducing formulas with their certainty levels, or formulas having a certainty level greater than some threshold.

2.1.1. Necessity and possibility measures

A necessity measure N is a function from the set of logical formulas to a totally ordered bounded scale, which is characterized by the axioms

i) $N(\top) = 1$,

ii) $N(\bot) = 0$

where $\top$ and $\bot$ stand for tautology and contradiction respectively, and 0 and 1 are the bottom and the top element of the scale L,

iii) $N(\varphi \wedge \psi) = \min(N(\varphi), N(\psi))$.

We use the real interval $[0, 1]$ as the range of necessity measures in the following, but this is not compulsory. A (finite or not) totally ordered scale bounded by a bottom and a top element is enough. A possibility measure Π is associated by duality with N, namely

$$\Pi(\varphi) = 1 - N(\neg\varphi)$$

where $1 - ()$ is the order-reversing map of the scale. It expresses that the absence of certainty in favor of $\neg\varphi$ leaves φ possible. Π satisfies the characteristic property $\Pi(\varphi \vee \psi) = \max(\Pi(\varphi), \Pi(\varphi))$.

2.1.2. Syntactic aspects

The min-decomposability of necessity measures allows us to work with weighted clauses without lack of generality, since $N(\bigwedge_{i=1,k} \varphi_i) \geq m \Leftrightarrow \forall i, N(\varphi_i) \geq m$, i.e., $(\bigwedge_{i=1,k} \varphi_i, m) \Leftrightarrow \bigwedge_{i=1,k}(\varphi_i, m)$. So we will call (weighted) CNF of a possibilistic knowledge base K a set of weighted clauses that is equivalent to K, i.e. that corresponds to the same necessity measure[1] .

Let $\vdash$ denote the syntactic inference in possibilistic logic. The basic inference rule in possibilistic logic is the following one:

- $(\neg\varphi \vee \psi, m)\ (\varphi \vee \rho, n) \vdash (\psi \vee \rho, \min(m, n))$ (resolution rule)

This rule is enough to make proofs using refutation. It implements an old principle claiming that the validity of a chain of inferences is the validity of its weakest link. Let K be a knowledge base made of possibilistic logic formulas. Proving (φ, m) from K amounts to adding $(\neg\varphi, 1)$, put in clausal form, to K, and using the above rule repeatedly to show that $K \cup (\neg\varphi, 1) \vdash (\bot, m)$.

The natural following inference rules are also easily retrieved:

- for $n \leq m$ $(\varphi, m) \vdash (\varphi, n)$ (weight weakening)
- if $\varphi \vdash_{CL} \psi$, then $(\varphi, m) \vdash (\psi, m)$ (formula weakening)
- $(\varphi, m)\ (\varphi, n) \vdash (\varphi, max(m, n))$ (weight fusion)

Classical resolution is retrieved when all the weights are equal to 1. Moreover

$$K \vdash (\varphi, m) \text{ if and only if } K^*_m \vdash_{CL} \varphi \quad (1)$$

where K_m is the so-called m-cut of the possibilistic base K and is defined as $K_m = \{(\varphi, n) \mid (\varphi, n) \in K \text{ with } n \geq m\}$, and K^* is called the classical projection of K and is defined as the set of classical formulas obtained from a possibilistic knowledge base K by forgetting the weights : $K^* = \{\varphi \mid (\varphi, n) \in K\}$. Note that formulas of the form $(\varphi, 0)$ which do not contain any information ($\forall\varphi, N(\varphi) \geq 0$ always holds) are never written.

2.1.3. Semantic aspects

From a semantic point of view, a possibilistic knowledge base $K = \{(\varphi_i, m_i)\}_{i=1,k}$ is associated with the possibility distribution π_K representing the fuzzy set of models u of K:

$$\pi_K(u) = \min_{i=1,k} \max(\mu_{[\varphi_i]}(u), 1 - m_i) \quad (2)$$

where $[\varphi_i]$ denotes the sets of models of φ_i and $\mu_{[\varphi_i]}$ its characteristic function. It can be shown that π_K is the largest possibility distribution such that $N_K(\varphi_i) \geq n_i, \forall i = 1, k$, i.e., the possibility distribution which allocates the greatest possible possibility degree to each interpretation in agreement with the constraints induced by K (where N_K is the necessity measure associated with π_K, namely $\Pi_K(\neg\varphi) = \max_{u\in[\neg\varphi]} \pi_K(u)$ and thus $N_K(\varphi) = \min_{u\in[\neg\varphi]}(1 - \pi_K(u)))$.

[1] Such a weighted CNF form always exists and gives the same inferences that the original knowledge base [8].

Thus, a possibilistic logic base is associated with a fuzzy set of models. This represents the set of more or less plausible states of the world (according to the available information), when dealing with uncertainty. A possibility distribution which rank-orders possible states is thus semantically equivalent to a possibilistic logic base. The semantic entailment is then defined by

$$K \models (\varphi, m) \text{ if and only if } N_K(\varphi) \geq m$$

$$(\Leftrightarrow \forall u\ \pi_K(u) \leq \max(\mu_{[\varphi]}(u), 1 - m))$$

2.1.4. Inconsistency level

An important feature of possibilistic logic is its ability to deal with inconsistency. The level of inconsistency of a possibilistic logic base is defined as $inc(K) = \max\{m \mid K \vdash (\bot, m)\}$ (by convention $max\ \emptyset = 0$). We can explain this inconsistency level with the m-cuts: the inconsistency level of a base is the greatest m such that the corresponding m-cut is classically inconsistent. Clearly, any entailment $K \vdash (\varphi, m)$ with $m > inc(K)$ can be rewritten as $K_m^{cons} \vdash (\varphi, m)$, where $K_m^{cons} = \{(\varphi_i, m_i) \in K^{cons}$ with $m_i \geq m\}$ and $K^{cons} = K - \{(\varphi_i, m_i)$ with $m_i \leq inc(K)\}$ is the set of formulas whose weights are above the level of inconsistency and which are thus not involved in the inconsistency. Indeed, $inc(K^{cons}) = 0$. More generally, $inc(K) = 0$ if and only if K^* is consistent in the usual sense. Moreover, it can be shown that

$$inc(K) = 1 - \max_u \pi_K(u) \tag{3}$$

The syntactic inference machinery of possibilistic logic, using resolution and refutation, has been proved to be sound and complete with respect to the semantics [8, 7]. Soundness and completeness are expressed by

$$K \vdash (\varphi, m) \Leftrightarrow K \models (\varphi, m) \text{ for } m > inc(K)$$

It is important to observe that formulas in K whose certainty level is strictly smaller than $inc(K)$ are *"drowned"* in the sense that they cannot be inferred nor be used in a valid proof. A way to escape the drowning effect is presented in Section 2.1.6.

2.1.5. Measure of imprecision

Information measures have been introduced for a long time in probability theory. They also exist in the propositional logic setting, where Lozinskii [26, 27] has proposed $|PS| - \log_2 |[K]|$ as a measure of precision (information) of a consistent knowledge base K. Information measures have been also introduced in other uncertainty frameworks such as possibility theory and belief function theory [9]. In particular, a possibility distribution can be associated with a measure of imprecision which generalizes Hartley measure of information [12] (which is itself a particular case of Shannon entropy), as recalled in the following.

A possibility distribution π can be associated with an information measure $Imp(\pi)$ which evaluates its imprecision (e.g., [13, 9]). It is defined by

$$Imp(\pi) = \sum_{j=1,s} (\pi_j - \pi_{j+1}).log_2 j \tag{4}$$

where π is a normalized possibility distribution (i.e. $\max_u \pi(u) = 1$), defined on a set of interpretations $\{u_1, \ldots, u_s\}$ with s elements u_j which are assumed to be ranked in such a way that the $\pi(u_j) = \pi_j$ form a non-increasing sequence $\pi_1 = 1 \geq \ldots \geq \pi_j \geq \ldots \geq \pi_s \geq \pi_{s+1} = 0$. Note that $\{u_1, \ldots, u_j\}$ is the set of interpretations whose possibility degree is greater or equal to π_j and has cardinality j. Introducing the set $M_j = \{u_1, \ldots, uj\}$, $Imp(\pi)$ is still equal to

$$Imp(\pi) = \sum_{j=1,s} p(M_j).log_2|Mj| \tag{5}$$

where $p(M_j) = \pi_j - \pi_{j+1}$. Observe that $\sum_{j=1,s} p(M_j) = 1$ and more generally that $\pi(u_i) = \sum_{j=i,s} p(M_j)$. It can be shown that if $\pi \models \pi'$ (i.e. $\pi \leq \pi'$) then $Imp(\pi) \leq Imp(\pi')$, which agrees with the idea that (semantic) entailment favors imprecision. Clearly, interpretations with high possibility degrees contribute more to imprecision than interpretations with low possibility degrees. Axiomatic justifications for (4) have been provided by Higashi and Klir [13] and by Ramer [30]. Note that if π is the characteristic function of a subset with r elements, i.e. $\pi_1 = \ldots = \pi_r = 1$ and $\pi_{r+1} = \ldots = \pi_s = 0$, $Imp(\pi) = log_2 r$, which is known as the Hartley [12] entropy of a subset. When $r = 1$, there is one interpretation and $Imp(\pi) = 0$, which indeed means that there is no imprecision.

