Merging with Integrity Constraints *

Sébastien Konieczny and Ramón Pino Pérez

Laboratoire d'Informatique Fondamentale de Lille
Université de Lille 1, 59655 Villeneuve d'Ascq, France
{konieczn,pino}@lifl.fr

Abstract. We consider, in this paper, the problem of knowledge base merging with integrity constraints. We propose a logical characterization of those operators and give a representation theorem in terms of preorders on interpretations. We show the close connection between belief revision and merging operators and we show that our proposal extends the pure merging case (i.e. without integrity constraints) we study in a previous work. Finally we show that Liberatore and Schaerf commutative revision operators can be seen as a special case of merging.

1 Introduction

An important issue of distributed knowledge systems is to be able to determine a global consistent state (knowledge) of the system. Consider, for example, the problem of the combination of several expert systems. Suppose that each expert system codes the knowledge of an human expert. To build an expert system it is reasonable to try to combine all these knowledge bases in a single knowledge base that expresses the knowledge of the expert group. This process allows to discover new pieces of knowledge distributing among the sources. For example if an expert knows that a is true and another knows that $a \rightarrow b$ holds, then the "synthesized" knowledge knows that b is true whereas none of the expert knows it. This was called *implicit knowledge* in [8]. However, simply put these knowledge bases together is a wrong way since there could be contradictions between some experts.

Some logical characterizations of merging have been proposed [18, 19, 13, 14, 16, 15, 11]. In this paper we extend these works by proposing a logical characterization when the result of the merging has to obey to a set of integrity constraints.

We define two subclasses of merging operators, namely majority merging and arbitration operators. The former striving to satisfy a maximum of protagonists, the latter trying to satisfy each protagonist to the best possible degree. In other words majority operators try to minimize global dissatisfaction whereas arbitration operators try to minimize individual dissatisfaction.

* The proofs have been omitted for space requirements but can be found in the extended version of this work [12].

We also provide a representation theorem *à la* Katsuno Mendelzon [9] and we show the close connections between belief revision and merging operators.

In section 2 we state some notations. In section 3 we propose a logical definition of merging operators with integrity constraints, we define majority and arbitration operators and give a model-theoretic representation of those operators. In section 4 we define two families of merging operators illustrating the logical definition. In section 5 we show the connections with other related works, first we show the close connection between belief revision and merging operators, then we show that this work extends the one of [11]. Finally we show that Liberatore and Schaerf commutative revision operators can be seen as a special case of merging operators. In section 6 we give some conclusions and discuss future work.

2 Preliminaries

We consider a propositional language $\mathcal{L}$ over a finite alphabet $\mathcal{P}$ of propositional atoms. An interpretation is a function from $\mathcal{P}$ to $\{0,1\}$. The set of all the interpretations is denoted $\mathcal{W}$. An interpretation I is a model of a formula if and only if it makes it true in the usual classical truth functional way. Let φ be a formula, $mod(\varphi)$ denotes the set of models of φ, *i.e.* $mod(\varphi) = \{I \in \mathcal{W}\ I \models \varphi\}$.

A *knowledge base* K is a finite set of propositional formulae which can be seen as the formula φ which is the conjunction of the formulae of K.

Let $\varphi_1, \ldots, \varphi_n$ be n knowledge bases (not necessarily different). We call *knowledge set* the multi-set Ψ consisting of those n knowledge bases: $\Psi = \{\varphi_1, \ldots, \varphi_n\}$. We note $\bigwedge \Psi$ the conjunction of the knowledge bases of Ψ, i.e. $\bigwedge \Psi = \varphi_1 \wedge \cdots \wedge \varphi_n$. The union of multi-sets will be noted $\sqcup$. Knowledge bases will be denoted by lower case Greek letters and knowledge sets by upper case Greek letters.

Since an inconsistent knowledge base gives no information for the merging process, we will suppose in the rest of the paper that the knowledge bases are consistent.

Definition 1. *A knowledge set Ψ is consistent if and only if $\bigwedge \Psi$ is consistent. We will use $mod(\Psi)$ to denote $mod(\bigwedge \Psi)$ and write $I \models \Psi$ for $I \in mod(\Psi)$.*

Definition 2. *Let Ψ_1, Ψ_2 be two knowledge sets. Ψ_1 and Ψ_2 are equivalent, noted $\Psi_1 \leftrightarrow \Psi_2$, iff there exists a bijection f from $\Psi_1 = \{\varphi^1_1, \ldots, \varphi^1_n\}$ to $\Psi_2 = \{\varphi^2_1, \ldots, \varphi^2_n\}$ such that $\vdash f(\varphi) \leftrightarrow \varphi$.*

A pre-order $\leq$ over $\mathcal{W}$ is a reflexive and transitive relation on $\mathcal{W}$. A pre-order is total if $\forall I, J \in \mathcal{W}\ I \leq J$ or $J \leq I$. Let $\leq$ be a pre-order over $\mathcal{W}$, we define $<$ as follows: $I < J$ iff $I \leq J$ and $J \not\leq I$, and $\simeq$ as $I \simeq J$ iff $I \leq J$ and $J \leq I$. We wrote $I \in min(mod(\varphi), \leq)$ iff $I \models \varphi$ and $\forall J \in mod(\varphi)\ I \leq J$.

