Contraction in Propositional Logic

Thomas Caridroit, Sébastien Konieczny, and Pierre Marquis

CRIL, CNRS – Université d'Artois, France
{caridroit,konieczny,marquis}@cril.fr

Abstract. The AGM model for the revision and contraction of belief sets provides rationality postulates for each of the two cases. In the context of finite propositional logic, Katsuno and Mendelzon pointed out postulates for the revision of belief bases which correspond to the AGM postulates for the revision of beliefs sets. In this paper, we present postulates for the contraction of propositional belief bases which correspond to the AGM postulates for the contraction of belief sets. We highlight the existing connections with the revision of belief bases in the sense of Katsuno and Mendelzon thanks to Levi and Harper identities and present a representation theorem for operators of contraction of belief bases.

1 Introduction

Belief change has been studied for many years in philosophy, databases, and artificial intelligence. The AGM model, named after its three initiators Carlos Alchourrón, Peter Gärdenfors and David Makinson, is the main formal framework for modeling belief change [1]. Its key concepts and constructs have been the subject of significant developments [5, 6, 13]. Alchourrón, Gärdenfors and Makinson pointed out some postulates and representation theorems thereby establishing the basis for a framework suited to the belief change issue when beliefs are expressed using the language of any Tarskian logic. Tarskian logics consider abstract consequence relations, that satisfy inclusion, monotony and idempotence (and the AGM framework adds also to them the supraclassicality, compacity and deduction conditions).

Katsuno and Mendelzon [11] presented a set of postulates for revision operators in the framework of finite propositional logic and a representation theorem in terms of faithful assignments.[1] This representation theorem is important because it is at the origin of the main approaches to iterated belief revision [4].

Revision and contraction operators are closely related, as reflected by Levi and Harper identities. These identities can be used to define contraction operators from revision operators and vice versa. So the existence of work on contraction in the context of finite propositional logic might be expected. However, as far as we know, this issue has not been investigated.

The objective of this paper is to define operators of propositional contraction matching Katsuno and Mendelzon's revision operators and to check that these operators offer the expected properties. In the following, we present a set of postulates for contraction operators in the framework of finite propositional logic, and establish a corresponding

[1] Such assignments correspond to a specific case of Grove's systems of spheres [7].

representation theorem. The obtained results are not very surprising, but they are new nevertheless, and they appear as a first important step in the study of iterated contraction.

The rest of the paper is organized as follows. In Section 2, some formal preliminaries are presented. In Section 3, the AGM and KM frameworks for belief contraction and revision are recalled. In Section 4, a connection between belief sets and belief bases is pointed out. In Section 5, we define postulates that a contraction operator on belief bases should satisfy. In Section 6 the correspondence between contraction of belief sets and contraction of belief bases is investigated; we check, using Levi and Harper identities, that there is a connection between propositional revision operators satisfying Katsuno and Mendelzon postulates and propositional contraction operators satisfying our postulates. Section 7 gives a representation theorem for the contraction of belief bases. We conclude and discuss some perspectives for future work in Section 8. For space reasons, several proofs are not included, they can be found in the corresponding technical report [3].

2 Preliminaries

We consider a finite propositional language L built up from a (finite) set of symbols P and the usual connectives. $\bot$ (resp. $\top$) is the Boolean constant `false` (resp. `true`). Formulas are interpreted in the standard way, and $Cn(\varphi) = \{\psi \in L \mid \varphi \vdash \psi\}$ denotes the deductive closure of $\varphi \in L$.

A belief base is a set of propositional formulas $\{\varphi_1, \ldots, \varphi_n\}$. We suppose in this paper that a belief base is represented by $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_n$ (This is a usual harmless assumption xhen one supposes irrelevance of syntax[2] (cf. postulate (C5)).

A belief set K is a deductively closed set of formulas. Obviously one can associate with any belief base φ a belief set that is the set of all its consequences $K = Cn(\varphi)$.

If φ is a formula, then $Mod(\varphi)$ denotes the set of its models. Conversely if M is a set of interpretations, then α_M denotes the formula (unique, up to logical equivalence) the models of which are those of M.