Thus, a measure of imprecision $Imp(\pi_K)$ can be associated with a fully consistent possibilistic logic base K, where π_K is defined by (2) (the consistency of K, i.e. $inc(K) = 0$ ensures the normalization of π_K). Let $m_1 \geq \ldots \geq m_t$ be the different certainty levels associated with formulas in K. Then, it can be checked that

$$Imp(\pi_K) = m_t.log_2|[K_{m_t}]| + (m_{t-1} - m_t).log_2|[K_{m_{t-1}}]| + \ldots + (1 - m_1).log_2\mathcal{N}_K \tag{6}$$

where $\mathcal{N}_K$ is the number of interpretations induced by the language. Note that $m_t + (m_{t-1} - m_t) + \ldots + (1 - m_1) = 1$. So, if we introduce k further propositional symbols in the language, then $Imp(\pi_K)$ is changed into $Imp(\pi_K) + k$, i.e. imprecision is increased as expected, but by a fixed amount. In particular, when all the formulas in K are fully certain, $Imp(\pi_K) = log_2|[K]|$ since $t = 1$ and $K = K_{m_1}$. When all the formulas in K are fully certain, this measure of imprecision reduces to Lozinskii's measure of information (precision), up to a reversing of the scale. There exist other noticeable measures, such that Yager's specificity index [32], which rather estimates precision, and is defined by $Spe(\pi) = \sum_{j=1,s} p(M_j).(1/|M_j|)$.

Lastly, observe that the measure of imprecision $Imp(\pi_K)$ is defined only when K is consistent (i.e. when π_K is normalized). It could be extended to the inconsistent case, by noticing that in general $\pi_1 \leq 1$ and $\sum_{j=1,s} p(M_j) = 1 - inc(K)$, which leads to change $Imp(\pi_K)$ given by (5) into $Imp(\pi_K)/(1 - inc(K))$. In such a case, both the inconsistency level $inc(K)$ and the imprecision measure $Imp(\pi_K)/(1 - inc(K))$ should be provided to the user.

But we may think of other ways to renormalize π_K, and the problem of defining the imprecision (information) of a possibilistic base in the presence of hard conflicts (i.e. when π_K is not normalized) has not been studied yet. The coherence functions introduced in Section 3.2 can be seen as candidate definitions for taking into account imprecision and conflicts together.

2.1.6. Handling paraconsistent information

An extension of the possibilistic inference was proposed for handling paraconsistent information [3]. It is defined as follows. First, for each formula φ such that (φ, m) is in K, compute (φ, p, q) where p (resp. q) is the highest degree with which φ (resp. $\neg\varphi$) is supported in K. More precisely φ is said to be supported in K at least at degree r if there is a consistent sub-base of K^*_r which entails φ. Let K^o be the set of bi-weighted formulas which is thus obtained.

Example 2.1. For instance, take $K = \{(a, 0.8), (\neg a \vee b, 0.6), (\neg a, 0.5), (\neg c, 0.3), (c, 0.2), (\neg c \vee b, 0.1)\}$. Then $K^o = \{(a, 0.8, 0.5), (\neg a \vee b, 0.6, 0), (\neg a, 0.5, 0.8), (\neg c, 0.3, 0.2), (c, 0.2, 0.3), (\neg c \vee b, 0.6, 0)\}$.

A formula (φ, p, q) is said to have a paraconsistency degree equal to $\min(p, q)$. For defining an inference relation from K^o, we introduce two measures:

- the undefeasibility degree of a consistent set A of formulas:

$$UD(A) = \min\{p \mid (\varphi, p, q) \in K^o \text{ and } \varphi \in A\}$$

- the unsafeness degree of a consistent set A of formulas:

$$US(A) = \max\{q \mid (\varphi, p, q) \in K^o \text{ and } \varphi \in A\}$$

We say that A *is a reason for* ψ if A is a minimal (for set inclusion) consistent subset of K that implies ψ, i.e. :

- $A \subseteq K$
- $A^* \nvdash_{CL} \bot$
- $A^* \vdash_{CL} \psi$
- $\forall B \subset A, B^* \nvdash_{CL} \psi$

Let $label(\psi) = \{(A, UD(A), US(A)) \mid A \text{ is a reason for } \psi\}$, and $label(\psi)^* = \{A \mid (A, UD(A), US(A)) \in label(\psi)\}$. Then $(\psi, UD(A'), US(A'))$ is said to be a DS-consequence of K^o (or K), denoted by $K \vdash_{DS} (\psi, UD(A'), US(A'))$, if and only if $UD(A') > US(A')$, where A' is maximizing $UD(A)$ in $label(\psi)^*$ and in case of several such A', the one which minimizes $US(A')$. It can be checked that $\vdash_{DS}$ extends the entailment in possibilistic logic [3].

Example 2.1. (continued) In the above example, $label(\psi) = \{(A, 0.6, 0.5), (B, 0.2, 0.3)\}$ with $A = \{(a, 0.8, 0.5), (\neg a \vee b, 0.6, 0)\}$ and $B = \{(c, 0.2, 0.3), (\neg c \vee b, 0.6, 0)\}$. Then, $K \vdash_{DS} (b, 0.6, 0.5)$.

If we first minimize $US(A)$ and then maximize $UD(A')$, the entailment would not extend the possibilistic entailment. Indeed in the above example, we would select $(B, 0.2, 0.3)$ but $0.2 > 0.3$ does not hold, while $K \vdash (b, 0.6)$ since $0.6 > inc(K) = 0.5$. Note that $\vdash_{DS}$ is more productive than the possibilistic entailment, as seen on the example, e.g., $K \vdash_{DS} (\neg c, 0.3, 0.2)$, while $K \vdash (\neg c, 0.3)$ does not hold since $0.3 < inc(K) = 0.5$.

An entailment denoted $\vdash_{SS}$, named safely supported-consequence relation, less demanding than $\vdash_{DS}$, is defined by $K \vdash_{SS} \psi$ if and only if $\exists A \in label(\psi)$ such that $UD(A) > US(A)$. It can be shown that the set $\{\psi \mid K \vdash_{SS} \psi\}$ is classically consistent [3].

2.2. Quasi-classical logic

In [4, 15] Besnard and Hunter define a new paraconsistent logic. This logic has several very nice features, in particular the connectives behave classically, and when the knowledge base is classically consistent, then quasi-classical logic gives almost the same conclusions as classical logic[2]. Moreover it has been proved in [28] that inference in quasi-classical logic has a very low computational complexity. It is only coNP-complete. That is much less than most approaches to reasoning under inconsistency, which are typically at the second level of the polynomial hierarchy [29], as all methods based on maximal (for set inclusion) consistent subsets of formulas for example. In [28] a linear time translation from inference in quasi-classical logic to inference in classical logic is also provided. It allows us to use existing automated reasoning techniques developed for classical entailment. Finally, one of the major features of this logic is that it has a nice and intuitive semantics, that is not the case of most paraconsistent logics.

The basic ideas behind this logic is to use all rules of classical logic proof theory, but to forbid the use of resolution after the introduction of a disjunction (it allows to get rid of the *ex falso quodlibet sequitur*). So the rules of quasi-classical logic are split into two classes: composition and decomposition rules, and the proofs cannot use decomposition rules once a composition rule has been used. Intuitively speaking, this means that we may have resolution-based proofs both for a and $\neg a$. We also have as additional valid consequences the disjunctions build from the previous consequences (e.g. $\neg a \vee b$). But it is forbidden to reuse such additional consequences for building further proofs. For details on quasi-classical logic proof theory see [15]. We will present only the semantic side of this logic in the following.

For the sake of simplicity we will restrict ourselves to the class of CNF formulas in this section. Generalization for the class of all formulas can be found in [15], but it requires more conditions for the satisfaction relations (de Morgan laws, double negation elimination, etc). So choosing CNF formulas does not lose generality but ease the definitions.

2.2.1. Quasi-classical consequence

Definition 2.1. Let $\mathcal{O}_{\mathcal{PS}}$ be the set defined as follows:

$$\mathcal{O}_{\mathcal{PS}} = \{+a \mid a \in PS\} \cup \{-a \mid a \in PS\}$$

We call any $X \subseteq \mathcal{O}_{\mathcal{PS}}$ a QC interpretation.

In such an interpretation X, $+a \in X$ means that X provides a reason for a and a reason against $\neg a$. Similarly $-a \in X$ states that X provides a reason for $\neg a$ and a reason against a.

This definition is close to a form of Herbrand interpretation. It allows to localize the conflicts to propositional symbols. For example if $PS = \{a, b, c, d\}$, then the interpretation $X = \{+a, +b, -b\}$ means that we have a reason for a, we have a conflict on b and we have no information about c and d.

In the same way, it can also be considered as a four-valued semantics *à la* Belnap [1]. Let T, F, B, N be four truth values whose intuitive meaning is respectively *True*, *False*, *Both*, *Neither*. Then we can translate a QC interpretation X to a four truth values interpretation as follows : for each symbol a of PS

[2]In fact only tautologies or formulas containing tautologies cannot be recovered.