By abuse if φ is a knowledge base, φ will also denote the knowledge set $\Psi = \{\varphi\}$. For a positive integer n we will denote Ψ^n the multi-set $\{\underbrace{\Psi, \ldots, \Psi}_{n}\}$.

3 Merging with Integrity Constraints

We first state a logical definition for merging with integrity constraints operators (IC merging operators for now on), that is we give a set of properties an operator has to satisfy in order to have a rational behaviour concerning the merging.

In this work, the result of the merging has to obey a set of integrity constraints where each integrity constraint is a formula. We suppose that these constraints do not contradict each others and we bring them together in a knowledge base μ. This knowledge base represents the constraints for the result of merging the knowledge set Ψ, not for the knowledge bases in Ψ. Thus a knowledge base φ in Ψ does not obey necessarily to the constraints. We consider that integrity constraints has to be true in the knowledge base resulting of merging Ψ, that is the result is not only consistent with the constraints (as it is often the case in databases), but it has to imply the constraints.

We will consider operators $\triangle$ mapping a knowledge set Ψ and a knowledge base μ to a knowledge base $\triangle_\mu(\Psi)$ that represents the merging of the knowledge set Ψ according to the integrity constraints μ.

Definition 3. *$\triangle$ is an IC merging operator if and only if it satisfies the following postulates:*

(IC0) $\triangle_\mu(\Psi) \vdash \mu$
(IC1) *If μ is consistent, then $\triangle_\mu(\Psi)$ is consistent*
(IC2) *If $\bigwedge \Psi$ is consistent with μ, then $\triangle_\mu(\Psi) = \bigwedge \Psi \wedge \mu$*
(IC3) *If $\Psi_1 \leftrightarrow \Psi_2$ and $\mu_1 \leftrightarrow \mu_2$, then $\triangle_{\mu_1}(\Psi_1) \leftrightarrow \triangle_{\mu_2}(\Psi_2)$*
(IC4) *If $\varphi \vdash \mu$ and $\varphi' \vdash \mu$, then $\triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi \nvdash \bot \Rightarrow \triangle_\mu(\varphi \sqcup \varphi') \wedge \varphi' \nvdash \bot$*
(IC5) $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2) \vdash \triangle_\mu(\Psi_1 \sqcup \Psi_2)$
(IC6) *If $\triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$ is consistent, then $\triangle_\mu(\Psi_1 \sqcup \Psi_2) \vdash \triangle_\mu(\Psi_1) \wedge \triangle_\mu(\Psi_2)$*
(IC7) $\triangle_{\mu_1}(\Psi) \wedge \mu_2 \vdash \triangle_{\mu_1 \wedge \mu_2}(\Psi)$
(IC8) *If $\triangle_{\mu_1}(\Psi) \wedge \mu_2$ is consistent, then $\triangle_{\mu_1 \wedge \mu_2}(\Psi) \vdash \triangle_{\mu_1}(\Psi)$*

The meaning of the postulates is the following: *(IC0)* assures that the result of the merging satisfies the integrity constraints. *(IC1)* states that if the integrity constraints are consistent, then the result of the merging will be consistent. *(IC2)* states that if possible, the result of the merging is simply the conjunction of the knowledge bases with the integrity constraints. *(IC3)* is the principle of irrelevance of syntax, i.e. if two knowledge sets are equivalent and two integrity constraints bases are logically equivalent then the knowledge bases result of the two merging will be logically equivalent. *(IC4)* is the fairness postulate, the point is that when we merge two knowledge bases, merging operators must not give preference to one of them. *(IC5)* expresses the following idea: if a group Ψ_1 compromises on a set of alternatives which I belongs to, and another group Ψ_2 compromises on another set of alternatives which contains I, so I has to be in the chosen alternatives if we join the two groups. *(IC5)* and *(IC6)* together state that if you could find two subgroups which agree on at least one alternative, then the result of the global arbitration will be exactly those alternatives the two groups agree on. *(IC5)* and *(IC6)* have been proposed by Revesz [19] for weighted model-fitting operators. *(IC7)* and *(IC8)* are a direct generalization of the *(R5-R6)* postulates for revision. They states that the notion of closeness is well-behaved (see [9] for a full justification).

