Propositional merging operators based on set-theoretic closeness

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Abstract. In the propositional setting, a well-studied family of merging operators are distance-based ones: the models of the merged base are the closest interpretations to the given profile. Closeness is, in this context, measured as a number resulting from the aggregation of the distances to each base of the profile. In this work we define a new familly of propositional merging operators, close to such distance-based merging operators, but relying on a set-theoretic definition of closeness, already at work in several revision/update operators from the literature. We study a specific merging operator of this family, obtained by considering set-product as the aggregation function.

1 Introduction

Information merging is a very important task in artificial intelligence: the issue is to determine the beliefs, or the goals, of a group of agents from their individual points of view. Much work has been devoted to the definition of merging operators in the propositional case [11, 9, 1, 8, 10].

In [8] a set of postulates is proposed to characterize different families of merging operators, and several families of operators satisfying those postulates are defined. Such operators are called model-based merging operators because basically they select the models of a given integrity constraint (i.e. a formula encoding laws, norms, etc., used for constraining the result of the merging) that are the closest ones to the given profile of belief/goal bases of the group. Often, those operators are defined from a distance between interpretations, which intuitively indicates how conflicting they are. This distance between interpretations induce a distance between an interpretation and a base, which indicates how plausible/satisfactory the interpretation is with respect to the base. Once such distances are computed, an aggregation function is used to define the overall distance of each model (of the integrity constraints) to the profile. Semantically, the models of the result of the merging are the closest models of the integrity constraints to the profile.

A commonly-used distance between interpretations is the Hamming distance (also called Dalal distance [3]). The Hamming distance between two interpretations is the number of propositional variables the two interpretations disagree on. The amount of conflict between two interpretations is thus assessed as the number of atoms whose truth values must be flipped in one interpretation in order to make it identical to the second one. Such a distance is very meaningful when no extra-information on the epistemic states of the agents are available.

The major problem with distance-based merging operators is that evaluating the closeness between two interpretations as a number may lead to lose too many information.Thus, the conflicting variables themselves (and not only how many they are) can prove significant. Especially, when variables express real-world properties, it can be the case that some variables are more important than others, or that some variables are logically connected. In those cases, distances are not fully satisfactory.

As an alternative to distance, an interesting measure used to evaluate the closeness of two interpretations is diff, the symetrical difference between them. Instead of evaluating the degree of conflict between two interpretations as the number of variables on which they differ (as it is the case with the Hamming distance), the diff measure assesses it as the set of such variables.

In this work, we consider the family of propositional merging operators based on the diff measure. We specifically focus on the operator $\Delta^{\mathsf{diff},\oplus}$ from this family obtained by considering set-product as the aggregation function. We evaluate it with respect to three criteria: logical properties, strategy-proofness and complexity.

2 A Diff-Based Merging Operator: $\Delta^{\mathsf{diff},\oplus}$

The key idea underlying our approach consists in evaluating the degree of conflict between two interpretations ω and ω' as the set of variables on which they differ:

$$\mathsf{diff}(\omega,\omega') = \{p \in \mathcal{P} \mid \omega(p) \neq \omega'(p)\}.$$

This definition has already been used in the belief revision/update literature in order to define a number of operators [6, 13, 12, 2, 14].

As for distances, we can straightforwardly define, using diff, a notion of closeness between an interpretation and a base, as the minimum closeness between the interpretation and the models of the base. Of course, since diff gives as output a set instead of a number, set-inclusion has to be considered as minimality criterion:

$$\mathsf{diff}(\omega, K) = min(\{\mathsf{diff}(\omega,\omega') \mid \omega' \models K\}, \subseteq).$$

So the closeness between an interpretation ω and a base K is measured as the set of the minimal sets (for set inclusion) of propositional variables which have to be flipped in ω to make it a model of K.

Now, we need to aggregate those measures in order to define a global notion of closeness between an interpretation and a profile. This is the aim of the aggregation functions. Of course, usual functions at work for distance-based operators cannot be used here simply because we do not deal with numbers, but with sets.

Several aggregation functions can be considered in our setting. For space reasons, we focus on a single one in this paper. We consider

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set-product $\oplus$ as an aggregation function: for two sets of sets E and E', $E \oplus E' = \{c \cup c' \mid c \in E \text{ and } c' \in E'\}$.

Definition 1 *Let $E = \{K_1, \ldots, K_n\}$ be a profile and ω an interpretation. The closeness between ω and E is given by:*

$$\mathsf{diff}(\omega, E) = \min(\{\oplus_{K_i \in E}\, \mathsf{diff}(\omega, K_i)\}, \subseteq).$$

By construction, each element of $\mathsf{diff}(\omega, E)$ is a minimal set c of variables (a conflict set) such that for each base K_i, ω can be transformed into a model of K_i by flipping in ω the variables of c.