- a is T if $+a \in X$ and $-a \notin X$
- a is F if $+a \notin X$ and $-a \in X$
- a is B if $+a \in X$ and $-a \in X$
- a is N if $+a \notin X$ and $-a \notin X$

For a detailed comparison between quasi-classical entailment and Belnap entailment see [17].

Definition 2.2. Let $l_1 \vee \ldots \vee l_n$ be a clause, then $literals(l_1 \vee \ldots \vee l_n)$ is the set of literals $\{l_1, \ldots, l_n\}$ that are in the clause. Let $l_1 \vee \ldots \vee l_n$ be a clause and let l_i be a literal such that $l_i \in literals(l_1 \vee \ldots \vee l_n)$, then $Focus(l_1 \vee \ldots \vee l_n, l_i)$ is the clause without the disjunct l_i, i.e. $Focus(l_1 \vee \ldots \vee l_n, l_i) = l_1 \vee \ldots l_{i-1} \vee l_{i+1} \vee \ldots l_n$. Let $Focus(l, l) = \perp$.

Definition 2.3. Let a be a propositional symbol, $\sim$ is the complementation operation defined as $\sim a$ is $\neg a$ and $\sim (\neg a)$ is a. This operation is not in the object language but will be used to make definitions clearer.

Let us now define the notion of strong satisfaction:

Definition 2.4. For a model X, we define the strong satisfaction relation $\models_S$ as follows. Let a be a propositional symbol, let $l_1, \ldots l_n$ be literals, and let φ and ψ be two formulas:

- $X \models_S a$ iff $+a \in X$
- $X \models_S \neg a$ iff $-a \in X$
- $X \models_S \varphi \wedge \psi$ iff $X \models_S \varphi$ and $X \models_S \psi$
- $X \models_S l_1 \vee \ldots \vee l_n$ iff ($X \models_S l_1$ or … or $X \models_S l_n$) and
  $\forall i \in \{1, \ldots, n\}$ (if $X \models_S \sim l_i$, then $X \models_S Focus(l_1 \vee \ldots \vee l_n, l_i)$)

So the quasi-classical disjunction semantics (for strong satisfaction) is more demanding than the classical one, since it has to cope with conflicting pieces of information. One can note that it is this semantics for disjunction that captures the resolution principle on which QC proof theory relies.

A characteristic property of disjunction is the following one, let $\{a_i, \ldots, a_n\}$ be a subset of PS [15]:

Proposition 2.1. $X \models_S a_1 \vee \ldots \vee a_n$ iff
- there exists a_i such that $+a_i \in X$ and $-a_i \notin X$, or
- for all a_i, $+a_i \in X$ and $-a_i \in X$.

Let us note also that reduced to binary clauses the previous definition gives :

$X \models_S a \vee b$ iff ($X \models_S a$ or $X \models_S b$) and
if $X \models_S \neg a$, then $X \models_S b$ and
if $X \models_S \neg b$, then $X \models_S a$

The notion of strong model is easily extended to knowledge bases (sets of formulas) by stating that a model X is a strong model of K if X is a strong model of all the formulas of K.

So let us now introduce an example that we will use through this paper to illustrate the definitions.

Example 2.2. $K = \{a, \neg a \wedge b, c \vee d, c \vee e, \neg a \vee f\}$. $X = \{+a, -a, +b, +c, +f\}$, $Y = \{+a, -a, +b, +d, +e, +f\}$ and $Z = \{+a, -a, +b, -b, +c, -c, +d, -d, +e, -e, +f, -f\}$ are three strong models of K.

Note that the model $X = \mathcal{O}_{\mathcal{PS}}$ is a strong model of all formulas of $\mathcal{L}_{\mathcal{PS}}$. So every formula of $\mathcal{L}_{\mathcal{PS}}$ always has at least one strong model.

Similarly, a notion of weak satisfaction is defined :

Definition 2.5. For a model X, we define the weak satisfaction relation $\models_w$ as follows :

- $X \models_w a$ iff $+a \in X$
- $X \models_w \neg a$ iff $-a \in X$
- $X \models_w \varphi \wedge \psi$ iff $X \models_w \varphi$ and $X \models_w \psi$
- $X \models_S l_1 \vee \ldots \vee l_n$ iff ($X \models_S l_1$ or ... or $X \models_S l_n$)

Example 2.2. (continued) $K = \{a, \neg a \wedge b, c \vee d, c \vee e, \neg a \vee f\}$. $X_1 = \{+a, -a, +b, +c\}$ and $X_2 = \{+a, -a, +b, -c, +d, +e, +f, -f\}$ are two weak models of K (which are not strong models).

Note that straightforwardly all strong models of a formula are also weak models, and that weak satisfaction is close to the satisfaction relation in classical logic.

Now we can define the consequence relation as :

Definition 2.6. We say that a formula ψ is a (quasi-classical) consequence of φ if and only if all the strong models of φ are weak models of ψ :

$\varphi \models_{QC} \psi$ iff $\forall X(X \models_S \varphi \Rightarrow X \models_w \psi)$

Strong satisfaction is used for the assumptions. It allows to capture semantically the resolution process, since it forces the resolvent b of a clause $a \vee b$ to hold if $\sim a$ holds, whereas weak satisfaction is used for the conclusions and allows for the introduction of disjunctions. Note that this kind of definition, using a more restrictive satisfaction relation for assumptions than for conclusions is usual in other logics for AI. For example nonmonotonic inference relations use a preferential entailment relation for assumptions, while the satisfaction relation for conclusions remains classical [22].

The extension of the definition to a consequence of a knowledge base (set of formulas) is straightforwardly done by considering the set conjunctively, that is a formula ψ is a consequence of a knowledge base if each interpretation that is a strong model of each formula of the knowledge base is a weak model of ψ.

Example 2.2. (continued) $K = \{a, \neg a \wedge b, c \vee d, c \vee e, \neg a \vee f\}$. Examples of consequences of K are $\neg a$, $c \vee d$, $b \vee c$, $f \wedge b$.

The following result illustrates the link with the syntactical intuition we gave at the beginning of this section, i.e. that the idea behind QCL (on the proof-theoretic side) is to forbid the use of resolution after introduction of a disjunction : notice that a, $\neg a$ and $\neg a \vee \neg f$ (by disjunction introduction) are consequences of K, but that $\neg f$ is not.

Note that if K is in CNF and is consistent, then Definition 2.6 yields classical entailment. Formally, we have the following proposition, where φ denotes a CNF formula, $\tilde{\varphi}$ the formula after deletion of tautological clauses.

Proposition 2.2. If K and φ are in CNF and if K is classically consistent, then $K \models_{QC} \tilde{\varphi}$ iff $K \models_{CL} \varphi$.

This proposition is a direct consequence of results of [28]. Note also that the strong QC models of K are in this case (a rewriting of) terms of a DNF of the knowledge base K.

2.2.2. Minimal QC models - Coherence function

Let us denote by $\mathsf{QC}(K)$ the set of strong models of K. In the following we will mainly work with strong QC models, so the term *QC model* will mean *strong* QC models, and we will explicitly use the term *weak* when requested.

Let us now define the notion of minimal QC model.

Definition 2.7. The set of minimal (strong) models of K is defined as :
$\mathsf{MQC}(K) = \{X \in \mathsf{QC}(K) \mid \text{ if } Y \subset X, \text{ then } Y \notin \mathsf{QC}(K)\}$

For example if $K = \{a \vee b\}$, then the QC models of K are $\mathsf{QC}(K) = \{\{+a\}, \{+b\}, \{+a, +b\}, \{+a, -a, +b\}, \{+a, +b, -b\}, \{+a, -a, +b, -b\}$. Whereas the mininal QC models of K are $\mathsf{MQC}(K) = \{\{+a\}, \{+b\}\}$. They can be viewed as a concise representation of classical models of K when K is classically consistent.

We will now define a measure of consistency, called coherence defined in [16]. Let us first define the notions of $\mathsf{Conflictbase}_{\mathsf{QC}}$ and of $\mathsf{Opinionbase}_{\mathsf{QC}}$.

Definition 2.8. Let X be a QC interpretation,
$\mathsf{Conflictbase}_{\mathsf{QC}}(X) = \{a \mid +a \in X \text{ and } -a \in X\}$
$\mathsf{Opinionbase}_{\mathsf{QC}}(X) = \{a \mid +a \in X \text{ or } -a \in X\}$

Intuitively the conflict base is the set of propositional symbols on which there is conflictual information, and the opinion base is the set of propositional symbols on which the interpretation provides some information.

Now the degree of coherence of an interpretation is defined as

Definition 2.9. $\mathsf{Coherence}_{\mathsf{QC}}$ is a function from the sets of interpretations to $[0, 1]$ defined as:
$\mathsf{Coherence}_{\mathsf{QC}}(\emptyset) = 1$ and $\forall X \neq \emptyset$

$$\mathsf{Coherence}_{\mathsf{QC}}(X) = 1 - \frac{|\mathsf{Conflictbase}_{\mathsf{QC}}(X)|}{|\mathsf{Opinionbase}_{\mathsf{QC}}(X)|}$$

If $\mathsf{Coherence}_{\mathsf{QC}}(X) = 1$, then X is totally coherent, and if $\mathsf{Coherence}_{\mathsf{QC}}(X) = 0$, then X is totally incoherent.