Now we define two merging operators subclasses, namely majority merging operators and arbitration operators.

A majority merging operator is an IC merging operator that satisfies the following *majority* postulate:

(Maj) $\exists n \;\; \triangle_\mu\,(\Psi_1 \sqcup {\Psi_2}^n) \vdash \triangle_\mu(\Psi_2)$

This postulate expresses the fact that if an opinion has a large audience, it will be the opinion of the group.

An arbitration operator is an IC merging operator that satisfies the following postulate:

$$\textbf{(Arb)} \quad \left.\begin{array}{l} \triangle_{\mu_1}(\varphi_1) \leftrightarrow \triangle_{\mu_2}(\varphi_2) \\ \triangle_{\mu_1 \leftrightarrow \neg\mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow (\mu_1 \leftrightarrow \neg\mu_2) \\ \mu_1 \nvdash \mu_2 \\ \mu_2 \nvdash \mu_1 \end{array}\right\} \Rightarrow \triangle_{\mu_1 \vee \mu_2}(\varphi_1 \sqcup \varphi_2) \leftrightarrow \triangle_{\mu_1}(\varphi_1)$$

This postulate ensures that this is the median possible choices that are preferred.

Now that we have a logical definition of IC merging operators, we will define a representation theorem that give a more intuitive way to define IC merging operators. More precisely we show that to each IC merging operator corresponds a family of pre-orders on possible worlds. Let's first define the following:

Definition 4. *A* syncretic assignment *is a function mapping each knowledge set Ψ to a total pre-order $\leq_\Psi$ over interpretations such that for any knowledge sets Ψ, Ψ_1, Ψ_2 and for any knowledge bases φ_1, φ_2:*

1. *If $I \models \Psi$ and $J \models \Psi$, then $I \simeq_\Psi J$*
2. *If $I \models \Psi$ and $J \not\models \Psi$, then $I <_\Psi J$*
3. *If $\Psi_1 \equiv \Psi_2$, then $\leq_{\Psi_1} = \leq_{\Psi_2}$*
4. *$\forall I \models \varphi_1 \; \exists J \models \varphi_2 \; J \leq_{\varphi_1 \sqcup \varphi_2} I$*
5. *If $I \leq_{\Psi_1} J$ and $I \leq_{\Psi_2} J$, then $I \leq_{\Psi_1 \sqcup \Psi_2} J$*
6. *If $I <_{\Psi_1} J$ and $I \leq_{\Psi_2} J$, then $I <_{\Psi_1 \sqcup \Psi_2} J$*

A majority syncretic assignment *is a syncretic assignment which satisfies the following:*

7. *If $I <_{\Psi_2} J$, then $\exists n \; I <_{\Psi_1 \sqcup {\Psi_2}^n} J$*

A fair syncretic assignment *is a syncretic assignment which satisfies the following:*

$$8. \quad \left.\begin{array}{l} I <_{\varphi_1} J \\ I <_{\varphi_2} J' \\ J \simeq_{\varphi_1 \sqcup \varphi_2} J' \end{array}\right\} \Rightarrow I <_{\varphi_1 \sqcup \varphi_2} J$$

The following theorem states that these conditions on the assignment corresponds to the properties of the merging operator:

Theorem 1. *An operator is an IC merging operator (respectively IC majority merging operator or IC arbitration operator) if and only if there exists a syncretic assignment (respectively majority syncretic assignment or fair syncretic assignment) that maps each knowledge set* Ψ *to a total pre-order* $\leq_\Psi$ *such that*

$$mod(\triangle_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi).$$

As pointed out by D. Makinson (personal communication), this definition of merging operators from such assignments can be compared to the framework of Social Choice Theory [2, 10, 17]. The aim of Social Choice Theory is to aggregate individual choices into a social choice, *i.e.* to find, for a given set of agents (corresponding to our knowledge sets) with individual preference relations, a social preference relation which reflects the preferences of the set of agents. It turns out that the conditions 5 and 6 of the syncretic assignment are known in this framework as the Pareto conditions and are widely seen as desirable. This bring an additional support to postulates (IC5) and (IC6) that correspond respectively to conditions 5 and 6.

We will show in the next section that the set of postulates (IC0-IC8) is consistent by given two families of operators satisfying these postulates. That is we do not demand to much to merging operators. On the other hand these postulates are sufficiently strong to rule out basic merging methods. For example we can define a merging operator *à la* full meet revision, that is:

$$\triangle_\mu(\Psi) = \begin{cases} \Psi \wedge \mu & \text{if consistent} \\ \mu & \text{otherwise} \end{cases}$$

But this operator is not an IC merging operator since it does not satisfy (IC6).