Given a preorder (i.e., a reflexive and transitive relation) $\leq_\varphi$ over the set of interpretations, $<_\varphi$ is its strict part defined by $I <_\varphi J$ if and only if $I \leq_\varphi J$ and $J \not\leq_\varphi I$ and $\simeq_\varphi$ is the associated equivalence relation defined by $I \simeq_\varphi J$ if and only if $I \leq_\varphi J$ and $J \leq_\varphi I$. $\min(X, \leq_\varphi)$ denotes the set of minimal elements of X for $\leq_\varphi$, i.e., $\min(X, \leq_\varphi) = \{x \in X \mid \nexists y \in X \text{ such that } y <_\varphi x\}$.

3 AGM and KM Belief Revision and Contraction

Alchourrón, Gärdenfors and Makinson [1, 5] pointed out the following postulates for the contraction of belief sets. These postulates are formulated in a very general framework, but here we limit the discussion to the case of finite propositional logic. Given a belief set K and a formula μ, $K \div \mu$ denotes the contraction of K by μ. $+$ is the

[2] Note that in some works the term "belief base" is just used for syntax-dependent belief change [8]. Here this term denotes a non-deductively closed set of formulas (as in [11]).

expansion operator, the result it gives is just the set of consequences of the union of the two theories (i.e. $K + \mu = Cn(K \cup \{\mu\})$).

(K÷1) $K \div \mu$ is a belief set
(K÷2) $K \div \mu \subseteq K$
(K÷3) If $\mu \notin K$, then $K \div \mu = K$
(K÷4) If $\nvdash \mu$, then $\mu \notin K \div \mu$
(K÷5) If $\mu \in K$, then $K \subseteq (K \div \mu) + \mu$
(K÷6) If $\mu \equiv \beta$, then $K \div \mu = K \div \beta$
(K÷7) $(K \div \mu) \cap (K \div \beta) \subseteq K \div (\mu \wedge \beta)$
(K÷8) If $\mu \notin K \div (\mu \wedge \beta)$, then $K \div (\mu \wedge \beta) \subseteq K \div \mu$

See [5] for detailed explanations on these postulates (we will comment their propositional counterpart later). Alchourrón, Gärdenfors and Makinson also provided postulates (**(K⋆1)-(K⋆8)**) for belief revision. We will focus on their propositional counterpart proposed by Katsuno and Mendelzon [11]. But let us first recall that AGM belief revision and belief contraction are closely related. Actually every belief revision operator induces a belief contraction one, and vice versa:

(Levi Identity) $K \star \mu = (K \div \neg\mu) + \mu$
(Harper Identity) $K \div \mu = K \cap (K \star \neg\mu)$

Let us now recall the Katsuno and Mendelzon propositional counterpart to belief revision postulates and their representation theorem in terms of faithful assignment [11]. Let φ and μ be two propositional formulas where φ represents the current belief base of the agent and μ is the new piece of information (i.e., the change formula). The revision of φ by μ, denoted by $\varphi \circ \mu$, must satisfy the following postulates [11]:

(R1) $\varphi \circ \mu \vdash \mu$
(R2) If $\varphi \wedge \mu$ is consistent, then $\varphi \circ \mu \equiv \varphi \wedge \mu$
(R3) If μ is consistent, then $\varphi \circ \mu$ is consistent
(R4) If $\varphi_1 \equiv \varphi_2$ and $\mu_1 \equiv \mu_2$, then $\varphi_1 \circ \mu_1 \equiv \varphi_2 \circ \mu_2$
(R5) $(\varphi \circ \mu) \wedge \psi \vdash \varphi \circ (\mu \wedge \psi)$
(R6) If $(\varphi \circ \mu) \wedge \psi$ is consistent, then $\varphi \circ (\mu \wedge \psi) \vdash (\varphi \circ \mu) \wedge \psi$

A representation theorem is a way to associate with a set of postulates a constructive approach to build the corresponding family of operators. Katsuno and Mendelzon presented such a theorem in terms of faithful assignment, which associates with each belief base a pre-order that ranks the interpretations from the most plausible ones to the least plausible ones.