Example 2.2. (continued) For example, let $X = \{+a, -a, +b, +c, +f\}$, and $Y = \{+a, -a, +b, +d, +e, +f\}$, then $\mathsf{Coherence}_{\mathsf{QC}}(X) = 3/4$ and $\mathsf{Coherence}_{\mathsf{QC}}(Y) = 4/5$.

Now we can define the degree of coherence of a knowledge base :

Definition 2.10. Let K be a knowledge base, then $\mathsf{Coherence}_{\mathsf{QC}}(K)$ is defined as :

$$\mathsf{Coherence}_{\mathsf{QC}}(K) = \max_{X \in \mathsf{MQC}(K)} \mathsf{Coherence}_{\mathsf{QC}}(X)$$

Note that taking the $mean$ or the min instead of the max (or some refinement of those ones) may also lead to other meaningful measures. This “optimistic” choice of the max seems to be meaningful when one wants to compare several knowledge bases in order to decide which one is the least problematic. Using min instead can be seen as its “pessimistic” counterpart. It makes sense when we are interested in the worst case, for example when we want to know how much effort is needed to be sure to recover consistency. This point of view is closer to the one adopted in [20]. Taking the $mean$ is usually meaningful when one allows for compensations between alternatives, so in this case it means that a lot of models with a high coherence could compensate a model with a very low one.

Example 2.2. (continued) $K = \{a, \neg a \wedge b, c \vee d, c \vee e, \neg a \vee f\}$. Then $\mathsf{Coherence}_{\mathsf{QC}}(K) = 4/5$.

2.2.3. Significance function

In [18], Hunter also defines a degree of Significance for a contradiction. This degree of significance requires an additional meta-information that gives the relative importance of conflicts that affects sets of propositional atoms (that can be seen as a “topic” of the base), and it computes the significance of the contradiction embedded in a given QC model. This degree is defined as follows:

Definition 2.11. A mass assignment μ is a function from $2^{\mathcal{O}_{\mathcal{PS}}}$ into $[0, 1]$ such that :

- If $\mathsf{Coherence}_{\mathsf{QC}}(X) = 1$, then $\mu(X) = 0$
- $\sum_X \mu(X) = 1$

To illustrate what this mass assignment and this significance function mean, consider the following example [18]: suppose that we have some information about a soccer match coming from different news reports. If those pieces of information are conflicting about the nationality of the referee, it will be hardly noticed because it is not very important. It is not the same story if the conflicting information is about the outcome of the match. If one report says that team A won the match, whereas another says that it is team B, then it will be a more significant conflict. An example of mass assignment in this case could be to give a very small mass to the conflict on the nationality of the referee, and a big mass to the conflict on the outcome of the game (the remaining mass will be used to weight other possible conflicts on the information about the match).

Definition 2.12. A significance function (induced by a mass assignment μ), denoted S^{μ}_{QC} is a function from $2^{\mathcal{O}_{\mathcal{PS}}}$ into $[0,1]$ defined as :

$$S^{\mu}_{\mathsf{QC}}(X) = \sum_{W \subseteq X} \mu(W)$$

The explanation for performing the sum of all masses of subsets of X relies on the intuition that the significance of conflicts in an interpretation X can come from different independent sources of conflicts, and that some conflicts can be more problematic when we have more information. For example, usually a conflict on the weather for the day is not very important for me (as I work in my office), but if I know also that today I would like to go out (it is a non-working day), then the significance of a conflict on the weather is much more important.

Then the mass-based significance function is extended to knowledge base as follows:

Definition 2.13. Let K be a knowledge base, then the significance is defined as:

$$S^{\mu}_{\mathsf{QC}}(K) = \min(\{S^{\mu}_{\mathsf{QC}}(X) | X \in \mathsf{MQC}(K)\})$$

Example 2.2. (continued) Let $K = \{a, \neg a \wedge b, c \vee d, c \vee e, \neg a \vee f\}$. And let $\mu(\{+a, -a\}) = 0.1$, $\mu(\{+a, -a, +c\}) = 0.4$, $\mu(\{+c, -c\}) = 0.3$, $\mu(\{+f, -f\}) = 0.2$ be the corresponding mass assignment. Then we have $S_{\mathsf{QC}}(\{+a, -a, +b, +c, +f\}) = 0.5$, and $S_{\mathsf{QC}}(\{+a, -a, +b, +d, +e, +f\}) = 0.1$, so finally $S_{\mathsf{QC}}(K) = 0.1$.

3. Quasi-possibilistic logic

In the semantics of possibilistic logic, classical interpretations receive possibility weights. In quasi-classical logic the semantics is based in terms of reasons for or against propositional symbols. Unsurprisingly, in quasi-possibilistic logic reasons for or against propositional symbols become weighted.

In this section we will suppose that all the formulas are (weighted) CNF. As in the quasi-classical logic case, it does not lead to a lost of generality and ease the definitions.

3.1. Quasi-possibilistic consequence

Let us define the following notion of models:

Definition 3.1. Let $\mathcal{O}_{\mathcal{PS},\mathcal{L}}$ be the set defined as follows:

$$\mathcal{O}_{\mathcal{PS},\mathcal{L}} = \{(+a\ n) \mid a \in PS, n \in L\} \cup \{(-a\ n) \mid a \in PS, n \in L\}$$

We call any $X \subseteq \mathcal{O}_{\mathcal{PS},\mathcal{L}}$ a $\mathsf{Q}\Pi$ interpretation.

Notice that from the definition an interpretation X can contain both $(+a\ n_1)$ and $(-a\ n_2)$ for some atom a. Similarly, note that the definition allows for several $(+a\ n_1), \ldots, (+a\ n_i)$ in an interpretation. But we will see that only the occurrence with the greatest n_k needs to be taken into consideration (the others are in some sense subsumed by this one), so we will suppose that only one such occurrence can occur in an interpretation.

The meaning of such an interpretation X is that $(+a\ n) \in X$ means that X provides a reason for a with confidence n and a reason against $\neg a$ with confidence n. Similarly $(-a\ n) \in X$ states that X provides a reason for $\neg a$ with confidence n and a reason against a with confidence n.

Let us now define the notion of strong satisfaction:

Definition 3.2. For a model X, we define the strong satisfaction relation $\models_S$ as follows. Let a be a propositional symbol, let $l_1, \ldots l_n$ be literals, and let φ and ψ be two formulas: :

- $X \models_S (a, n)$ iff $(+a\ m) \in X$ with $m \geq n$
- $X \models_S (\neg a, n)$ iff $(-a\ m) \in X$ with $m \geq n$
- $X \models_S (\varphi \wedge \psi, n)$ iff $X \models_S (\varphi, n)$ and $X \models_S (\psi, n)$
- $X \models_S (l_1 \vee \ldots \vee l_n, n)$ iff
  $(X \models_S (l_1, n)$ or ... or $X \models_S (l_n, n))$ and
  $\forall i \in \{1, \ldots, n\}$ (if $X \models_S (\sim l_i, n)$, then $X \models_S (Focus(l_1 \vee \ldots \vee l_n, l_i), n)$)

The notion of a strong model is easily extended to knowledge bases (sets of formulas) by stating that a model X is a strong model of K if X is a strong model of all the formulas of K.

Example 3.1. $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3), (\neg a \vee f, 0.2)\}$. $X = \{(+a\ 0.8),$ $(-a\ 0.6), (+b\ 0.6), (+c\ 0.5), (+f\ 0.2)\}$, $Y = \{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+d\ 0.5),\ (+e\ 0.3),$ $(+f\ 0.2)\}$ and $Z = \{(+a\ 0.8), (-a\ 0.9), (+b\ 0.6), (+c\ 1), (+f\ 0.4), (-f\ 0.7)\}$ are three strong models of K.

Note that the model $X = \bigcup_{a_i \in PS} \{(+a_i\ 1), (-a_i\ 1)\}$ is a strong model of all formulas of $\mathcal{L}_{\mathcal{PS}}$. So every formula of $\mathcal{L}_{\mathcal{PS}}$ always has at least one strong model.

We also define the following notion of weak satisfaction:

Definition 3.3. For a model X, we define the weak satisfaction relation $\models_w$ as follows :

- $X \models_w (a, n)$ iff $(+a\ m) \in X$ with $m \geq n$
- $X \models_w (\neg a, n)$ iff $(-a\ m) \in X$ with $m \geq n$
- $X \models_w (\varphi \wedge \psi, n)$ iff $X \models_w (\varphi, n)$ and $X \models_w (\psi, n)$
- $X \models_S (l_1 \vee \ldots \vee l_n, n)$ iff $(X \models_S (l_1, n)$ or ... or $X \models_S (l_n, n))$

Example 3.1. (continued) $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3), (\neg a \vee f, 0.2)\}$. $X =$ $\{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+c\ 0.5)\}$ and $Y = \{(+a\ 0.8), (-a\ 0.9), (+b\ 0.9), (+d\ 0.5), (+e\ 0.3),$ $(+f\ 0.2)\}$ are two weak models of K.

Note that straighforwardly all strong models of a formula are also weak models thereof.