An other basic merging method generally accepted is the conjunction of the knowledge bases if consistent and their disjunction otherwise, the generalization of this operator in the presence of integrity constraints is the following, if $\Psi = \{\varphi_1, \ldots, \varphi_n\}$:

$$\triangle_\mu(\Psi) = \begin{cases} \bigwedge \varphi_i \wedge \mu & \text{if consistent, else} \\ \bigvee \varphi_i \wedge \mu & \text{if consistent} \\ \mu & \text{otherwise} \end{cases}$$

This operator is not an IC merging operator since it does not satisfy (IC6).

In [5] Benferhat et al. proposed merging operators in the possibility theory framework and gave their syntactic counterpart. Their operators merge two possibility distributions in a new one. Therefore the nature of the information they merge is very different from knowledge sets. Nevertheless one can identify their set of possibility distributions with a knowledge set in a natural way. In this case, their operators do not satisfy (IC4) nor (IC6). However with some strong constraints on the possibility distributions their LUK operator is a majority merging operator.

4 Examples of Operators

We define in this section two families of operators. The first one, the Σ family, is a family of majority merging operators. The second one, the *Gmax* family, gives arbitration operators.

We will suppose here that we dispose of a distance between interpretations (possible worlds), that is a function $d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{N}$ such that $d(I, J) = d(J, I)$ and $d(I, J) = 0$ iff $I = J$.

From now on we define the distance between an interpretation I and a knowledge base φ in the following way: $d(I, \varphi) = \min_{J \models \varphi} d(I, J)$

Definition 5. *Let Ψ be a knowledge set and let I be an interpretation we define the distance between an interpretation and a knowledge set as: $d_\Sigma(I, \Psi) = \sum_{\varphi \in \Psi} d(I, \varphi)$. Then we have the following pre-order: $I \leq_\Psi^\Sigma J$ iff $d_\Sigma(I, \Psi) \leq d_\Sigma(J, \Psi)$. And the operator $\triangle^\Sigma$ is defined by: $mod(\triangle_\mu^\Sigma(\Psi)) = \min(mod(\mu), \leq_\Psi^\Sigma)$.*

Theorem 2. *$\triangle^\Sigma$ is an IC majority merging operator.*

Definition 6. *Let Ψ be a knowledge set. Suppose $\Psi = \{\varphi_1 \ldots \varphi_n\}$. For each interpretation I we build the list $(d_1^I \ldots d_n^I)$ of distances between this interpretation and the n knowledge bases in Ψ, i.e. $d_j^I = d(I, \varphi_j)$. Let L_I^Ψ be the list obtained from $(d_1^I \ldots d_n^I)$ by sorting it in descending order. Let $\leq_{lex}$ be the lexicographical order between sequences of integers (of the same length), we define the following pre-order: $I \leq_\Psi^{GMax} J$ iff $L_I^\Psi \leq_{lex} L_J^\Psi$. And the operator $\triangle^{GMax}$ is defined by: $mod(\triangle_\mu^{GMax}(\Psi)) = \min(mod(\mu), \leq_\Psi^{GMax})$.*

Theorem 3. *$\triangle^{GMax}$ is an IC arbitration operator.*

We now give a "concrete" merging example and illustrate the behaviour of the two families of operators defined above it. We will choose as distance for the operators the Dalal distance [6]. The Dalal distance between two interpretations is the number of propositional letters on which the two interpretations differ.

Example: At a meeting of a block of flats co-owners, the chairman proposes for the coming year the construction of a swimming-pool, a tennis-court and a private-car-park. But if two of these three items are build, the rent will increase significantly. We will denote by S, T, P respectively the construction of the swimming-pool, the tennis-court and the private-car-park. We will denote I the rent increase.

The chairman outlines that build two items or more will have an important impact on the rent: $\mu = ((S \wedge T) \vee (S \wedge P) \vee (T \wedge P)) \rightarrow I$

There is four co-owners $\Psi = \{\varphi_1 \sqcup \varphi_2 \sqcup \varphi_3 \sqcup \varphi_4\}$. Two of the co-owners want to build the three items and don't care about the rent increase: $\varphi_1 = \varphi_2 = S \wedge T \wedge P$. The third one thinks that build any item will cause at some time an increase of the rent and want to pay the lowest rent so he is opposed to any construction: $\varphi_3 = \neg S \wedge \neg T \wedge \neg P \wedge \neg I$. The last one thinks that the flat really needs a tennis-court and a private-car-park but don't want a high rent increase : $\varphi_4 = T \wedge P \wedge \neg I$.