Definition 1. *A faithful assignment is a mapping that associates with any belief base φ a pre-order $\leq_\varphi$ on the set of all interpretations such that:*

1. *If $I \models \varphi$ and $J \models \varphi$, then $I \simeq_\varphi J$*
2. *If $I \models \varphi$ and $J \not\models \varphi$, then $I <_\varphi J$*
3. *If $\varphi \equiv \varphi'$, then $\leq_\varphi = \leq_{\varphi'}$*

Theorem 1. [11] *A revision operator $\circ$ satisfies the postulates (R1)-(R6) if and only if there exists a faithful assignment that associates with each belief base φ a total pre-order $\leq_\varphi$ such that*

$$Mod(\varphi \circ \mu) = min(Mod(\mu), \leq_\varphi)$$

4 From Belief Sets to Belief Bases

Our purpose is now to define contraction operators on belief bases in the framework of finite propositional logic. Let φ and μ be two formulas. $\varphi - \mu$ denotes the contraction of φ by μ, which is the new formula obtained by removing the piece of beliefs μ from the (consequences of the) belief base φ of the agent.

In order to relate AGM belief set contraction and our notion of propositional belief base contraction, we first have to formalize the link between belief sets and belief bases.

Proposition 1 shows that a belief set is always the deductive closure (C_n) of a belief base:

Proposition 1. *For any belief set K, there is a belief base φ_K such that $K = Cn(\varphi_K)$ and conversely, for any belief base φ, there is a belief set $K_\varphi = Cn(\varphi)$.*

Indeed, *Cn* is a bijection from E to F where F is the set of belief sets and E is the set of belief bases considered up to logical equivalence. Thus, for a belief base φ , the notation $K_\varphi = Cn(\varphi)$ and for a belief set K, the notation $\varphi_K = Cn^{-1}(K)$ are safe.

On this ground a correspondence between AGM contraction operators on belief sets and the contraction operators on belief bases can be established:

Definition 2. *Given a contraction operator on belief sets $\div$, the operator $-_{(\div)}$ on belief bases is defined by: $\varphi -_{(\div)} \mu = \varphi_{K_\varphi \div \mu}$. Conversely, given a contraction operator on belief bases $-$, the operator $\div_{(-)}$ on belief sets is defined by: $K \div_{(-)} \mu = K_{\varphi_K - \mu}$.*

Finally, the following proposition shows that if we use a contraction operator on belief sets $\div$ to define, via Definition 2, a contraction operator on belief bases $-_{(\div)}$, then the contraction operator on belief sets defined via Definition 2 is the initial contraction operator $\div$ (and vice versa):

Proposition 2. *We have $-_{(\div_{(-)})} = -$. Similarly we have $\div_{(-_{(\div)})} = \div$*

Let a contraction operator $\div$ on belief sets and $-$ a contraction operator on belief bases. The operators $\div$ and $-$ are said to correspond to each other if $\div = \div_{(-)}$ and $- = -_{(\div)}$.

5 Postulates for Propositional Contraction

We now define the following set of postulates for contraction of propositional belief bases:

(C1) $\varphi \vdash \varphi - \mu$
(C2) If $\varphi \nvdash \mu$, then $\varphi - \mu \vdash \varphi$
(C3) If $\varphi - \mu \vdash \mu$, then $\vdash \mu$
(C4) If $\varphi \vdash \mu$, then $(\varphi - \mu) \wedge \mu \vdash \varphi$
(C5) If $\varphi_1 \equiv \varphi_2$ and $\mu_1 \equiv \mu_2$, then $\varphi_1 - \mu_1 \equiv \varphi_2 - \mu_2$
(C6) $\varphi - (\mu \wedge \beta) \vdash (\varphi - \mu) \vee (\varphi - \beta)$
(C7) If $\varphi - (\mu \wedge \beta) \nvdash \mu$, then $\varphi - \mu \vdash \varphi - (\mu \wedge \beta)$

The intuitive meaning of these postulates is as follows: (C1) ensures that after contraction, no new information is added to the belief base. (C2) indicates that if μ is not deducible from φ, then no change is made during the contraction. (C3) ensures that the only possibility for the contraction of φ by μ to fail is that μ is a tautology. (C4) says us that the conjunction of the contraction of φ by μ and μ gives a propositional formula which is equivalent to φ (the converse implication is a consequence of (C1)). (C5) reflects the principle of independence of syntax. (C6) and (C7) express the minimality of change for the conjunction. (C6) says that the contraction by a conjunction always implies the disjunction of the contractions by the conjuncts. (C7) says that if μ has not been removed during the contraction by $\mu \wedge \beta$, then the contraction by μ must imply the contraction by the conjunction.