Now we can define the consequence relation as follows :

Definition 3.4. A formula (ψ, n) is a (quasi-possibilistic) consequence of (φ, m) if and only if all the strong models of (φ, m) are weak models of (ψ, n) :
$(\varphi, m) \models_{Q\Pi L} (\psi, n)$ iff $\forall X\ (X \models_S (\varphi, m) \Rightarrow X \models_w (\psi, n))$

The extension of the definition of a consequence to a knowledge base is straightforwardly done by considering it conjunctively, that is a formula (ψ, n) is a consequence of a knowledge base if all models that are strong models of each formula of the knowledge base are weak models of (ψ, n).

Example 3.1. (continued) $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3), (\neg a \vee f, 0.2)\}$. Consequences of K are for example $(a, 0.8)$, $(f, 0.2)$, $(b \vee g, 0.6)$, etc.

Let us stress now that the entailment so defined is a true generalization of quasi-classical entailment and possibilistic entailment:

Proposition 3.1. If $L = \{0, 1\}$, then $K \models_{Q\Pi L} (\varphi, 1)$ if and only if $K^* \models_{QC} \varphi$

We have also the following consequence :

Corollary 3.1. If $K \models_{Q\Pi L} (\psi, m)$, then $K^* \models_{QC} \psi$

This relation between quasi-possibilistic logic and quasi-classical logic can be illustrated also directly on the corresponding satisfaction relation :

Let us note X^* the “classical” QC model derived from X, that is X where we “forget” the weights, i.e. $X^* = \{+a \mid (+a\ m) \in X\} \cup \{-a \mid (-a\ m) \in X\}$. Let us denote also X_m the m-cut of the QC-model X, i.e $X_m = \{(+a\ n) \in X \mid n \geq m\} \cup \{(-a\ n) \in X \mid n \geq m\}$.

Lemma 3.1. If $X \models_S K$, then $X^* \models_S K^*$
If $X \models_w K$, then $X^* \models_w K^*$

This is straightforwardly obtained from the definitions of the satisfaction relations. Note that in this lemma we use the satisfaction relation of quasi-classical logic and of quasi-possibilistic, for the sake of simplicity we do not put a subscript for making the distinction between the two. It cannot be ambiguous since they do not work on the same formulas, quasi-classical satisfaction relation are used with QC interpretations and classical formulas, whereas quasi-possibilistic satisfaction relation are used with QΠ interpretations and possibilistic formulas.

So quasi-possibilistic entailment is a generalization of quasi-classical entailment, when we allow a finer scale than $\{0, 1\}$. In the same way, quasi-possibilistic entailment can be seen as a generalization of possibilistic entailment as shown in the following proposition.

First we can state a useful lemma.

Lemma 3.2. $X \models_S K$ iff $\forall m\ X_m^* \models_S K_m^*$

Proof:
We will prove this by induction on the m-cuts. Let us note $m_1, m_2, \ldots, m_q$ the m-cuts, with m_1 the biggest m-cut and $m_q > 0$ the lowest one.

We first show that $X_{m_1} \models_S K_{m_1}$ iff $X^*_{m_1} \models_S X^*_{m_1}$, then we will use the induction assumption $X_{m_i} \models_S K_{m_i}$ iff $\forall m \leq m_i\ X^*_m \models_S K^*_m$. Then finally we will be able to conclude $X_{m_q} \models_S K_{m_q}$ iff $\forall m \leq m_q\ X^*_m \models_S K^*_m$, but as m_q is the lowest cut, $X_{m_q} \models_S K_{m_q}$ means $X \models_S K$, that gives the conclusion.

So let us begin with $X_{m_1} \models_S K_{m_1}$ iff $X^*_{m_1} \models_S X^*_{m_1}$. This is obtained directly by noticing that the only weight in X_{m_1} and K_{m_1} is m_1, so by using the scale $L = \{0, m_1\}$, this is a consequence of proposition 3.1.

Now suppose that $X_{m_{i-1}} \models_S K_{m_{i-1}}$ iff $\forall m \leq m_{i-1}\ X^*_m \models_S K^*_m$. We want to prove that $X_{m_i} \models_S K_{m_i}$ iff $\forall m \leq m_i\ X^*_m \models_S K^*_m$ holds. The *only if* part is given directly by lemma 3.1. For the *if* part we want to show $X_{m_i} \models_S K_{m_i}$. We know from the hypothesis that all formulas in $K_{m_{i-1}}$ are satisfied by $X_{m_{i-1}}$, so by X_{m_i}. It remains to show that all formulas in $K_{m_i} - K_{m_{i-1}}$ are satisfied by X_{m_i}. Note that all those formulas are of the form (φ, m_i). Let us prove it by induction on the length of the formulas. Assume $length(\varphi) = 1$, i.e. $\varphi = a$ or $\varphi = \neg a$, with $a \in PS$. Suppose w.l.g that $\varphi = a$. Then from definition of strong satisfaction $X_{m_i} \models_S (\varphi, m_i)$ holds iff $(+a\ n) \in X_{m_i}$, with $n \geq m_i$. But it means that it holds iff $+a \in X^*_{m_i}$, i.e. iff $X^*_{m_i} \models \varphi$, that is the case by hypothesis. Now suppose that we have for $length(\varphi) = n - 1$ that $X_{m_i} \models_S (\varphi, m_i)$ if $X^*_{m_i} \models_S \varphi$. Let us show that this property holds also for $length(\varphi) = n$. So if $length(\varphi) = n$, then $\varphi = l \wedge \varphi'$ or $\varphi = l \vee \varphi'$, with $l = a$ or $l = \neg a$ s.t. $a \in PS$ and $length(\varphi') = n - 1$ (w.l.o.g. we will suppose in the following that $l = a$). So if $\varphi = a \wedge \varphi'$, then $X_{m_i} \models_S (\varphi, m_i)$ holds iff $X_{m_i} \models_S (a, m_i)$ and $X_{m_i} \models_S (\varphi', m_i)$. We have already shown that $X_{m_i} \models_S (a, m_i)$ if $X^*_{m_i} \models a$ and by induction hypothesis we now that $X_{m_i} \models_S (\varphi', m_i)$ if $X^*_{m_i} \models_S \varphi'$. So that gives that $X_{m_i} \models_S (\varphi, m_i)$ holds if $X^*_{m_i} \models a$ and $X^*_{m_i} \models_S \varphi'$, that is by definition $X^*_{m_i} \models a \wedge \varphi'$, that holds by hypothesis. The proof is similar for $\varphi = a \vee \varphi'$.

□

We can also prove the same result for weak models in the same way :

Lemma 3.3. $X \models_w K$ iff $\forall m\ X^*_m \models_w K^*_m$

Proposition 3.2. If K is consistent (i.e. K^* has a classical model), then $K \models_{Q\Pi L} (\tilde{\varphi}, n)$ if and only if $K \models_{\Pi L} (\varphi, n)$, where $\tilde{\varphi}$ is φ with tautologies deleted.

Proof:

$K \models_{Q\Pi L} (\tilde{\varphi}, n)$ iff $\forall X\ (X \models_S K \Rightarrow X \models_w (\tilde{\varphi}, n))$

(definition 3.4)

iff $\forall X\ ((\forall m\ X^*_m \models_S K^*_m) \Rightarrow (\forall m\ X^*_m \models_w (\tilde{\varphi}, n)^*_m)$

(lemma 3.2 and 3.3)

iff $\forall m\, \forall X\ (X^*_m \models_S K^*_m \Rightarrow X^*_m \models_w (\tilde{\varphi}, n)^*_m)$

iff $\forall m\ K^*_m \models_{QC} (\tilde{\varphi}, n)^*_m$

(definition 2.6)

iff $\forall m\ K^*_m \models_{CL} (\varphi, n)^*_m$

(equation 1)

iff $K \models_{\Pi L} (\varphi, n)$

□

So quasi-possibilistic entailment coincides with possibilistic entailment when the possibility distribution is normalized, but still provides meaningful results in the case of conflicts (non-normalized possibility distribution).

Concerning computational complexity, one can note that the good computational complexity properties of both quasi-classical logic and possibilistic logic are preserved for quasi-possibilistic logic.

Proposition 3.3. The complexity of the inference problem for quasi-possibilistic logic stated as follows

- **Input :** A CNF knowledge base K and a CNF formula (φ, m)
- **Output :** Does $K \models_{Q\Pi L} (\varphi, m)$ hold ?

is coNP-complete.

Proof:
This proof is a direct generalization of the proof that inference for quasi-classical logic is coNP-complete (proposition 3 of [28]). The proof goes exactly the same way, by increasing the size of the language and translating the QC interpretations into classical ones. In our case QΠ interpretations can be translated into possibilistic ones.

The membership proof holds since the translation transforms quasi-possibilistic inference into possibilistic inference, that is coNP-complete [8, 23].

The hardness proof is straightforward since inference in quasi-classical logic is a particular case of inference in quasi-possibilistic logic (cf proposition 3.1). □

Note finally that the entailment relation so defined is different from the $\vdash_{DS}$ and the $\vdash_{SS}$ entailment relations of Section 2.1.6.