The propositional letters S, T, P, I will be considered in that order for the valuations:

$$mod(\mu) = \mathcal{W} \setminus \{(0,1,1,0),(1,0,1,0),(1,1,0,0),(1,1,1,0)\}$$

$mod(\varphi_1) = \{(1,1,1,1),(1,1,1,0)\}$ $\quad$ $mod(\varphi_2) = \{(1,1,1,1),(1,1,1,0)\}$

$mod(\varphi_3) = \{(0,0,0,0)\}$ $\quad$ $mod(\varphi_4) = \{(1,1,1,0),(0,1,1,0)\}$

We sum up the calculations in table 1. The lines shadowed correspond to the interpretations rejected by the integrity constraints. Thus the result has to be found among the interpretations that are not shadowed.

Table 1. Distances

	φ_1	φ_2	φ_3	φ_4	$\mathbf{dist}_{\mathbf{\Sigma}}$	$\mathbf{dist}_{\mathbf{GMax}}$
$(0,0,0,0)$	3	3	0	2	8	(3,3,2,0)
$(0,0,0,1)$	3	3	1	3	10	(3,3,3,1)
$(0,0,1,0)$	2	2	1	1	6	(2,2,1,1)
$(0,0,1,1)$	2	2	2	2	8	(2,2,2,2)
$(0,1,0,0)$	2	2	1	1	6	(2,2,1,1)
$(0,1,0,1)$	2	2	2	2	8	(2,2,2,2)
$(0,1,1,0)$	1	1	2	0	4	(2,1,1,0)
$(0,1,1,1)$	1	1	3	1	6	(3,1,1,1)
$(1,0,0,0)$	2	2	1	2	7	(2,2,2,1)
$(1,0,0,1)$	2	2	2	3	9	(3,2,2,2)
$(1,0,1,0)$	1	1	2	1	5	(2,1,1,1)
$(1,0,1,1)$	1	1	3	2	7	(3,2,1,1)
$(1,1,0,0)$	1	1	2	1	5	(2,1,1,1)
$(1,1,0,1)$	1	1	3	2	7	(3,2,1,1)
$(1,1,1,0)$	0	0	3	0	3	(3,0,0,0)
$(1,1,1,1)$	0	0	4	1	5	(4,1,0,0)

With $\triangle^{GMax}$ as merging criterion $mod(\triangle_\mu^{GMax}(\Psi)) = \{(0,0,1,0),(0,1,0,0)\}$, so the decisions that best fit the group and that are allowed by the integrity constraints are to build either the tennis-court or the private-car-park, without increase the rent. Whereas if one takes the decision according to the majority wishes then with the $\triangle^{\Sigma}$ operator we have $mod(\triangle_\mu^{\Sigma}(\Psi)) = \{(1,1,1,1)\}$, and the decision that satisfies the majority of the group is to build the three items and to increase the rent.

This majority "vote" seems to be more "democratic" that the other method. For example in this case it works only if φ_3 accepts to conform to the majority wishes that is strongly opposed to its own. But φ_3 could decide to quit the co-owners committee, and the works will perhaps not carry on because of a lack of money. So if a decision, like in this example or like in a peace agreement or in a price agreement in a competitive market, requires the approval of all the members an arbitration method like $\triangle^{GMax}$ seems more adequate.

5 Connections with Related Works

5.1 Belief Revision

We show in this section that merging operators are related to AGM belief revision operators [1, 7, 9], the first result is easy to prove:

Theorem 4. *If $\triangle$ is an IC merging operator, then the operator $\circ$, defined as $\varphi \circ \mu = \triangle_\mu(\varphi)$, is an AGM revision operator.*

Conversely, we can wonder if we can build a merging operator from a given revision operator. We propose the following definition of a merging operator from a given revision operator $\circ$:

Definition 7.

- *Consider the faithful assignment* [1] *corresponding to the revision operator $\circ$.*
- *Define $f^\circ_\varphi(I) = n$ where n is the level where the interpretation I appears in the $\leq_\varphi$ pre-order. More formally n is the length of the longest chain of strict inequalities $I_0 <_\varphi \ldots <_\varphi I_n$ with $I_0 \models \varphi$ and $I_n = I$.*
- *Define $f^\circ_\Psi(I)$ from the $f^\circ_{\varphi_i}(I)$ with some given merging method (for example $f^\circ_\Psi(I) = \sum_{\varphi \in \Psi}(f^\circ_\varphi(I))$ if the chosen method is the Σ method).*
- *Define $I \leq_\Psi J$ iff $f^\circ_\Psi(I) \leq f^\circ_\Psi(J)$.*
- *Finally $mod(\triangle^\circ_\mu(\Psi)) = \min(mod(\mu), \leq_\Psi)$.*