The following proposition shows that the contraction operators satisfying postulates (C1)-(C7) correspond to the contraction operators satisfying the AGM postulates (K÷1)-(K÷8).

Proposition 3. *Let $\div$ be a contraction operator on belief sets and $-$ ($= -_{(\div)}$) its corresponding operator on belief bases. Then $\div$ satisfies (K÷1)-(K÷8) if and only if $-$ satisfies (C1)-(C7).*

Furthermore, it turns out that the contraction of φ by a conjunction $(\mu \wedge \beta)$ can have only three different outcomes (up to logical equivalence). Such a trichotomy result is similar to the one in the classical AGM framework [5].

Proposition 4. *In the presence of (C1) - (C5), (C6) and (C7) are equivalent to (Tri):*

$$\textbf{\textit{(Tri)}} \quad \varphi - (\mu \wedge \beta) \equiv \begin{cases} \varphi - \mu \text{ or} \\ \varphi - \beta \text{ or} \\ (\varphi - \mu) \vee (\varphi - \beta) \end{cases}$$

In fact, looking at the proof of this proposition, we also know that if $\varphi - (\mu \wedge \beta) \vdash \beta$, then $\varphi - (\mu \wedge \beta) \equiv \varphi - \mu$. This means that when β is more entenched (i.e, more important/plausible) than μ, then when we are asked to remove $\mu \wedge \beta$ if we prefer to keep β (and to remove μ), then the contraction by the conjunction is exactly the contraction by μ alone.

6 A Correspondence between Contraction and Revision

Now that we have defined postulates for contraction operators on belief bases, we can check that the contraction operators satisfying these postulates correspond to revision operators in the sense of Katsuno and Mendelzon [11].

We first show that Levi and Harper identities hold also in this propositional setting. We note $\circ_{(-)}$ the revision operator on belief bases defined from $-$ via Levi identity and $-_{(\circ)}$ the contraction operator on belief bases defined from $\circ$ via Harper identity.

Definition 3. *Levi and Harper identities for belief bases can be expressed as follows:*
$\varphi \circ_{(-)} \mu \equiv (\varphi - \neg\mu) \wedge \mu$ ***(Levi identity)***
$\varphi -_{(\circ)} \mu \equiv \varphi \vee (\varphi \circ \neg\mu)$ ***(Harper identity)***

Operators obtained by means of these identities satisfy the expected properties:

Proposition 5. *If the contraction operator $-$ satisfies (C1)-(C5) then the revision operator $\circ$ $(= \circ_{(-)})$ defined using Levi identity satisfies (R1)-(R4). Furthermore if (C6) is satisfied by $-$, then (R5) is satisfied by $\circ$, and if (C7) is satisfied by $-$, then (R6) is satisfied $\circ$.*

Therefore, the KM revision operators for propositional belief bases can be defined using Levi identity from the contraction operators for propositional belief bases we have introduced. Reciprocally, contraction operators for propositional belief bases can be defined using Harper identity, from KM revision operators for belief bases.

Proposition 6. *If the revision operator $\circ$ satisfies (R1)-(R4) then the contraction operator $-$ $(= -_{(\circ)})$ defined using Harper identity satisfies (C1)-(C5). Furthermore, if (R5) is satisfied by $\circ$, then (C6) is satisfied by $-$ and if (R6) is satisfied by $\circ$, then (C7) is satisfied by $-$.*

The following proposition shows that if we use a revision operator $\circ$ to define, via Harper identity, a contraction operator $-_{(\circ)}$, then the revision operator defined via Levi identity, from $-_{(\circ)}$ is the initial revision operator $\circ$. The other way around, if we use a contraction operator $-$ to define, via Levi identity a revision operator $\circ_{(-)}$, then the contraction operator defined via Harper identity from $\circ_{(-)}$ is the initial contraction operator $-$.

Proposition 7.
- *if $\circ$ is a revision operator, then $\circ_{(-_{(\circ)})} = \circ$*
- *if $-$ is a contraction operator, then $-_{(\circ_{(-)})} = -$*

Our postulates for contraction of belief bases are thus in close correspondence with the revision postulates for belief bases defined by Katsuno and Mendelzon.