Finally, we define a merging operator $\Delta^{\mathsf{diff},\oplus}$ which picks up the models of the integrity constraints whose closeness to the profile E contains at least one of the minimal (w.r.t. $\subseteq$) conflict set:

Definition 2 *Let $E = \{K_1, K_2, \ldots, K_n\}$ be a profile, μ an integrity constraint. Then $\mathsf{diff}_\mu(E) = \min(\{\mathsf{diff}(\omega, E) \mid \omega \models \mu\}, \subseteq)$ and $[\Delta^{\mathsf{diff},\oplus}_\mu(E)] = \{\omega \models \mu \mid \exists c \in \mathsf{diff}(\omega, E) \text{ s.t. } c \in \mathsf{diff}_\mu(E)\}$.*

3 Properties of $\Delta^{\mathsf{diff},\oplus}$

$\Delta^{\mathsf{diff},\oplus}$ satisfies most of the logical properties proposed in [8]:

Proposition 1 *$\Delta^{\mathsf{diff},\oplus}$ satisfies* **(IC0)***,* **(IC1)***,* **(IC2)***,* **(IC3)***,* **(IC4)** *and* **(IC7)***. It does not satisfy* **(IC5)***,* **(IC6)** *and* **(IC8)***.*

$\Delta^{\mathsf{diff},\oplus}$ does not satisfy **(IC5)** and **(IC6)**, which are postulates capturing aggregation properties. This is not surprising since, unlike distance-based operators (as the ones based on Hamming distance), $\Delta^{\mathsf{diff},\oplus}$ keeps a justification of the minimality of an interpretation (as a conflict set).

Beyond the IC postulates, $\Delta^{\mathsf{diff},\oplus}$ satisfies also an interesting additional logical property:

Definition 3 *A merging operator Δ satisfies the* temperance *property iff for every profile $\{K_1, \ldots, K_n\}$:*
$\Delta_\top(\{K_1, \ldots, K_n\})$ is consistent with each K_i **(temperance)**

Proposition 2 *$\Delta^{\mathsf{diff},\oplus}$ satisfies* **(temperance)***.*

This proposition shows that the merged base obtained using $\Delta^{\mathsf{diff},\oplus}$ is consistent with every base of the profile (when there is no integrity constraint). This proposition also gives an additional explanation to the fact that $\Delta^{\mathsf{diff},\oplus}$ does not satisfy **(IC6)**, since temperance is not compatible with this postulate.

Proposition 3 *There is no merging operator satisfying both* **(IC2)***,* **(IC6)***, and* **(temperance)***.*

It is worth noting that the temperance property is not satisfied by many merging operators. In particular, as implied by the previous proposition, none of the IC merging operators satisfies **(temperance)**. Interestingly, the temperance property shows that $\Delta^{\mathsf{diff},\oplus}$ can be viewed as a kind of negotiation operator, which can be used for determining the most consensual parts of the bases of all agents.

Let us now investigate how robust $\Delta^{\mathsf{diff},\oplus}$ is with respect to manipulation. Intuitively, a merging operator is strategy-proof if and only if, given the beliefs/goals of the other agents, reporting untruthful beliefs/goals does not enable an agent to improve her satisfaction. A formal counterpart of this idea is given in [4, 5]:

Proposition 4 *In the general case $\Delta^{\mathsf{diff},\oplus}$ is not strategy-proof for any of the three indexes i_{d_w}, i_{d_s} and i_p. When there is no integrity constraint (i.e., $\mu \equiv \top$), $\Delta^{\mathsf{diff},\oplus}$ is strategy-proof for i_{d_w}, but still not strategy-proof for i_{d_s} or i_p.*

Most of the model-based operators are not strategy-proof, even in very restricted situations [5]. For example, $\Delta^{d_H,\Sigma}$ or $\Delta^{d_H,Gmin}$, which are the best model-based operators with respect to strategy-proofness, are not strategy-proof for i_{d_w}, even if $\mu \equiv \top$. $\Delta^{\mathsf{diff},\oplus}$ performs better than any of them with this respect.

Let us consider now the complexity issue for the inference problem from a $\Delta^{\mathsf{diff},\oplus}$-merged base.

Proposition 5 MERGE($\Delta^{\mathsf{diff},\oplus}$) *is Π^p_2-complete. Hardness still holds under the restriction where E contains a single base K consisting of a conjunction of propositional variables, and α is a propositional variable.*

This result shows that $\Delta^{\mathsf{diff},\oplus}$ is computationally harder than usual distance-based operators, but is at the same complexity level as many formula-based operators [7].

4 Conclusion

In this work we have introduced a family of model-based merging operators, relying on a set-theoretic measure of conflict. We focused on set-product as an aggregation function and considered the corresponding operator $\Delta^{\mathsf{diff},\oplus}$. A feature of this operator, typically not shared by existing model-based operators, is that it satisfies the temperance property, and as a consequence, it is strategy-proof for the weak drastic index when there are no integrity constraints. The price to be paid is a higher complexity than usual model-based operators (but similar to the one of formula-based merging operators [5]).

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