The question now is to find the properties of the operator defined if we choose a particular merging method, for example if we choose the Σ method (we get similar results with a method *à la* $\triangle^{GMax}$), that is $f^\circ_\Psi(I) = \sum_{\varphi \in \Psi}(f^\circ_\varphi(I))$, we get the following results:

Theorem 5. *If a merging operator $\triangle^\circ$ is defined from a revision operator $\circ$ and from the Σ merging method according to definition 7, then the operator $\triangle^\circ$ satisfies (IC0-IC3), (IC5-IC8) and (Maj).*

Definition 8. *We define $f^\circ_\varphi(\varphi')$ by putting $f^\circ_\varphi(\varphi') = \min_{I \models \varphi'}(f^\circ_\varphi(I))$*

Theorem 6. *If a merging operator $\triangle^\circ$ is defined from a revision operator $\circ$ using Σ merging method according to definition 7, then the operator $\triangle^\circ$ is an IC majority merging operator if and only if the faithful assignment satisfies the following "symmetry"property: $f^\circ_\varphi(\varphi') = f^\circ_{\varphi'}(\varphi)$.*

We say that a revision operator $\circ$ is defined from a distance d if

[1] *i.e.* an assignment mapping each knowledge base to a pre-order satisfying conditions 1-3 of the syncretic assignment but with knowledge bases instead of knowledge sets (*cf* [9]).

- d is a distance, that is d is a function $d : \mathcal{W} \times \mathcal{W} \mapsto \mathbb{N}$ that satisfies $d(I, J) = d(J, I)$ and $d(I, J) = 0$ iff $I = J$.
- Let φ be a knowledge base and I be an interpretation: $d(I, \varphi) = \min_{J \models K} d(I, J)$
- $I \leq_\varphi J$ iff $d(I, \varphi) \leq d(J, \varphi)$
- $mod(\varphi \circ \mu) = \min(mod(\mu), \leq_\varphi)$

We can show that the only revision operators satisfying the symmetry property are those defined from a distance. So as a corollary we have the following:

Theorem 7. *A merging operator $\triangle$ defined from a revision operator $\circ$ and the Σ merging method is an IC merging operator if and only if $\circ$ is defined from a distance.*

5.2 Pure Merging

A logical characterization of merging operators in the case where there is no integrity constraints was proposed in [11]. We will call this case the pure merging case.

Definition 9. *Let $\triangle$ be an operator mapping a knowledge set Ψ to a knowledge base $\triangle(\Psi)$. $\triangle$ is a pure merging operator if and only if it satisfies the following postulates:*

(A1) $\triangle(\Psi)$ *is consistent*
(A2) *If* Ψ *is consistent, then* $\triangle(\Psi) = \bigwedge \Psi$
(A3) *If* $\Psi_1 \leftrightarrow \Psi_2$*, then* $\triangle(\Psi_1) \leftrightarrow \triangle(\Psi_2)$
(A4) *If* $\varphi \wedge \varphi'$ *is not consistent, then* $\triangle(\varphi \sqcup \varphi') \nvdash \varphi$
(A5) $\triangle(\Psi_1) \wedge \triangle(\Psi_2) \vdash \triangle(\Psi_1 \sqcup \Psi_2)$
(A6) *If* $\triangle(\Psi_1) \wedge \triangle(\Psi_2)$ *is consistent, then* $\triangle(\Psi_1 \sqcup \Psi_2) \vdash \triangle(\Psi_1) \wedge \triangle(\Psi_2)$

A pure merging operator is a pure majority operator if it satisfies $(M7)$:

(M7) $\forall \varphi\ \exists n\ \ \triangle(\Psi \sqcup \varphi^n) \vdash \varphi$

A pure merging operator is a pure arbitration operator if it satisfies $(A7)$:

(A7) $\forall \varphi'\ \exists \varphi\ \varphi' \nvdash \varphi\ \forall n\ \ \triangle(\varphi' \sqcup \varphi^n) = \triangle(\varphi' \sqcup \varphi)$

First it is easy to see that the postulates obtain from (ICi) ones when $\mu = \top$ are nearly the same that those given in [11]. The main differences is that postulate (IC4) is stronger than (A4) and that postulate (Maj) is stronger than (M7). Notice also that postulate (Arb) is not expressible when $\mu = \top$. So there is no direct relationship between arbitration in the sense of [11] and IC arbitration. Notice that (A7) expresses only a kind of non-majority rule and thus is not a direct characterization of arbitration, whereas (Arb) defines in a more positive manner the arbitration behaviour.