7 Representation Theorem

Let us now check that we can state a representation theorem for contraction within the framework of finite propositional logic, which is a counterpart of the representation theorem of Katsuno and Mendelzon for revision.

Lemma 1. *Let $-$ be a contraction operator satisfying (C1)-(C7).*

$$\text{If } \alpha_{\{I\}} \nvdash \varphi \text{ then } \varphi - \neg\alpha_{\{I\}} \equiv \varphi \vee \alpha_{\{I\}}$$

This lemma indicates that if a formula α, with only one model, does not imply a formula φ, then the contraction of φ by the negation of α is equivalent to the disjunction of φ and α.

The idea of the representation theorem is to express the set of models of the contraction of a base φ by a change formula μ as the union of the models of φ and of the minimal counter-models of μ with respect to $\leq_\varphi$.

Theorem 2. *A contraction operator $-$ satisfies the postulates **(C1) - (C7)** if and only if there exists a faithful assignment that associates with each belief base φ a total pre-order $\leq_\varphi$ on the set of all interpretations such that*

$$Mod(\varphi - \mu) = Mod(\varphi) \cup min(Mod(\neg\mu), \leq_\varphi)$$

Proof. The only-if part of the proof consists mainly in checking the **(C1) - (C7)** properties. For space reasons we focus only on the if part which is more tricky. Let $-$ be a contraction operator which satisfies the postulates (C1) to (C7).

For each formula φ, we define a total pre-order $\leq_\varphi$ using the operator $-$: $\forall I, I'$ two interpretations, we define the relation $\leq_\varphi$ by $I \leq_\varphi I'$ if and only if $I \in Mod(\varphi - \neg\alpha_{\{I,I'\}})$.
We first show that $\leq_\varphi$ is a total pre-order.