This is easily shown for $\vdash_{SS}$, since the set of the consequences $\{\varphi \mid K \vdash_{SS} \varphi\}$ obtained from this relation is a classically consistent set [3]. This is not the case with our paraconsistent inference relation given by the quasi-possibilistic logic, since for example with the base $K = \{(a, 0.8), (\neg a, 0.8)\}$ we can deduce both $(a, 0.8)$ and $(\neg a, 0.8)$. We can also note that quasi-possibilistic entailment is less syntax sensitive than $\vdash_{SS}$. For example from $K = \{(a \wedge \neg a \wedge b, 0.8)\}$ and $K' = \{(a, 0.8), (\neg a, 0.8), (b, 0.8)\}$ quasi-possibilistic entailment lead to the same conclusions, for example that $(b, 0.8)$ holds. Whereas with $\vdash_{SS}$ we can deduce b from K' but not from K.

In order to show the difference between the quasi-possibilistic inference relation and $\vdash_{DS}$, it is enough to note that on example 2.1 $\vdash_{DS}$ does not allow us to infer anything on c, whereas with the quasi-possibilistic inference relation we can infer $(c, 0.2)$.

Example 3.2. Let us restate the example of the introduction in propositional logic (we will use a pseudo first-order logic for conciseness, but we still in the propositional logic framework). Peter works in Grenoble ; Peter lives in Marseilles ; Peter is in his forties. Grenoble and Marseilles are distant places. Moreover we have the general knowledge that if somebody works in a place, (s)he cannot live in another place if the two places are distant. This situation can be logically encoded in the following way ($1 > \lambda > \delta$):

$$
\begin{aligned}
K = \{ & (\forall x \forall y (\neg work(x,y) \vee \neg dist(y,z) \vee \neg live(x,z), 1); (dist(G,M), 1) \\
& (live(P,M), \lambda) \\
& (fort(P), \delta); (work(P,G), \delta)\}
\end{aligned}
$$

If we use quasi-classical logic for finding consequences of this knowledge base, we cannot take the weight into account (so we work with K^*), and the consequences are for example $\{live(P, M),$ $\neg work(P, G), work(P, G), \neg live(P, M), fort(P)\}$, illustrating the conflict in the knowledge base. But it does not allow us to take into account the fact that $live(P, M)$ is more reliable than $work(P, G)$.

With possibilistic logic, degrees of certainty are taken into account and we can deduce for example $\{(live(P, M), \lambda), (\neg work(P, G), \lambda)\}$, but it does not allow us to derive the conclusion $(fort(P), \delta)$ that has nothing to do with the conflict but is drown below the inconsistency level. Moreover it does not distinguish between formulas on which we have no conflicting pieces of information and formulas that are challenged by (less reliable) pieces of information.

Taking quasi-possibilistic logic we can obtain $\{(live(P, M), \lambda), (\neg work(P, G), \lambda), (\neg live(P, M),$ $\delta), (work(P, G), \delta), (fort(P), \delta)\}$. So there is no more any *drowning effect*, and we can see that even if $live(P, M)$ is one of the most reliable conclusions, there is some piece of information against it. We think that making a distinction between unchallenged conclusions and plausible but conflicting ones is an important feature of quasi-possibilistic logic.

3.2. Coherence function

Let us define $\pm a$ as a notation for $+a$ or $-a$.

Now define the set of models that subsume a model X (that is the set of models that strongly satisfy at least as many formulas as model X):

$$subsum(X) = \{Y \mid Y \neq X, Y^* \subseteq X^* \text{ and } \forall(+a_i\ n) \in X, \exists(+a_i\ m) \in Y \text{ with } m \leq n, \text{ and } \forall(-a_i\ n) \in X, \exists(-a_i\ m) \in Y \text{ with } m \leq n\}$$

Let us denote by $\mathsf{Q}\Pi(K)$ the set of strong models of K. Let us now define the notion of minimal $\mathsf{Q}\Pi$ model.

Definition 3.5. The set of minimal (strong) models of K is defined as :
$\mathsf{MQ}\Pi(K) = \{X \in \mathsf{Q}\Pi(K) \mid \text{ if } Y \in subsum(X), \text{ then } Y \notin \mathsf{Q}\Pi(K)\}$

We will now define a measure of consistency, called coherence, extending the one defined in [16]. It is a natural extension where fuzzy scalar cardinality replaces cardinality.

Let us first define the notions of $\mathsf{Conflictbase}_{\mathsf{Q}\Pi}$ and of $\mathsf{Opinionbase}_{\mathsf{Q}\Pi}$. As we need to define a measure here, we will from now on consider that the set of weights will be the $[0, 1]$ interval.

Definition 3.6. Let X be a model,
$\mathsf{Conflictbase}_{\mathsf{Q}\Pi}(X) = \{(a\ n) \mid (+a\ n_1) \in X \text{ and } (-a\ n_2) \in X \text{ and } n = min(n_1, n_2)\}$
$\mathsf{Opinionbase}_{\mathsf{Q}\Pi}(X) = \{(a\ n) \mid (\pm a\ n) \in X \text{ and } \nexists(\pm a\ m) \in X \text{ with } m > n\}$

Let us define now the amount of conflict and opinion corresponding to those two sets, that is a fuzzy scalar cardinality of those sets :

Definition 3.7. Let B be a set of pairs $(a\ n)$, where $a \in PS$ and $n \in [0, 1]$, then $\mathcal{A}(B) = \sum_{(a\ n) \in B} n$

Now the degree of coherence of a model is defined as

Definition 3.8. $\mathsf{Coherence}_{Q\Pi}$ is a function from the sets of interpretations to $[0,1]$ defined as:

$$\mathsf{Coherence}_{Q\Pi}(X) = 1 - \frac{\mathcal{A}(\mathsf{Conflictbase}_{Q\Pi}(X))}{\mathcal{A}(\mathsf{Opinionbase}_{Q\Pi}(X))}$$

If $\mathsf{Coherence}_{Q\Pi}(X) = 1$, then X is totally coherent, and if $\mathsf{Coherence}_{Q\Pi}(X) = 0$, then X is totally incoherent.

Example 3.1. (continued) If $X = \{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+c\ 0.5), (+f\ 0.2)\}$, and $Y = \{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+d\ 0.5), (+e\ 0.3), (+f\ 0.2)\}$. Then $\mathsf{Coherence}_{Q\Pi}(X) = 0.71$ and $\mathsf{Coherence}_{Q\Pi}(Y) = 0.75$.

Now we can define the degree of coherence of a knowledge base :

Definition 3.9. Let K be a knowledge base, then $\mathsf{Coherence}_{Q\Pi}(K)$ is defined as :

$$\mathsf{Coherence}_{Q\Pi}(K) = \max_{X \in \mathsf{MQ\Pi}(K)} \mathsf{Coherence}_{Q\Pi}(X)$$

Example 3.1. (continued) $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3), (\neg a \vee f, 0.2)\}$. $\mathsf{Coherence}_{Q\Pi}(K) = 0.75$

3.3. Coherence distribution

The Coherence function of the previous section allows us to concisely reflect the amount of conflict of a possibilistic knowledge base. But the drawback is that it slightly departs from the purely qualitative framework, since now conflicts are matter of degree, and several small conflicts can be as important as a big one.

We can figure out another generalization of the coherence function, based on m-cuts, that allows us to draw a more precise picture of the amount of conflicts in the knowledge base. More formally, the coherence distribution of a possibilistic knowledge base K is defined as the set :

$$\mathsf{DistCoherence}(K) = \{(K_m, \mathsf{Coherence}_{QC}(K_m^*)), m \in [0,1]\}$$

Example 3.1. (continued) $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3), (\neg a \vee f, 0.2)\}$.
$\mathsf{Coherence}_{QC}(K_{0.8}) = 1$, $\mathsf{Coherence}_{QC}(K_{0.6}) = 1/2$, $\mathsf{Coherence}_{QC}(K_{0.5}) = 2/3$,
$\mathsf{Coherence}_{QC}(K_{0.3}) = 3/4$, $\mathsf{Coherence}_{QC}(K_{0.2}) = 4/5$.

This example illustrates the interest of a more precise picture than the measure given by $\mathsf{Coherence}_{Q\Pi}$, since the coherence measure is not monotonic with respect to the level-cutting of the knowledge base. Figure 1 shows that in our example it is the 0.6 layer that is responsible for most of the conflicts in the base. Non-monotonicity of the coherence function is quite natural since adding new formulas in a knowledge base can bring more information (in this case Coherence increase), or bring more conflict (in this case Coherence descrease). When the new formulas bring both, it depends of their relative amount. See [15] for more explanations.

From this coherence distribution we can compute a Coherence degree that is derived from the distribution by using the certainty gaps between the m-cuts normalized as weights. This is the same kind of idea as in Section 2.1.5, and it is, technically speaking, a Choquet integral.

Definition 3.10.

$$\mathsf{Coherence}_{\mathrm{Q\Pi D}}(K) = \sum_{j=1,s} (m_j - m_{j+1}).\mathsf{Coherence}_{\mathrm{QC}}(K^*_{m_j})$$

where K_{m_j} is the m-cut of K and $m_1 = 1$.