Theorem 8. *If $\triangle$ is an IC merging operator, then $\triangle_\top$ is a pure merging operator (i.e. it satisfies (A1-A6)). Furthermore if $\triangle$ is an IC majority merging operator, then $\triangle_\top$ is a pure majority merging operator.*

5.3 Liberatore and Schaerf Commutative Revision

This section addresses the links between merging operators and those defined by Liberatore and Schaerf. The postulates given by Liberatore and Schaerf [13, 14] for commutative revision are the following:

(LS1) $\varphi \diamond \mu \leftrightarrow \mu \diamond \varphi$
(LS2) $\varphi \wedge \mu$ implies $\varphi \diamond \mu$
(LS3) If $\varphi \wedge \mu$ is satisfiable then $\varphi \diamond \mu$ implies $\varphi \wedge \mu$
(LS4) $\varphi \diamond \mu$ is unsatisfiable iff both φ and μ are unsatisfiable
(LS5) If $\varphi_1 \leftrightarrow \varphi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\varphi_1 \diamond \mu_1 \leftrightarrow \varphi_2 \diamond \mu_2$
(LS6) $\varphi \diamond (\mu \vee \theta) = \begin{cases} \varphi \diamond \mu & \text{or} \\ \varphi \diamond \theta & \text{or} \\ (\varphi \diamond \mu) \vee (\varphi \diamond \theta) & \end{cases}$
(LS7) $\varphi \diamond \mu$ implies $\varphi \vee \mu$
(LS8) If φ is satisfiable then $\varphi \wedge (\varphi \diamond \mu)$ is also satisfiable

This definition of commutative revision operators is very close to the one of belief revision operators. But it suffers two drawbacks from a merging point of view. First it allows to merge only two knowledge bases. And it forces the result to be in the disjunction of the two given knowledge bases. We argue in [11, 12] that it has not to be always the case.

Definition 10. *If $\triangle$ is an IC merging operator we define a commutative revision operator $\diamond_\triangle$ by $\varphi \diamond_\triangle \mu = \triangle_{\varphi \vee \mu}(\varphi \sqcup \mu)$. We will say that $\diamond_\triangle$ is the commutative revision operator associated with $\triangle$.*

Theorem 9. *If $\triangle$ is an IC merging operator, then the operator $\diamond_\triangle$ associated with it satisfies (LS1-LS5),(LS7) and (LS8).*

By definitions $\diamond_\triangle$ operators are commutative, but the following property shows that they can be consider as "double revision operators" (we recall that $\varphi \circ \mu = \triangle_\mu(\varphi)$ is an AGM revision operator).

Theorem 10. *If $\triangle$ is an IC merging operator then it satisfies*

$$\triangle_{\varphi \vee \mu}(\varphi \sqcup \mu) \leftrightarrow \triangle_\varphi(\mu) \vee \triangle_\mu(\varphi)$$

In order to obtain systematically a commutative revision operator from an IC merging operator using definition 10, IC merging operators need to satisfy an additional property:

$$\triangle_\varphi(\mu \vee \theta) = \begin{cases} \triangle_\varphi(\mu) & \text{if } \triangle_{\mu \vee \theta}(\varphi) \vdash \neg\theta \\ \triangle_\varphi(\theta) & \text{if } \triangle_{\mu \vee \theta}(\varphi) \vdash \neg\mu \\ \triangle_\varphi(\mu) \vee \triangle_\varphi(\theta) & \text{otherwise} \end{cases} \tag{1}$$

Theorem 11. *If $\triangle$ is an IC merging operator, then the operator $\diamond$ defined as $\varphi \diamond \mu = \triangle_{\varphi \vee \mu}(\varphi \sqcup \mu)$ satisfies (LS1-LS8) if and only if $\triangle$ satisfies property (1).*

Remark 1. Property (1) implies $\triangle_A(\varphi) = \triangle_A(\triangle_\varphi(A))$

This remark shows that property (1) is quite a topological one since $\triangle_A(\varphi) = \varphi \circ A = (A \circ \varphi) \circ A$. That is to say that the result of the revision of φ by A depends only of the models of φ that are the closest to A. Revisions defined from a distance satisfy this property.

A serious drawback of commutative revision definition is that it does not allow to merge more than two knowledge bases since it is not associative (see [13, 14]), but the idea that the result of the merging has to implies the disjunction of the knowledge bases can be very useful in a lot of applications. IC merging operators allow to generalize Liberatore and Schaerf operators to n knowledge bases, by defining the merging of a knowledge set $\{\varphi_1 \sqcup \ldots \sqcup \varphi_n\}$ as:

$$\triangle_{\varphi_1 \vee \ldots \vee \varphi_n}(\varphi_1 \sqcup \ldots \sqcup \varphi_n)$$

The logical properties of these operators are worth more study.