– **Total:** let I and I' be two interpretations. As $\alpha_{\{I,I'\}}$ has at least one model, $\neg\alpha_{\{I,I'\}}$ has at least one counter-model. We deduce that $\nvdash \neg\alpha_{\{I,I'\}}$, which allows us to conclude from (C3) that $\varphi - \neg\alpha_{\{I,I'\}} \nvdash \neg\alpha_{\{I,I'\}}$. So we know that there is $J \in Mod(\varphi - \neg\alpha_{\{I,I'\}})$ such that $J \in Mod(\alpha_{\{I,I'\}}) = \{I, I'\}$. Therefore, either $I \in Mod(\varphi - \neg\alpha_{\{I,I'\}})$ and thus $I \leq_\varphi I'$, or $I' \in Mod(\varphi - \neg\alpha_{\{I,I'\}})$ and thus $I' \leq_\varphi I$. Hence $\leq_\varphi$ is total.
– **Reflexive:** Every binary relation which is total necessarily is reflexive.
– **Transitive:** Suppose that $I \leq_\varphi J$ and $J \leq_\varphi L$. Let us consider the case when I, J and L are pairwise distinct, and none of them is a model of φ. Indeed, in the remaining case when at least two of them are equal, transitivity is trivially satisfied. If one of them is a model of φ, then the result also trivially holds by (C1). Indeed, if $L \models \varphi$, then by the assumptions and (C1) we deduce that I and J are also models of φ. Similarly, if $J \models \varphi$, then by (C1) $I \models \varphi$. And if $I \models \varphi$ then by construction $I \leq_\varphi I'$ forall I', so especially for $I' = L$.
So now let us consider the general case. Towards a contradiction, suppose $I \nleq_\varphi L$. As $\leq_\varphi$ is total, we have $L <_\varphi I$, therefore $L \models \varphi - \neg\alpha_{\{I,L\}}$ and $I \not\models \varphi - \neg\alpha_{\{I,L\}}$. By (Tri) we have that $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{I,L\}}$ or $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{J\}}$ or $\varphi - \neg\alpha_{\{I,J,L\}} \equiv (\varphi - \neg\alpha_{\{I,L\}}) \vee (\varphi - \neg\alpha_{\{J\}})$.
• Case (1) $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{I,L\}}$. From (C6) we have that $\varphi - \neg\alpha_{\{I,L\}} \vdash \varphi - \neg\alpha_{\{I\}} \vee \varphi - \neg\alpha_{\{L\}} \equiv \varphi \vee \alpha_{\{I\}} \vee \alpha_{\{L\}}$. Since $J \not\models \varphi \vee \alpha_{\{I\}} \vee \alpha_{\{L\}}$, we have $J \not\models \varphi - \neg\alpha_{\{I,L\}}$, so $J \not\models \varphi - \neg\alpha_{\{I,J,L\}}$. Since $L \models \varphi - \neg\alpha_{\{I,J,L\}}$ and $L \not\models \neg\alpha_{\{J,L\}}$, we deduce that $\varphi - \neg\alpha_{\{I,J,L\}} \nvdash \neg\alpha_{\{J,L\}}$. So by (C7) we have that $\varphi - \neg\alpha_{\{J,L\}} \vdash \varphi - \neg\alpha_{\{I,J,L\}}$. As $J \not\models \varphi - \neg\alpha_{\{I,J,L\}}$, we have $J \not\models \varphi - \neg\alpha_{\{J,L\}}$, which means by definition that $J \nleq_\varphi L$. Contradiction.
• Case (2) $\varphi - \neg\alpha_{\{I,J,L\}} \equiv \varphi - \neg\alpha_{\{J\}} \equiv \varphi \vee \alpha_{\{J\}}$. This means in particular that $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$ and $J \models \varphi - \neg\alpha_{\{I,J,L\}}$. So we know that $\varphi - \neg\alpha_{\{I,J,L\}} \nvdash \neg\alpha_{\{I,J\}}$. So by (C7) we have that $\varphi - \neg\alpha_{\{I,J\}} \vdash \varphi - \neg\alpha_{\{I,J,L\}}$. As $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$, we have $I \not\models \varphi - \neg\alpha_{\{I,J\}}$, which means by definition that $I \nleq_\varphi J$. Contradiction.
• Case (3) $\varphi - \neg\alpha_{\{I,J,L\}} \equiv (\varphi - \neg\alpha_{\{I,L\}}) \vee (\varphi - \neg\alpha_{\{J\}}) \equiv (\varphi - \neg\alpha_{\{I,L\}}) \vee (\varphi \vee \alpha_{\{J\}})$. This equivalence implies that $J \models \varphi - \neg\alpha_{\{I,J,L\}}$, $L \models \varphi - \neg\alpha_{\{I,J,L\}}$, and $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$. Since $J \models \varphi - \neg\alpha_{\{I,J,L\}}$ and $J \not\models \neg\alpha_{\{I,J\}}$, we deduce that $\varphi - \neg\alpha_{\{I,J,L\}} \nvdash \neg\alpha_{\{I,J\}}$. So by (C7) we have that $\varphi - \neg\alpha_{\{I,J\}} \vdash \varphi - \neg\alpha_{\{I,J,L\}}$. As $I \not\models \varphi - \neg\alpha_{\{I,J,L\}}$, we have $I \not\models \varphi - \neg\alpha_{\{I,J\}}$, which means by definition that $I \nleq_\varphi J$. Contradiction.

We have shown that $\leq_\varphi$ is a total, reflexive and transitive relation. It is therefore a total pre-order. Then we show that the mapping $\varphi \mapsto \leq_\varphi$ is a faithful assignment.

- The third condition (if $\varphi_1 \equiv \varphi_2$, then $\leq_{\varphi_1} = \leq_{\varphi_2}$) comes from (C5). Indeed, if $\varphi_1 \equiv \varphi_2$ then $\varphi_1 - \neg\alpha_{\{I_1,I_2\}} \equiv \varphi_2 - \neg\alpha_{\{I_1,I_2\}}$, hence $I_1 \leq_{\varphi_1} I_2$ iff $I_1 \leq_{\varphi_2} I_2$, so $\leq_{\varphi_1} = \leq_{\varphi_2}$.
- The first condition comes from (C1): $\varphi \vdash \varphi - \neg\mu$, so if $I_1 \in Mod(\varphi)$ then $I_1 \in Mod(\varphi - \neg\alpha_{\{I_1,I_2\}})$ and if $I_2 \in Mod(\varphi)$ then $I_2 \in Mod(\varphi - \neg\alpha_{\{I_1,I_2\}})$. So by definition, we have $I_1 \leq_\varphi I_2$ and $I_2 \leq_\varphi I_1$, hence $I_1 \simeq_\varphi I_2$.
- Let us now show that the second condition (if $I_1 \models \varphi$ and $I_2 \not\models \varphi$ then $I_1 <_\varphi I_2$) is satisfied. From the definition of $\leq_\varphi$ and (C1), we can deduce from $I_1 \models \varphi$ that $I_1 \leq_\varphi I_2$. It remains to show that $I_2 \not\leq_\varphi I_1$. We consider two cases:
  - If $\varphi \vdash \neg\alpha_{\{I_1,I_2\}}$, then we have $\varphi \vdash \neg\alpha_{\{I_1\}} \wedge \neg\alpha_{\{I_2\}}$. So, in particular, $\varphi \vdash \neg\alpha_{\{I_1\}}$, which contradicts the fact that $I_1 \models \varphi$, showing that this case is impossible.
  - If $\varphi \nvdash \neg\alpha_{\{I_1,I_2\}}$, then, from (C2), $\varphi - \neg\alpha_{\{I_1,I_2\}} \vdash \varphi$. We therefore deduce that $I_2 \not\models \varphi - \neg\alpha_{\{I_1,I_2\}}$, hence $I_2 \not\leq_\varphi I_1$.