Example 3.1. (continued) $\mathsf{Coherence}_{\mathrm{Q\Pi D}}(K) = (1-0.6)1+(0.6-0.5)1/2+(0.5-0.3)2/3+(0.3-0.2)3/4+(0.2)4/5 = 0.82$

We can see on this example that $\mathsf{Coherence}_{\mathrm{Q\Pi D}}(K)$ is greater than $\mathsf{Coherence}_{\mathrm{Q\Pi}}(K)$, but it is not always the case. For example if we take $K = \{(a,1), (\neg a,1), (b,0.6)\}$, then $\mathsf{Coherence}_{\mathrm{Q\Pi}}(K) = 0.375$ whereas $\mathsf{Coherence}_{\mathrm{Q\Pi D}}(K) = 0.3$.

But one can note that the two functions are true generalizations of the significance function in the quasi-classical case, so if all the formulas of the knowledge bases have the same weight, the two functions give the same result, and if this weight is 1, then they give the same result as quasi-classical significance. It is also interesting to note that the extreme cases are the same for the two functions, they both give 0 iff there is no conflict in the base (i.e. K^* is classically consistent), and they both gives 1 iff the weights of all the formulas of the base are 1 and $\mathsf{Coherence}_{\mathrm{QC}}(K^*) = 1$.

3.4. Significance function

We can also define counterparts of Hunter's significance function [18] in this quasi-possibilistic framework.

In this framework, the significance function must take into account the strength of the conflicts on literals. For example if $\mu(\{+a, -a\}) = 0.3$, then for the two bases $K = \{(a, 0.1), (\neg a, 1)\}$ and $K' =$

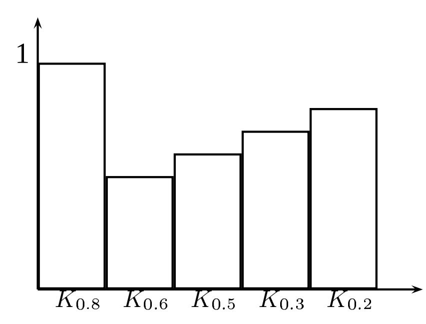

Figure 1. Coherence distribution of K

$\{(a,1),(\neg a,1)\}$, we notice that there is a big conflict in K' about a, whereas in K one of the two literals has a very weak support, so the conflict is a very mild one. So we will define significance functions that take this strength of conflicts into account for handling the meta-information given by the mass assignment. To explain this we can go back to the soccer match example of Section 2.2.3. Suppose that our news reports come from three different newspaper. The first news report says that the winner of the match is the A team. This newspaper is a sport newspaper, so we have a high confidence in its news report. The second newspaper is a main national one, says that the referee was Belgian and that team A won. The last newspaper is a little regional newspaper with a quickly written news report. It says that the referee was Italian and that team B won. So we can write it as : $K = \{(a, 0.8), (a \wedge b, 0.6), (\neg a \wedge \neg b, 0.4)\}$, where a denotes a win of team A and b the Belgian nationality of the referee. With the mass assignment $\mu(\{+a, -a\}) = 0.3$ and $\mu(\{+b, -b\}) = 0.01$ (the remaining mass is devoted to other potential conflicts not detailed here), it is then necessary to be able to define what is the significance of the conflicts, accounting for both the confidence in the information and the relative importance of the conflicts. The mass assignment is defined as in the classical framework, but as the models here are weighted, we will need to use the weight-forgetting function.

Let us state now some of the expected properties of a significance function S. After that we will try to generalize significance functions in the framework of quasi-possibilistic logic while satisfying those properties.

(S1) If $\mathsf{Coherence}(K) = 1$, then $S(K) = 0$

(S2) If $\mathsf{Coherence}(K) = 0$, then $\forall K'$ s.t. $Atoms(K') = Atoms(K)$ $S(K') \leq S(K)$

(S3) $S(K \cup K') \geq \max(S(K), S(K'))$

(S4) If $Atoms(K) \cap Atoms(K') = \emptyset$, then $S(K \cup K') \geq S(K) + S(K')$

The first property is the minimality, it states that if there is no conflict in a knowledge base, then the significance of the (non-existing) conflict is 0. The second property is maximality, it states that the greatest significance of a knowledge base is reached when it is fully conflictual. This property, along with the definition of the mass assignment, allows us to deduce that a knowledge base K with $\mathsf{Coherence}(K) = 0$ and $Atoms(K) = PS$ has a significance of 1. The third property states that when joining two knowledge bases, the significance of the conflict is at least as important as the one of each of the knowledge bases. This means that having more information can not decrease the significance of the existing conflicts. So significance is monotonic, which contrats with coherence (see Section 3.3). The last property is a kind of separability property. If two knowledge bases are independent in the sense that they do not share any propositional symbol, then the significance of the union is greater than the sum of the significance of the two bases. This is easily explained : when taking two bases together we get all the conflicts (so all the significance) that are in only one base, but we also get new conflicts (so potentially more significance).

One can check that those properties are satisfied by the significance function in the quasi-classical case. So one could expect them to hold also for its generalizations in the quasi-possibilistic framework. It is the case for the ones we define in the following, except for the *qualitative significance* function. Since its definition is no more based on a sum, but on a purely qualitative one, property (S4) is not satisfied as such. We have to change the "$+$" symbol by a $\max$ as usual, but in this case property (S4) is subsumed by property (S3).

3.4.1. Cardinal Significance

This first definition of significance function is a naive generalization of Hunter's definition, based on a fuzzy scalar cardinality :

Definition 3.11. Let B be a set of pairs $(a\ n)$, where $a \in PS$ and $n \in [0,1]$, then $\mathcal{C}(B) = \frac{\sum_{(a\ n)\in B} n}{|B|}$

Definition 3.12. A possibilistic cardinal significance function (induced by a mass assignment μ), denoted $S^{\mu}_{\mathrm{Q\Pi}_1}$ is a function from $2^{\mathcal{O}_{\mathcal{PS},\mathcal{L}}}$ into $[0,1]$ defined as :

$$S^{\mu}_{\mathrm{Q\Pi}_1}(X) = \sum_{W \subseteq X} \mu(W^*) \times \mathcal{C}(\textsf{Conflictbase}_{\mathrm{Q\Pi}}(W))$$

Example 3.1. (continued) Let $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3), (\neg a \vee f, 0.2)\}$. And let $\mu(\{+a, -a\}) = 0.1$, $\mu(\{+a, -a, +c\} = 0.4$, $\mu(\{+c, -c\}) = 0.3$, $\mu(\{+f, -f\}) = 0.2$ be the corresponding mass assignment. Then we have $S_{\mathrm{Q\Pi}_1}(\{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+c\ 0.5), (+f\ 0.2)\}) = 0.5 \times 0.6 = 0.3$, and $S_{\mathrm{Q\Pi}_1}(\{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+d\ 0.5), (+e\ 0.3), (+f\ 0.2)\}) = 0.1 \times 0.6 = 0.06$, and $S_{\mathrm{Q\Pi}_1}(K) = 0.06$.

This definition, based on a fuzzy scalar cardinality suffers from the fact that several small conflicts can be as important as a big one. So if one wants to avoid this situation, one has to seek for other indices.

3.4.2. Choquet Significance

This generalization of the significance function allows us to take into account the certainty gap between conflicts. It is based on a Choquet integral.

Definition 3.13. A possibilistic Choquet significance function (induced by a mass assignment μ), denoted $S^{\mu}_{\mathrm{Q\Pi}_3}$ is a function from $2^{\mathcal{O}_{\mathcal{PS},\mathcal{L}}}$ into $[0,1]$ defined as :

$$S^{\mu}_{\mathrm{Q\Pi}_2}(X) = \sum_{j=1,s} (m_j - m_{j+1}) S^{\mu}_{\mathsf{QC}}(X^*_{m_j})$$

Example 3.1. (continued) Let $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3), (\neg a \vee f, 0.2)\}$. And let $\mu(\{+a, -a\}) = 0.1$, $\mu(\{+a, -a, +c\} = 0.4$, $\mu(\{+c, -c\}) = 0.3$, $\mu(\{+f, -f\}) = 0.2$ be the corresponding mass assignment. Then we have $S_{\mathrm{Q\Pi}_2}(\{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+c\ 0.5), (+f\ 0.2)\}) = (0.8-0.6)0 + (0.6-0.5)0.1 + (0.5-0.2)0.5 + (0.2)0.5 = 0.26$ and $S_{\mathrm{Q\Pi}_2}(\{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+d\ 0.5), (+e\ 0.3), (+f\ 0.2)\}) = (0.6)0.1 = 0.06$, and $S_{\mathrm{Q\Pi}_2}(K) = 0.06$.

We can define a distribution of significance for a base, as done for the coherence function in Section 3.3. This allows us to have a more precise representation of this measure. But, conversely, computing directly $S_{\mathrm{Q\Pi}_2}$ gives a concise view of this distribution.

This definition is still quantitative since we use the certainty gaps between m-cuts. In the following section we will define a purely qualitative significance function.

3.4.3. Qualitative Significance

We need to redefine the mass assignment in order to get a more qualitative definition.

Definition 3.14. A qualitative mass assignment μ^* is a function from $2^{\mathcal{O}_{\mathcal{PS}}}$ to $[0,1]$ such that

- If $\mathsf{Coherence}_{\text{Q}\Pi}(X^*) = 1$, then $\mu^*(X^*) = 0$
- $\max_X \mu^*(X^*) = 1$

Definition 3.15. Let B be a set of pairs $(a\ n)$, where $a \in PS$ and $n \in [0,1]$, then $\mathcal{B}(B) = \max\{n \mid (a\ n) \in B\}$.