6 Conclusion

In this paper we have presented a logical framework for knowledge base merging in the presence of integrity constraints when there is no preference over the knowledge bases. We stated a set of properties an IC merging operator should satisfy in order to have a rational behaviour. This set of properties can then be used to classify particular merging methods.

We made a distinction between arbitration and majority operators, arbitration operators striving to minimise individual dissatisfaction and majority operators trying to minimise global dissatisfaction. An open question is to know if arbitration and majority merging are two distinct merging subclasses or if it is possible for a merging operator to be both an arbitration and a majority merging operator.

We provide a model-theoretic characterisation for IC merging operators. This characterisation is much more natural than the one in [11], due to the presence of integrity constraints.

Actually, in a committee, all the protagonists do not have the same weight on the final decision, so one generally needs to weight each knowledge base to reflect this. The idea behind weights is that the higher weight a knowledge base has, the more important it is. If the knowledge bases reflect the view of several people, weights could represent, for example, the cardinality of each group. We want to characterize logically the use of this weights. Majority operators are close to this idea of weighted operators since they allow to take cardinalities into account. But a more subtle treatment of weights in merging is still to do.

An on going work is the study of merging operators that adopt a coherence approach to theory merging. These operators are based on an union of all the knowledge bases and on the selection of some maximal subsets due to a given order (not necessarily the inclusion), see *e.g.* [3, 4]. An important drawback of coherence merging operators is that the source of each knowledge is lost in the merging process. So the problem is to take into account the source of each piece of information in order to allow subtler behaviours for merging operators, for example define majority or arbitration operators.

References

1. C. E. Alchourrón, P. Gärdenfors, and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50:510–530, 1985.
2. K. J. Arrow. *Social choice and individual values.* Wiley, New York, second edition, 1963.
3. C. Baral, S. Kraus, J. Minker, and V. S. Subrahmanian. Combining knowledge bases consisting of first-order theories. *Computational Intelligence*, 8(1):45–71, 1992.
4. S. Benferhat, D. Dubois, J. Lang, H. Prade, A. Saffioti, and P. Smets. A general approach for inconsistency handling and merging information in prioritized knowledge bases. In *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, pages 466–477.
5. S. Benferhat, D. Dubois, and H. Prade. A computational model for belief change and fusing ordered belief bases. In *Frontiers in Belief revision.* Kluwer, 1999. To appear.
6. M. Dalal. Investigations into a theory of knowledge base revision: preliminary report. In *Proceedings of AAAI-88*, pages 475–479, 1988.
7. P. Gärdenfors. *Knowledge in flux.* MIT Press, 1988.
8. J. Halpern and Y. Moses. A guide to completeness and complexity for modal logics of knowledge and belief. *Artificial Intelligence*, 54(3):319–379, 1992.
9. H. Katsuno and A. O. Mendelzon. Propositional knowledge base revision and minimal change. *Artificial Intelligence*, 52:263–294, 1991.
10. J. S. Kelly. *Arrow impossibility theorems.* Series in economic theory and mathematical economics. Academic Press, New York, 1978.
11. S. Konieczny and R. Pino Pérez. On the logic of merging. In *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, pages 488–498, 1998.
12. S. Konieczny and R. Pino Pérez. Merging with integrity constraints. Technical Report 99-01, LIFL, 1999. http://www.lifl.fr/GNOM/articles/it9901.ps.
13. P. Liberatore and M. Schaerf. Arbitration: A commutative operator for belief revision. In *Proceedings of the Second World Conference on the Fundamentals of Artificial Intelligence*, pages 217–228, 1995.
14. P. Liberatore and M. Schaerf. Arbitration (or how to merge knowledge bases). *IEEE Transactions on Knowledge and Data Engineering*, 10(1):76–90, 1998.
15. J. Lin. Integration of weighted knowledge bases. *Artificial Intelligence*, 83(2):363–378, 1996.
16. J. Lin and A. O. Mendelzon. Knowledge base merging by majority. Manuscript.
17. H. Moulin. *Axioms of cooperative decision making.* Monograph of the Econometric Society. Cambridge University Press, 1988.
18. P. Z. Revesz. On the semantics of theory change: arbitration between old and new information. In *Proceedings of the 12^{th} ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Databases*, pages 71–92, 1993.
19. P. Z. Revesz. On the semantics of arbitration. *International Journal of Algebra and Computation*, 7(2):133–160, 1997.