  The second condition for the assignment to be faithful is checked.

Finally, it remains to show that

$$Mod(\varphi - \mu) = Mod(\varphi) \cup min(Mod(\neg\mu), \leq_\varphi).$$

We consider two cases:

- If $\varphi \nvdash \mu$, then from (C1) and (C2), $Mod(\varphi - \mu) = Mod(\varphi)$. Furthermore, $\exists I \in Mod(\varphi)$ such that $I \in Mod(\neg\mu)$. The second condition on faithful assignment allows us to deduce that $min(Mod(\neg\mu), \leq_\varphi) \subseteq Mod(\varphi)$. The conclusion follows.
- If $\varphi \vdash \mu$, then we assume $\nvdash \mu$ without loss of generality. Indeed, if $\vdash \mu$ then $Mod(\varphi - \mu) = Mod(\varphi) \cup min(Mod(\neg\mu), \leq_\varphi)$ is trivially deduced from (C1) and (C4), which shows that $\varphi - \mu \vdash \varphi \vee \neg\mu$ since $Mod(\neg\mu) = \emptyset = min(Mod(\neg\mu), \leq_\varphi)$ when μ is valid. (C4) allows us to deduce that $Mod(\varphi - \mu) \subseteq Mod(\varphi) \cup min(Mod(\neg\mu), \leq_\varphi)$. Given an interpretation I such that $I \models \varphi - \mu$, we can deduce from (C4) that $I \models \varphi$ or $I \models \neg\mu$.
  - If $I \models \varphi$, then directly $I \in Mod(\varphi) \cup min(Mod(\neg\mu), \leq_\varphi)$.
  - If $I \models \neg\varphi$ and $I \models \neg\mu$, then we want to show that $I \in min(Mod(\neg\mu), \leq_\varphi)$. Towards a contradiction, suppose that there exists an interpretation $J \models \neg\mu$ such that $J <_\varphi I$. By definition of faithful assignment, we have $I \not\models \varphi - (\neg\alpha_{\{I,J\}})$. In addition, we know that $I \models \neg\mu$ and $J \models \neg\mu$, so $\mu \vdash \neg\alpha_{\{I,J\}}$. Therefore there exists β such that $I \models \beta$, $J \models \beta$ and $\mu \equiv (\neg\alpha_{\{I,J\}}) \wedge \beta$. By (C6), $\varphi - \mu \vdash (\varphi - (\neg\alpha_{\{I,J\}})) \vee (\varphi - \beta)$, we also know that $\varphi - (\neg\alpha_{\{I,J\}} \vee \neg\beta) \nvdash \neg\alpha_{\{I,J\}}$ by (C3). By (C6) and (C7), we have $\varphi - \mu \equiv \varphi - \neg\alpha_{\{I,J\}}$. This contradicts our assumption, $I \not\models \varphi - \neg\alpha_{\{I,J\}}$.

  Subsequently we have $Mod(\varphi - \mu) \subseteq Mod(\varphi) \cup min(Mod(\neg\mu), \leq_\varphi)$. Let us show now that $Mod(\varphi) \cup min(Mod(\neg\mu), \leq_\varphi) \subseteq Mod(\varphi - \mu)$.
  - If $I \in Mod(\varphi)$, then since from (C1), we have $\varphi \vdash \varphi - \mu$, we conclude that $I \in Mod(\varphi - \mu)$.