Definition 3.16. A qualitative possibilistic significance function (induced by a qualitative mass assignment μ^*), denoted $S^{\mu^*}_{\text{Q}\Pi_2}$ is a function from $2^{\mathcal{O}_{\mathcal{PS},\mathcal{L}}}$ into $[0,1]$ defined as :

$$S^{\mu^*}_{\text{Q}\Pi_3}(X) = \max_{W^* \in X^*} \min(\mu^*(W^*), \mathcal{B}(\mathsf{Conflictbase}_{\text{Q}\Pi}(W)))$$

Example 3.1. (continued) Let $K = \{(a, 0.8), (\neg a \wedge b, 0.6), (c \vee d, 0.5), (c \vee e, 0.3)$. And let $\mu^*(\{+a, -a\}) = 0.1$, $\mu^*(\{+a, -a, +c\} = 0.4$, $\mu^*(\{+c, -c\}) = 0.3$, $\mu^*(\{+f, -f\}) = 0.2$, $\mu^*(\{+b, -b\}) = 1$ be the corresponding mass assignment. Then we have $S_{\text{Q}\Pi_3}(\{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+c\ 0.5), (+f\ 0.2)\}) = \max(\min(\{0.1, 0.6\}, \min(\{0.4, 0.6\})) = 0.4$, and $S_{\text{Q}\Pi_3}(\{(+a\ 0.8), (-a\ 0.6), (+b\ 0.6), (+d\ 0.5), (+e\ 0.3), (+f\ 0.2)\}) = \max(\min(\{0.1, 0.6\})) = 0.1$, and $S_{\text{Q}\Pi_3}(K) = 0.1$.

Whereas cardinal significance and Choquet significance are true generalization of Hunter's significance in the sense that if all the weights are 0 or 1, then Hunter's significance is recovered, it is not the case with this qualitative significance function, that gives in the $\{0,1\}$ case (i.e. in the quasi-classical framework) a significance of 0 if there is no conflict, and 1 if there is one in the interpretation.

4. Concluding remarks

This paper has provided a first introduction of quasi-possibilistic logic, a logic aiming at handling both plain contradictions and priority (or certainty) levels of the pieces of information in a unified way. Quasi-possibilistic logic has still to be developed, in particular its syntactic counterpart (with its associated inference algorithms). The development of the syntactic machinery of $Q\Pi L$ may be discussed in the more general setting of the definition of possibility and necessity measures on non-classical logic structures, a question which has been already considered in [5]. However, in [5], only one particular logic, da Costa's C_1, which substentially differs from QCL, is taken as an example of paraconsitent logic associated with necessity measures. Moreover a more systematic comparison of this inference with other inconsistency-tolerant inference relations which go beyond standard possibilistic inference has to be carried on.

The paper has discussed several types of generalized measures of information and conflict, similar ideas could be thought of for another type of weighted logic, named penalty logic [10]. Indeed in penalty logic, the cost of interpretations can be related to the contour function of a belief structure. Information measures also exist in the latter framework and could be adapted to penalty logic.

Acknowlegments

The authors would like to thank Anthony Hunter and Pierre Marquis for their hepful comments on an earlier version of this work.

References

[1] Belnap, N.: A useful four-valued logic, *Modern uses of multiple-valued logic*, 1977, 8–37.

[2] Benferhat, S., Dubois, D., Prade, H.: Some syntactic approaches to the handling of inconsistent knowledge bases: a comparative study, part 1: the flat case, *Studia Logica*, **58**, 1997, 17–45.

[3] Benferhat, S., Dubois, D., Prade, H.: An overview of inconsistency-tolerant inferences in prioritized knowledge bases, in: *Fuzzy Sets, Logic and Reasoning about Knowledge*, vol. 15 of *Applied Logic Series*, Kluwer, 1999, 395–417.

[4] Besnard, P., Hunter, A.: Quasi-classical logic: Non-trivializable classical reasoning from inconsistent information, *Proceedings of the 3rd European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty (ECSQARU'95)*, LNAI 946, Springer Verlag, 1995.

[5] Besnard, P., Lang, J.: Possibility and necessity functions over non-classical logics, *Proceedings of the 10th Int. Conf. on Uncertainty in Artificial Intelligence (UAI'94)*, 1994.

[6] Cayrol, C., Lagasquie-Schiex, M.: Non-monotonic syntax-based entailment: a classification of consequence relations, *Proceedings of the 3rd European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty (ECSQARU'95)*, LNAI 946, Springer Verlag, 1995.

[7] Dubois, D., Lang, J., Prade, H.: Automated reasoning using possibilistic logic: Semantics, belief revision and variable certainty weights, *IEEE Trans. Data and Knowledge Engineering*, **6**, 1994, 64–71.

[8] Dubois, D., Lang, J., Prade, H.: *Handbook of Logic in Artificial Intelligence and Logic Programming*, vol. 3, chapter Possibilistic logic, Oxford University Press, 1994, 439–513.

[9] Dubois, D., Prade, H.: Properties of measures of information in evidence and possibility theories, *Fuzzy Sets and Systems*, **24**, 1987, 161–182, Reprinted in Fuzzy Sets and Systems, supplement to Vol. 100, 35-49, 1999.

[10] Dupin de Saint-Cyr, F., Lang, J., Schiex, T.: Penalty logic and its links with Dempster-Shafer theory, *Proceedings of the 10th Int. Conf. on Uncertainty in Artificial Intelligence (UAI'94)*, 1994.

[11] Gärdenfors, P.: *Knowledge in flux*, MIT Press, 1988.

[12] Hartley, R. V. L.: Transmission of information, *The Bell Systems Technical Journal*, 1928, 535–563.

[13] Higashi, M., Klir, G. J.: Measures of uncertainty and information based on possibility distributions, *International Journal of General Systems*, **9**, 1983, 43–58.

[14] Hunter, A.: *Handbook of Defeasible Reasoning and Uncertain Information*, vol. 2, chapter Paraconsistent logics, Kluwer, 1998.

[15] Hunter, A.: Reasoning with conflicting information using quasi-classical logic, *Journal of Logic and Computation*, **10**, 2000, 677–703.

[16] Hunter, A.: Measuring inconsistency in knowledge via quasi-classical models, *Proceedings of the American National Conference on Artificial Intelligence (AAAI'02)*, 2002.

[17] Hunter, A.: Evaluating coherence and compromise in inconsistent information, 2003, Manuscript. University College London.

[18] Hunter, A.: Evaluating significance of inconsistencies, *Proceedings of the 18th International Joint Conference in Artificial Intelligence (IJCAI'03)*, 2003.

[19] Katsuno, H., Mendelzon, A. O.: On the difference between updating a knowledge base and revising it, *Proceedings of the 2nd International Conference on Principles of Knowledge Representation and Reasoning (KR'91)*, 1991.

[20] Konieczny, S., Lang, J., Marquis, P.: Quantifying information and contradiction in propositional logic through test actions, *Proceedings of the 18th International Joint Conference in Artificial Intelligence (IJCAI'03)*, 2003.

[21] Konieczny, S., Pino Pérez, R.: Merging information under constraints: a qualitative framework, *Journal of Logic and Computation*, **12**(5), 2002, 773–808.

[22] Kraus, S., Lehmann, D., Magidor, M.: Nonmonotonic reasoning, preferential models and cumulative logics, *Artificial Intelligence*, **44**, 1990, 167–207.

[23] Lang, J.: *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, vol. 5, Kluwer, 2000.

[24] Lang, J., Marquis, P.: In Search of the Right Extension, *Proceedings of the 7th International Conference on Principles of Knowledge Representation and Reasoning (KR'00)*, 2000.

[25] Lang, J., Marquis, P.: Resolving Inconsistencies by Variable Forgetting, *Proceedings of the 8th International Conference on Principles of Knowledge Representation and Reasoning (KR'02)*, 2002.

[26] Lozinskii, E.: Information and evidence in logic systems, *Journal of Experimental and Theoretical Artificial Intelligence*, **6**, 1994, 163–193.

[27] Lozinskii, E.: Resolving contradictions: a plausible semantics for inconsistent systems, *Journal of Automated Reasoning*, **12**, 1994, 1–31.

[28] Marquis, P., Porquet, N.: Computational aspects of quasi-classical entailment, *Journal of Applied Non-Classical Logics*, **11**, 2001, 295–312.

[29] Nebel, B.: *Handbook of Defeasible Reasoning and Uncertainty Management Systems*, vol. 3, chapter How hard is it to revise a belief base?, Kluwer Academic Publishers, 1998, 77–145.

[30] Ramer, A.: Concepts of fuzzy information measures on continuous domains, *International Journal of General Systems*, **17**, 1990, 241–248.

[31] Rescher, N., Manor, R.: On inference from inconsistent premises, *Theory and Decision*, **1**, 1970, 179–219.

[32] Yager, R. R.: Entropy and specificity in a mathematical theory of evidence, *International Journal of General Systems*, **9**, 1983, 249–260.