- ∘ Suppose now that $I \notin Mod(\varphi)$ and $I \in min(Mod(\neg\mu), \leq_\varphi)$ and suppose that $I \notin Mod(\varphi - \mu)$. In this case, $min(Mod(\neg\mu), \leq_\varphi)$ is not empty, which means that $\nvdash \mu$. So, from (C3), $\varphi - \mu \nvdash \mu$. We can deduce that $\exists J \in Mod(\varphi - \mu)$ such that $J \in Mod(\neg\mu)$.
  Let us consider the two possible cases: $J \in Mod(\varphi)$ and $J \notin Mod(\varphi)$. If $J \in Mod(\varphi)$, then by the second condition of the faithful assignment we have that $J <_\varphi I$. But as $J \in Mod(\neg\mu)$, this means that $I \notin min(Mod(\neg\mu), \leq_\varphi)$. Contradiction. If $J \notin Mod(\varphi)$, then we have that $J \in Mod(\varphi - \mu)$ and $I \notin Mod(\varphi - \mu)$. So $\varphi - \mu \nvdash \neg\alpha_{\{I,J\}}$, hence by (C7) we have that $\varphi - \neg\alpha_{\{I,J\}} \vdash \varphi - \mu$. As $I \notin Mod(\varphi - \mu)$, we have $I \notin Mod(\varphi - \neg\alpha_{\{I,J\}})$. Then by definition (and (C3)) this means that $J <_\varphi I$. But we also know that $J \in Mod(\neg\mu)$, so this implies that $I \notin min(Mod(\neg\mu), \leq_\varphi)$. Contradiction. □

Note that a similar construction has been used in [14] for the contraction of Horn belief sets. [3]

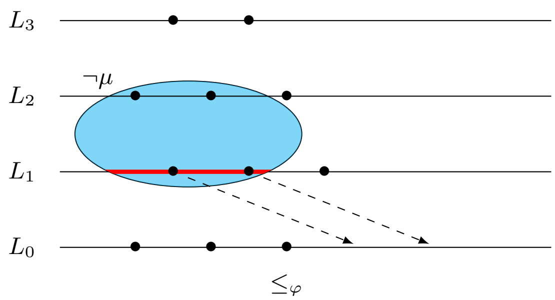

Fig. 1. Contraction of φ by μ

We illustrate the representation theorem on Figure 1. The interpretations (depicted as dots) are located at different levels L_i, two interpretations at the same level are equally plausible (i.e., $I \simeq_\varphi J$) and an interpretation I appearing at a lower level than another J is strictly more plausible (i.e., $I <_\varphi J$). The interpretations appearing at the lowest level (L_0) are the models of the belief base φ.

When φ is contracted by μ, the result consists of all models of φ to which are added to the most plausible models of $\neg\mu$ according to the pre-order of plausibility $\leq_\varphi$ associated with φ by the faithful assignment. This represents the minimal change required for not implying μ any longer. These interpretations are located at L_1 on Figure 1. The minimal interpretations of $\neg\mu$ (at L_1) are added next to the interpretations of φ (at L_0).

8 Conclusion and Perspectives

In this paper we investigated belief contraction in the framework of finite propositional logic. The aim was, like in Katsuno and Mendelzon work for revision, to define postulates for contraction operators. We have checked that the operators of contraction

[3] We thank a reviewer for pointing this paper to us.

characterized by our postulates correspond to the revision operators characterized by Katsuno and Mendelzon postulates. We have also given a representation theorem in terms of faithful assignment.

The aim of this work was to ensure that the translation of the AGM contraction in the finite propositional framework offers the expected properties. This is more than a technical exercice, since this step is necessary to define iterated contraction operators, which is the main perspective of this work. Indeed, the translation by Katsuno and Mendelzon of the AGM postulates is the basis of the study of iterated revision operators following Darwiche and Pearl [4, 2, 10, 12]. There has been very few work on iterated contraction: to the best of our knowledge, only one paper [9] addresses this problem, but in a different framework from the one of Darwiche and Pearl. Defining "Darwiche and Pearl"-like iterated contraction operators will be a first step in the investigation of the relationships between [9] and [4].

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